Spiking neural models: from point processes to partial differential equations.

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2016/06/09

Outline

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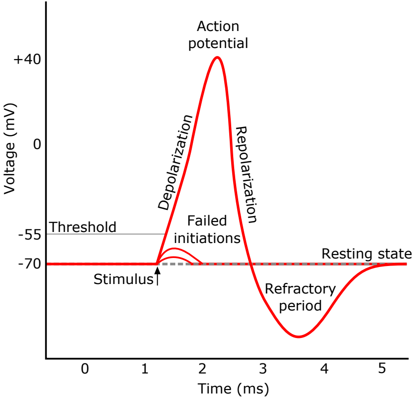
1. Introduction
    - Neurobiologic context
    - Microscopic modelling
    - Macroscopic modelling
2. A key tool: The thinning procedure
3. First approach: Mathematical expectation
4. Second approach: Mean-field interactions

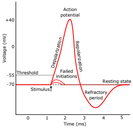
Biological context



- Neurons = electrically excitable cells.
- Action potential = spike of the electrical potential.
- Physiological constraint: refractory period.

Biological context

microscopic scale



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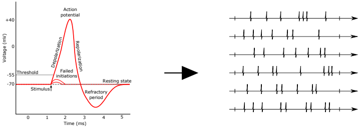

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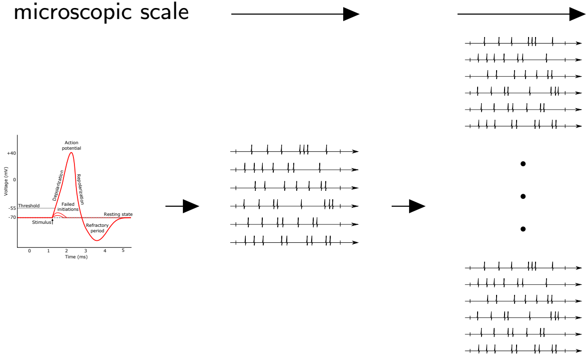

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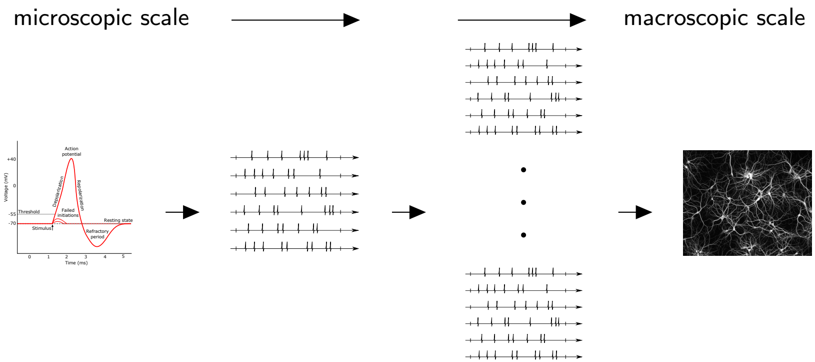
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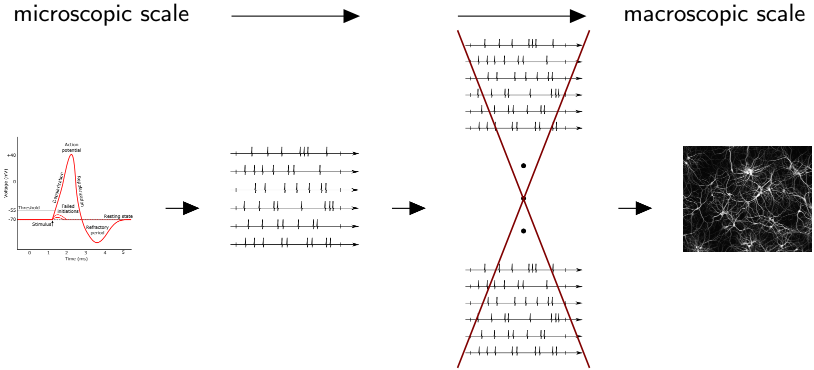

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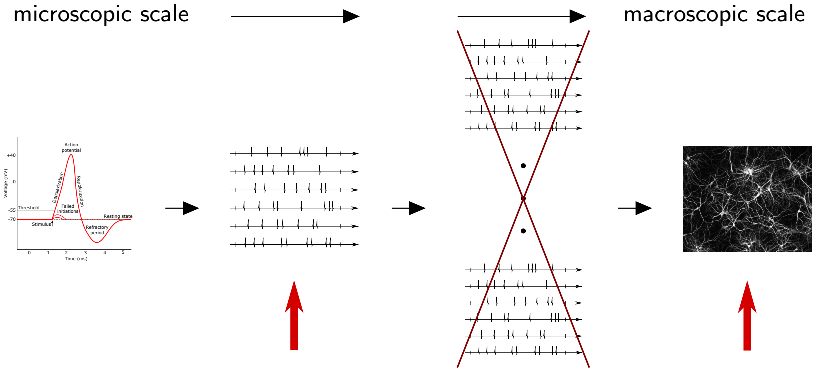
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Microscopic modelling

Microscopic modelling of spike trains

Time point processes = random countable sets of times (points of $\mathbb{R}$ or $\mathbb{R}_+$).

- Point process: $N = \{T_i, i \in \mathbb{Z}\}$ s.t. $\cdots < T_0 \leq 0 < T_1 < \cdots$.
- Point measure: $N(dt) = \sum_{i\in\mathbb{Z}} \delta_{T_i}(dt)$. Hence, $\int f(t)N(dt) = \sum_{i\in\mathbb{Z}} f(T_i)$.

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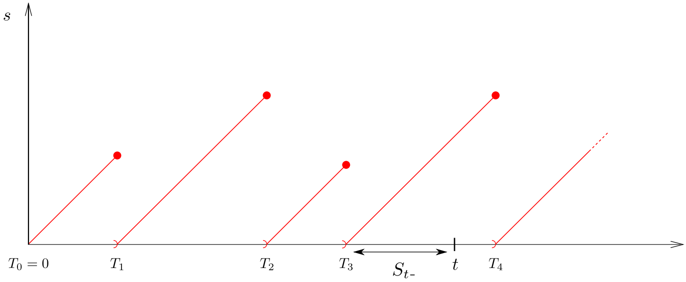

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Stochastic intensity

- Heuristically,

$$\lambda_t = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \mathbb{P}\left(N([t, t+\Delta t]) = 1 \,|\, \mathscr{F}^N_{t-} \right),$$

  where $\mathscr{F}^N_{t-}$ denotes the history of N before time t.
- Local behaviour: probability to find a new spike.
- May depend on the past (e.g. refractory period, aftershocks).

Some classical point processes in neuroscience

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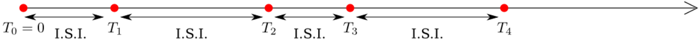

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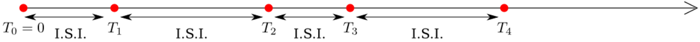

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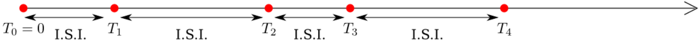

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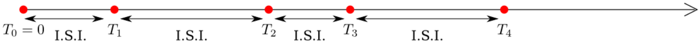
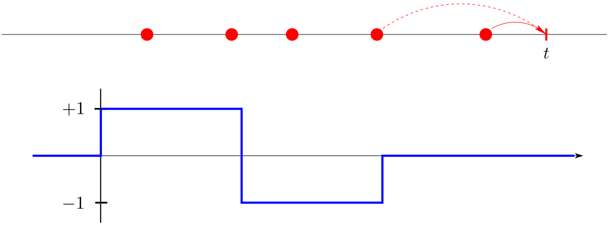


- Linear Hawkes process: $$\lambda_t = \mu + \underbrace{\int_0^{t-} {\color{blue}h}(t-z){\color{red}N(dz)}}_{\sum_{\substack{T\in N\\ T<t}} {\color{blue}h}(t-T)}, \qquad h \geq 0.$$



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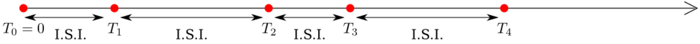

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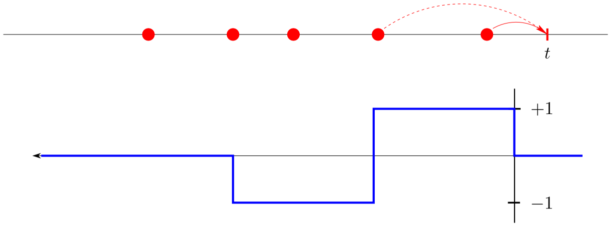

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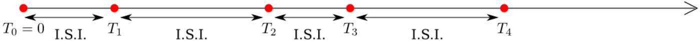

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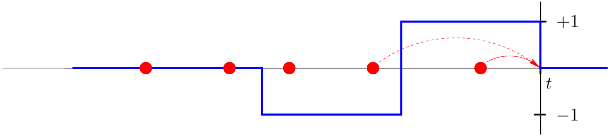

Age structured equations (K. Pakdaman, B. Perthame, D. Salort, 2010)

- Age = delay since last spike.
- $n(t,s) = \begin{cases} \text{probability density of finding a neuron with age } s \text{ at time } t. \\ \text{ratio of the neural population with age } s \text{ at time } t. \end{cases}$

$$\text{mean firing rate} \rightarrow \begin{cases} \dfrac{\partial n(t,s)}{\partial t} + \dfrac{\partial n(t,s)}{\partial s} + p\left(s, X\left(t\right)\right) n\left(t,s\right) = 0 \\ n(t,0) = \displaystyle\int_0^{+\infty} p\left(s, X\left(t\right)\right) n\left(t,s\right) ds. \end{cases} \qquad \text{(PPS)}$$

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Parameters

- rate function p. For example, $p(s,X) = \mathbb{1}_{\{s>\sigma(X)\}}$.

$$X(t) = \int_0^t d(t-x)n(x,0)dx \quad \text{(global neural activity)}$$

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Cornerstone: $X(t) \longleftrightarrow \displaystyle\int_0^{t-} h(t-x) N(dx)$.

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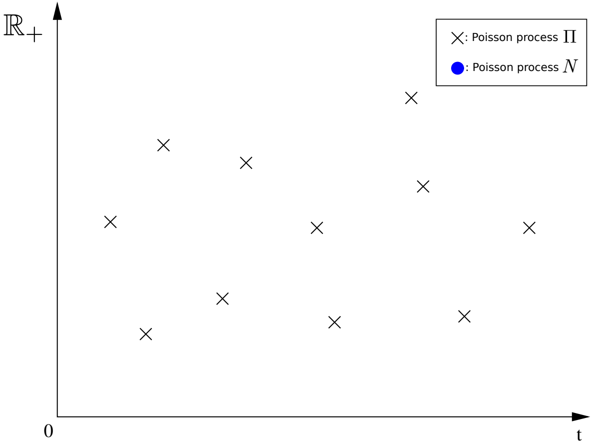
Lewis and Shedler's Thinning, 1979



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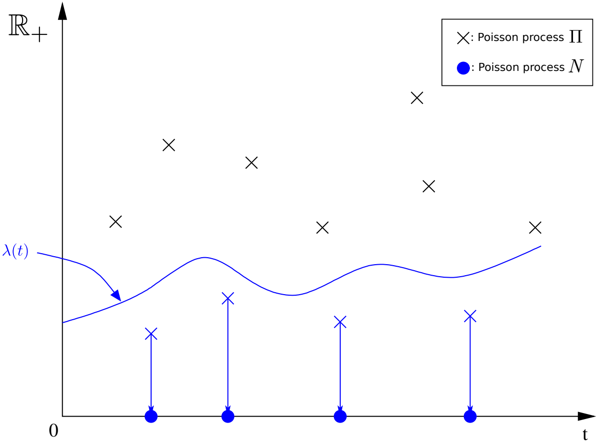

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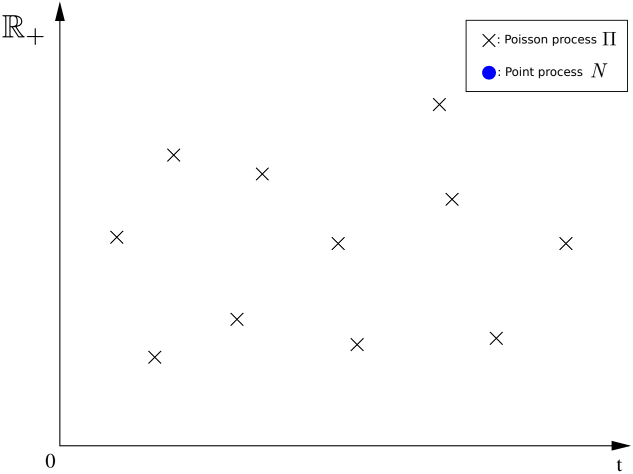

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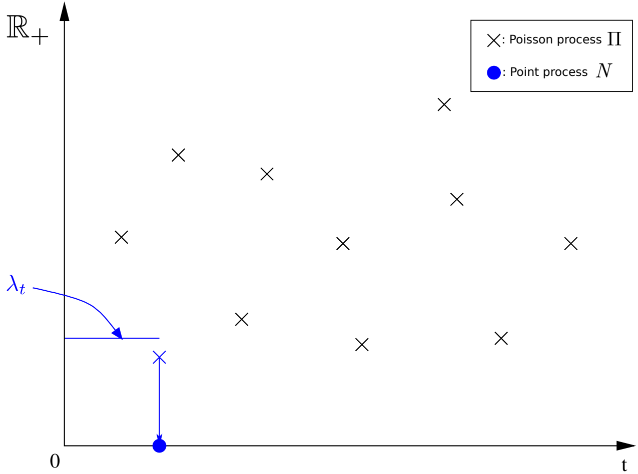

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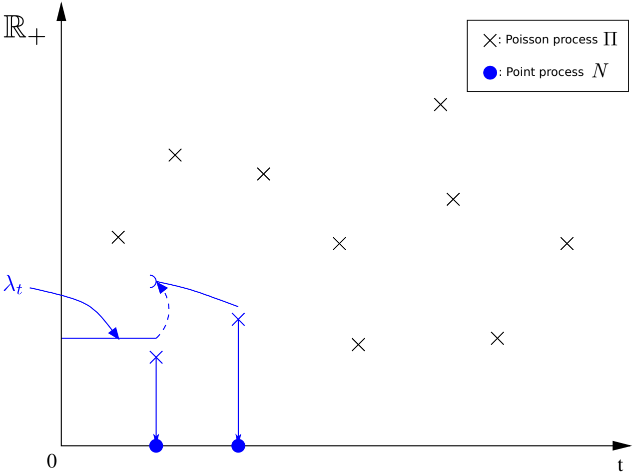

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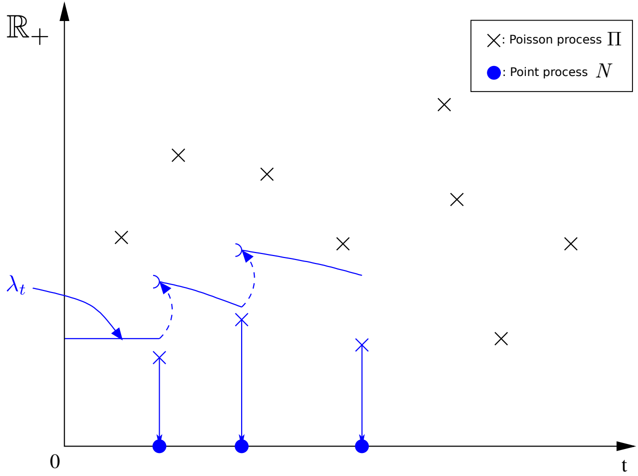

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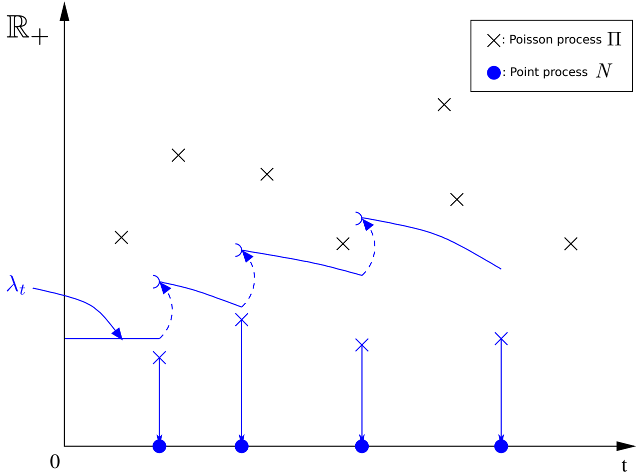

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- Non-Markovian case

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- Fokker-Planck equation gives the following PDE system:

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where $u(t,\cdot)$ is the distribution of S_{t-}.

System in expectation

Theorem (C., Caceres, Doumic, Reynaud-Bouret 15)

Let λ_t be some non negative predictable process which is L^1_{loc} in expectation. The distribution of S_{t-}, namely $u(t,\cdot)$, satisfies the following system,

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in the weak sense where $\rho_{\lambda,\mathbb{P}_0}(t,s)=\mathbb{E}\left[\lambda_t|S_{t-}=s\right]$ for almost every t.

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- Non-markovian $\Rightarrow$ $\rho_{\lambda,\mathbb{P}_0}(t,s)$ more complex.
- Linear Hawkes process: closed system for $v(t,s):=\int_s^{+\infty}u(t,\sigma)d\sigma$.

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Propagation of chaos: a tool to link the two scales

Mean field n-neurons system

- Weak dependence: homogeneous interactions scaled by $1/n$.
- Symmetry: the neurons are exchangeable.
- The dynamics is described by a growing system of equations.

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Mean-field

- Neuroscience: Intrinsic spiking (Stannat et al. 2014), I&F (Delarue et al. 2015), point processes models (Galves and Löcherbach 2015).
- Hawkes: Mean field approximation (Delattre et al., 2015), inference (Delattre et al., Bacry et al. 2016).

Propagation of chaos: a tool to link the two scales

Mean field n-neurons system

- Weak dependence: homogeneous interactions scaled by $1/n$.
- Symmetry: the neurons are exchangeable.
- The dynamics is described by a growing system of equations.

Asymptotic when $n \to +\infty$

- The neurons become independent (they are i.d.).
- Their distribution is described by one non-linear PDE.

Mean-field

- Neuroscience: Intrinsic spiking (Stannat et al. 2014), I&F (Delarue et al. 2015), point processes models (Galves and Löcherbach 2015).
- Hawkes: Mean field approximation (Delattre et al., 2015), inference (Delattre et al., Bacry et al. 2016).

- Here: Age dependent Hawkes processes.

Multivariate Hawkes processes

- Multivariate HP: $(i = 1, \ldots, n)$

$$\lambda_t^i = \Phi\left(\int_0^{t-} h_{i\to i}(t-x) N^i(dx) + \sum_{j\neq i} \int_0^{t-} h_{j\to i}(t-x) N^j(dx) \right).$$

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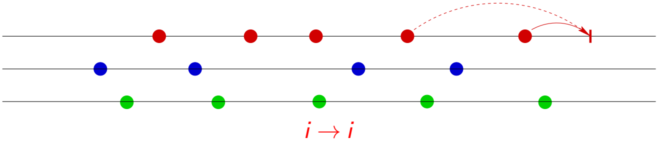

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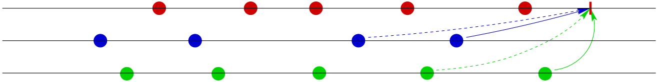

$j \neq i$

Interaction function $h_{j\to i}$ ↔ synaptic weight of neuron j over neuron i.

Generalized Hawkes processes

Renewal process

$$\lambda_t = f(S_{t-})$$

Multivariate HP

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Age dependent Hawkes process (n-neurons system)

It is a multivariate point process $(N^i)_{i=1,..,n}$ with intensity given for all $i = 1, \dots, n$ by

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- How to approximate them as $n \to +\infty$?
- LLN heuristics: they are close to independent copies of a *limit process*.

Scheme of the coupling method

Idea of coupling (Sznitman)

The idea is to find a suitable coupling between the particles of the n-particle system and n i.i.d. copies of a *limit process*.

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1' Use the PDE to find the distribution of the limit process.

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Recall the intensities of the *n*-neurons system

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- The blue terms should be close one from the other.
- The process $\overline{N}$ depends on its own distribution (McKean-Vlasov equation). Its existence is not trivial.
- The intensity of $\overline{N}$ depends on the time and the age ⇒ $\overline{S}_{t-}$ is Markovian.

1'/ Study the associated PDE system 1

If the limit process $\overline{N}$ exists, then the distribution of $\overline{S}_{t-}$, denoted by $u(t,\cdot)$ satisfies (Fokker-Planck equation):

$$\begin{cases} \dfrac{\partial u(t,s)}{\partial t} + \dfrac{\partial u(t,s)}{\partial s} + \Psi(s, X(t))\, u(t,s) = 0, \\ u(t,0) = \displaystyle\int_{s\in\mathbb{R}_+} \Psi(s, X(t))\, u(t,s)\, ds, \end{cases} \qquad \text{(PPS-NL)}$$

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Theorem (C. 15)

Assume that $h : \mathbb{R}_+ \to \mathbb{R}$ is locally integrable and that u^{in} is a non-negative function such that both $\int_0^{+\infty} u^{\mathrm{in}}(s) ds = 1$ and there exists $M > 0$ such that for all $s \geq 0$, $0 \leq u^{\mathrm{in}}(s) \leq M$.
Then, there exists a unique solution in the weak sense u such that $t \mapsto u(t,\cdot)$ belongs to $BC(\mathbb{R}_+, \mathscr{P}(\mathbb{R}_+))$ (Moreover, the solution is in $C(\mathbb{R}_+, L^1(\mathbb{R}_+))$.

2/ Show that the limit process is well-posed

Recall the intensity of the limit process

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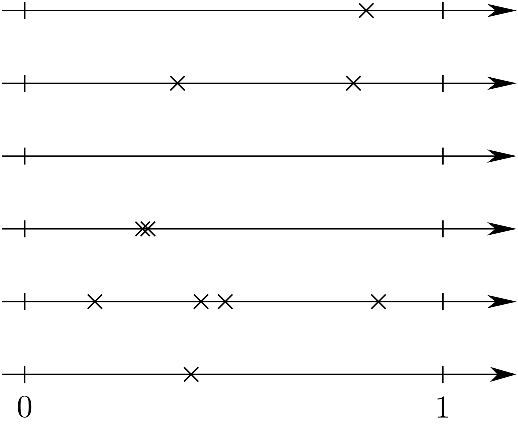
- Hence the limit process is well-defined.

3/ The coupling

Six realizations of a Poisson process with intensity 2 on $[0, 1]$.



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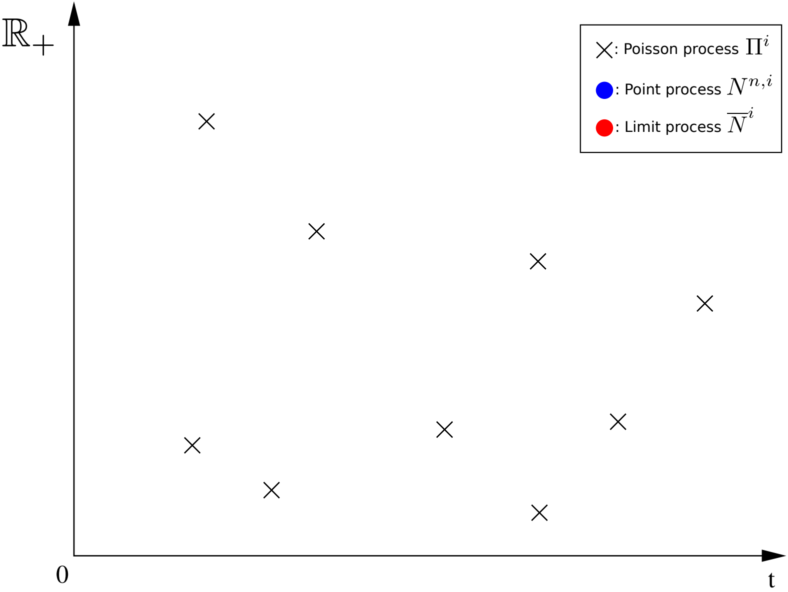

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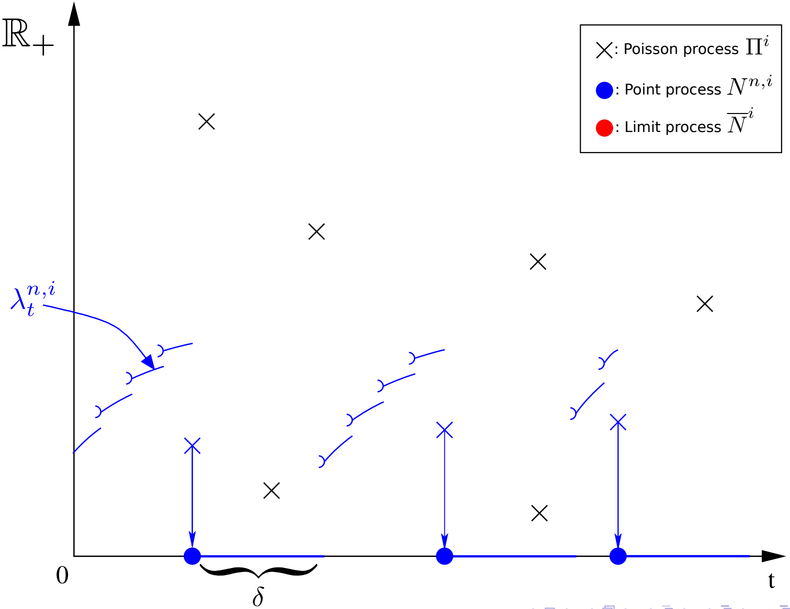

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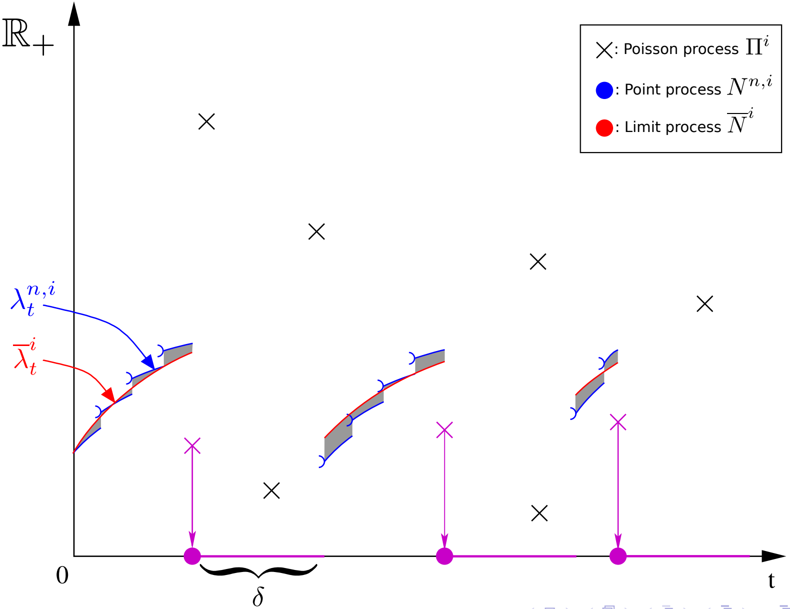

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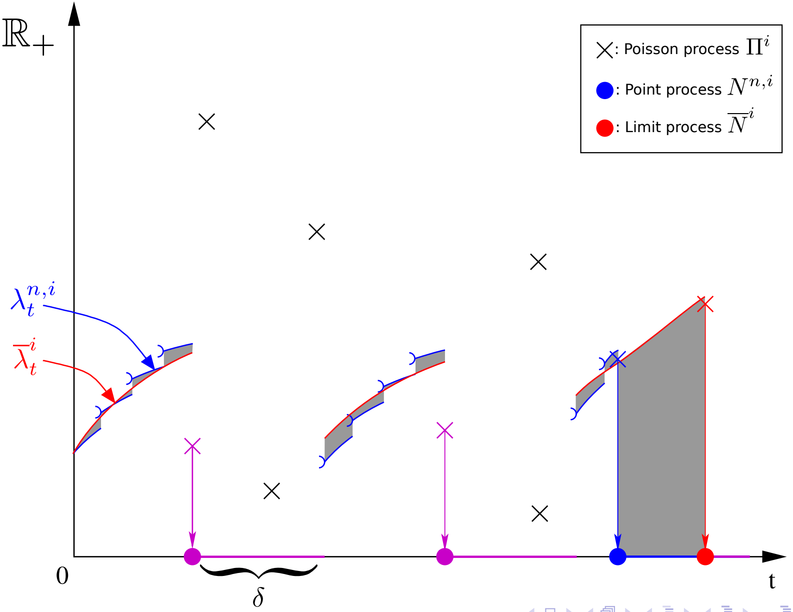

4/ Control/Convergence 1

Theorem (C. 15)

The coupling described in the previous slide is such that

$$\mathbb{E}\left[\underbrace{Card((N^i \triangle \overline{N}^i)\cap[0,\theta])}_{number\ of\ \times\ in}\right] = \mathbb{E}\left[\underbrace{\int_0^\theta |\lambda_t^i - \overline{\lambda}_t^i| dt}_{area\ of}\right] \lesssim n^{-1/2}.$$

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Corollary

If the distribution of the initial value of the age is bounded then the coupling described in the previous slide is such that

$$\mathbb{P}\left((S_t^i)_{t\in[0,\theta]} \neq (\overline{S}_t^i)_{t\in[0,\theta]}\right) \lesssim n^{-1/2}.$$

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Fix k in $\mathbb{N}$. If the initial conditions are i.i.d., then the processes $N^1, \dots, N^k$ of the n-neurons system behave (when $n \to +\infty$) as i.i.d. copies of the limit process $\overline{N}$.

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- Link between (PPS) and a well-designed microscopic model.
- Goodness-of fit tests: Renewal and Hawkes processes.

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  - Link with an i.i.d. network.
  - Ends up with (PPS) for Renewal or Poisson processes.
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  - Break independence with correlated synaptic weights (cf Faugeras and Maclaurin, 2014).