TD 1 - Temporal point processes

Poisson process

1 Basic exercises

Exercise 1. Let T be a positive continuous random variable. Show that T has the memoryless property, i.e.

$$\forall t, s \geq 0, \quad \mathbb{P}(T > t+s \mid T > t) = \mathbb{P}(T > s),$$

if and only if $T \sim \mathscr{E}(\lambda)$ for some $\lambda > 0$.

Exercise 2. Let T be a positive random variable with density. Show that $T \sim \mathscr{E}(\lambda)$ if and only if its hazard rate function is $q(t) = \lambda$ for all $t > 0$.

Exercise 3. In this exercise, we describe the times at which we receive a spam mail thanks to a non homogeneous Poisson process. We assume that :

- the average rate of spams received during the day (8am - 8pm) is 2 per hour,
- the average rate of spams received during the night (8pm - 8am) is 1 per hour.

1. What is the intensity function of the Poisson process ?
2. What is the probability to receive 0 spam between midnight and 1am ?
3. Given that we received 4 spams between 8am and 9am, what is the probability to receive 2 spams between 1pm and 2pm ?
4. Given that we received 4 spams between 8am and 9am, what is the probability to receive 5 spams between 8am and 10am ?
5. Given that we received 10 spams between 8am and 8pm, what is the probability to have received 0 spam between 1pm and 2pm ?
6. Given that we received 1 spam between 6pm et 8pm, what is the probability to receive 1 spam between 7pm and 9pm ?

Exercise 4. Let N be a non homogeneous Poisson process with intensity $\lambda(t)$. The two following questions are independent.

1. Show that $\lambda(t) = 0$ on $[s, u]$ implies that $N([s, u]) = 0$ almost surely.
2. Let $t \geq 0$ and assume that λ is right continuous at t. Prove that

$$\mathbb{P}(N([t, t+dt]) \geq 1) \sim_{dt \to 0^+} \lambda(t) dt$$

2 Intermediate exercises

Exercise 5. Write an algorithm which gives the simulation of the n first points of a homogeneous Poisson process via its renewal structure :

> Let $(S_1, \ldots, S_n)$ be a vector of i.i.d. random variables distributed according to $\mathscr{E}(\lambda)$. Define $T_k = \sum_{i=1}^{k} S_i$ for all $k = 1, \ldots, n$. Then, the vector $(T_1, \ldots, T_n)$ is distributed as the n first point of a Poisson process with rate λ.

Moreover : make a modified version of your algorithm such that it gives the simulation of a homogeneous Poisson process on $[0,T]$ for a fixed time horizon T.

1. Compare the trajectories of Poisson processes of rate $\lambda = 1/2$, $\lambda = 1$ and $\lambda = 2$. *Remark : Two graphical representations of point processes are standard : 1) the graph of the associated counting process, 2) a raster plot, i.e. a series of points plotted on the x-axis (more interesting for a multidimensional point process where the y-axis can describe the several dimensions).*
2. Let N denote a Poisson process with rate $\lambda = 2$.
   (a) Illustrate the fact that $\mathbb{E}[T_2] = 1$ by computing the empirical mean over a large number of realizations of N.
   (b) Illustrate the fact that $\mathbb{E}[N_{10}] = 20$ by computing the empirical mean over a large number of realizations of N.

Exercise 6. Write an algorithm which gives the simulation of a non homogeneous Poisson process on $[0,T]$ via Lewis-Shedler thinning method.

Exercise 7. Write an algorithm which gives the simulation of a non homogeneous Poisson process on $[0,T]$ via its conditional structure :

> Let $\lambda : [0,T] \to \mathbb{R}_+$ be a measurable function and denote $\Lambda(t) = \int_0^t \lambda(s)ds$. Let N be a Poisson process with intensity $\lambda(t)$. The total number of points of N, denoted N_T, is distributed according to $\mathscr{P}(\Lambda(T))$. Given that $N_T = n$, the n points of the process are distributed according to a vector of n i.i.d. real random variables with density
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> $$f(t) = \frac{\lambda(t)}{\Lambda(T)} \mathbf{1}_{[0,T]}(t).$$

Let $C, T, \alpha, \beta > 0$ and denote

$$\lambda(t) = C \cdot (t/T)^{\alpha}(1 - t/T)^{\beta}.$$

1. Compare the trajectories of Poisson processes with intensity $\lambda(t)$ with several values of C, T, α and β. *Remark : This form of intensity λ is linked with the Beta distribution.*

Exercise 8. Let $N = \{T_1 < \cdots < T_k < \ldots\}$ be a Poisson process with intensity $\lambda > 0$. The objective is to prove that N is a renewal process with inter event time distribution $\mathscr{E}(\lambda)$.

Let $n \in \mathbb{N}^*$, $0 < t_1 < \cdots < t_n < t_{n+1} = T$ and $h_1, \ldots, h_n > 0$ such that $t_k + h_k < t_{k+1}$.

1. Express the event
$$\{t_1 < T_1 \leq t_1 + h_1 < t_2 < \cdots < t_n < T_n \leq t_n + h_n\}$$
thanks to the counting process $(N_t)_{t \in [0,T]}$.

2. Prove that
$$\frac{\mathbb{P}\left(t_1 < T_1 \leq t_1 + h_1 < t_2 < \dots T_n \leq t_n + h_n\right)}{h_1 \dots h_n} \xrightarrow[h_1,\dots,h_n \to 0]{} \lambda^n e^{-\lambda t_n}. \tag{1}$$

Remark: Since $0 < t_1 < \dots < t_n$ and $h_1, \dots, h_n > 0$ are generic, Equation (1) proves[1] that $(T_1, \dots, T_n)$ follows the density $f(t_1, \dots, t_n) = \lambda^n e^{-\lambda t_n} \mathbf{1}_{0 < t_1 < \dots < t_n}$.

3. Prove that the inter event intervals $(S_n)_{n \geq 1}$ are independent and distributed as $\mathscr{E}(\lambda)$. *Hint: Change of variable*

Exercise 9. Let $(S_n)_{n \geq 1}$ be a sequence of i.i.d. r.v. distributed according to $\mathscr{E}(\lambda)$. Denote $T_n = \sum_{i=1}^N S_i$ and $N = \{T_n, n \in \mathbb{N}\}$ the associated point process. The objective is to prove that N is a Poisson process with intensity λ.

1. Prove that $(T_1, \dots, T_n)$ has density $f(t_1, \dots, t_n) = \lambda^n e^{-\lambda t_n} \mathbf{1}_{0 < t_1 < \dots < t_n}$.
2. Let $0 \leq s \leq t$ and $k, \ell \in \mathbb{N}$. Express the event $\{N_s = k, N_t - N_s = \ell\}$ thanks to the random variables $(T_n)_{n \geq 1}$.
3. Deduce that N_s and $N_t - N_s$ are independent and that $N_t - N_s \sim \mathscr{P}(\lambda(t-s))$.

Remark: This argument can be generalized from 2 inter event intervals to a generic number. Hence, it follows that N is a Poisson process with intensity λ.

Exercise 10. Let N be a non homogeneous Poisson process with right continuous intensity $\lambda(t)$ on $[0,T]$. Denote $\Lambda(t) = \int_0^t \lambda(s) ds$.

1. Following the lines of Exercise 8, prove that for all $0 < t_1 < \dots < t_n < T$,
$$\frac{\mathbb{P}\left(N_T = n, t_1 < T_1 \leq t_1 + h_1 < t_2 < \dots T_n \leq t_n + h_n\right)}{h_1 \dots h_n} \xrightarrow[h_1,\dots,h_n \to 0]{} \prod_{i=1}^n \lambda(t_i) e^{-\Lambda(T)}. \tag{2}$$

2. Deduce that
$$\frac{\mathbb{P}\left(t_1 < T_1 \leq t_1 + h_1 < t_2 < \dots T_n \leq t_n + h_n | N_T = n\right)}{h_1 \dots h_n} \xrightarrow[h_1,\dots,h_n \to 0]{} n! \prod_{i=1}^n f(t_i),$$
where $f(t) = \frac{\lambda(t)}{\Lambda(T)}$.

Remark: Since $0 < t_1 < \dots < t_n$ and $h_1, \dots, h_n > 0$ are generic, it proves that, conditionally on $\{N_T = n\}$, the density of $(T_1, \dots, T_n)$ with respect to the Lebesgue measure on $\mathbb{R}^n$ is given by
$$(t_1, \dots, t_n) \mapsto n! \prod_{i=1}^n f(t_i) \mathbf{1}_{t_1 < \dots < t_n < T}.$$

This is the density of the order statistics of n i.i.d. variables distributed according to the density f.

Exercise 11. Let N^1 and N^2 be two independent Poisson processes with intensities $\lambda^1(t)$ and $\lambda^2(t)$. The objective is to prove that $N_t = N_t^1 + N_t^2$ defines a Poisson process with intensity $\lambda(t) = \lambda^1(t) + \lambda^2(t)$. Let us define $\Lambda(t) = \int_0^t \lambda(s) ds$.

1. Let $0 < s < t$ and $k, \ell \in \mathbb{N}$. Compute the probability $\mathbb{P}\left(N_s = k, N_t - N_s = \ell\right)$.

[1] The proof relies on the monotone class Theorem

2. Deduce that N_s and $N_t - N_s$ are independent with respective distributions $\mathscr{P}(\Lambda(s))$ and $\mathscr{P}(\Lambda(t) - \Lambda(s))$.

Remark: This argument can be generalized from 2 inter event intervals to a generic number. Hence, it follows that N is a Poisson process with the prescribed intensity.

Exercise 12. Let N be a Poisson process with rate λ and $p \in]0,1[$. Let $(\varepsilon_n)_n$ be a sequence of i.i.d. random variables distributed as $\mathscr{B}(p)$ which is furthermore independent of N. Let

$$N^0 = \{T_i \in N,\, \varepsilon_i = 0\} \text{ and } N^1 = \{T_i \in N,\, \varepsilon_i = 1\}.$$

1. Let $t \geq 0$. Prove that N_t^0 and N_t^1 are two independent random variables distributed as $\mathscr{P}((1-p)\lambda t)$ and $\mathscr{P}(p\lambda t)$ respectively.

Remark: More generally, one can use the time independence of the Poisson processes to prove that N^0 and N^1 are two independent Poisson processes with rates $(1-p)\lambda$ and $p\lambda$ respectively.

3 Advanced exercises

Exercise 13. Let us define the *Poisson mixture* as follows. Let $\tilde{\lambda}$ be a positive random variable and denote $\tilde{P}$ its distribution. The Poisson process mixture N with rate $\tilde{\lambda}$ satisfies for instance

$$\mathbb{P}(N_t = k) = \int_\lambda \frac{(\lambda t)^k}{k!} e^{-\lambda t} d\tilde{P}(\lambda).$$

Prove that :

1. $\mathbb{E}[N_t] = \mathbb{E}\left[\tilde{\lambda}\right] t$,
2. $\mathrm{Var}(N_t) = \mathbb{E}\left[\tilde{\lambda}\right] t + \mathrm{Var}(\tilde{\lambda}) t^2 \geq \mathbb{E}[N_t]$ with strict inequality unless $\tilde{\lambda}$ is a.s. constant.

Exercise 14. Let us define the *compound Poisson process* as follows. Let $Y, Y_1, Y_2, \dots$ be i.i.d. non-negative integer-valued random variables with probability generating function $g(z) = \mathbb{E}\left[z^Y\right]$ ($|z| \leq 1$) and let them be independent of a Poisson process N of rate λ. Then, define the compound counting process as

$$X_t = \sum_{i=1}^{N_t} Y_i.$$

Prove that :

1. $\mathbb{E}\left[z^{X_t}\right] = \exp(\lambda t(g(z) - 1))$,
2. $\mathbb{E}[X_t] = \lambda t \mathbb{E}[Y]$,
3. $\mathrm{Var}(X_t) = \lambda t \mathbb{E}\left[Y^2\right] \geq \mathbb{E}[X_t]$ with strict inequality unless $Y = 0$ or 1 a.s.