Microscopic approach of a time elapsed neural model

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Outline

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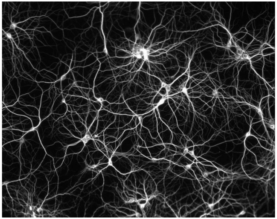
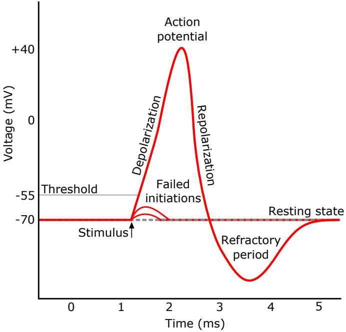
Biological context





- Action potential: brief and stereotyped phenomenon.
- Physiological constraint: refractory period.

Age structured equations (K. Pakdaman, B. Perthame, D. Salort, 2010)

- Age = delay since last spike.
- $n(t,s) = \begin{cases} \text{probability density of finding a neuron with age } s \text{ at time } t. \\ \text{ratio of the population with age } s \text{ at time } t. \end{cases}$

$$\begin{cases} \frac{\partial n(t,s)}{\partial t} + \frac{\partial n(t,s)}{\partial s} + p(s, X(t))\, n(t,s) = 0 \\ m(t) := n(t,0) = \int_0^{+\infty} p(s, X(t))\, n(t,s)\, ds \end{cases} \qquad \text{(PPS)}$$

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$$X(t) = \int_0^t d(x)m(t-x)dx \quad \text{(global neural activity)}$$

- Propagation time.
- d = delay function. For example, $d(x) = e^{-\tau x}$.

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- N is a random countable set of points of $\mathbb{R}$ (or $\mathbb{R}_+$) locally finite a.s.
- Denote $\dots < T_{-1} < T_0 \leq 0 < T_1 < \dots$ the ordered sequence of points of N.
- $N(A)$ = number of points of N in A.
- Point measure: $N(dt) = \sum_{i\in\mathbb{Z}} \delta_{T_i}(dt)$. Hence, $\int f(t)N(dt) = \sum_{i\in\mathbb{Z}} f(T_i)$.

Age process

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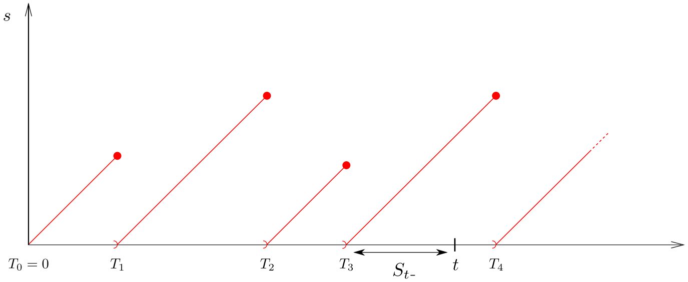

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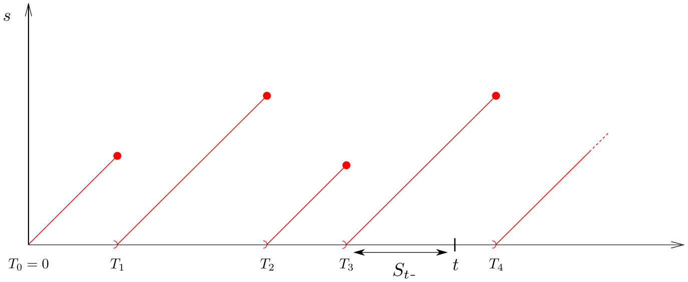

Microscopic age

- We consider the continuous to the left (hence predictable) version of the age.
- The age at time 0 depends on the spiking times before time 0.
- The dynamic is characterized by the spiking times after time 0.

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$$\lambda(t, \mathscr{F}_{t-}^N) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \mathbb{P}\left(N([t, t+\Delta t]) = 1 \,|\, \mathscr{F}_{t-}^N \right),$$

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- $p(s, X(t))$ and $\lambda(t, \mathscr{F}^N_{t-})$ are analogous.

Some classical point processes in neuroscience

- Poisson process: $\lambda(t, \mathscr{F}_{t-}^N) = \lambda(t) =$ deterministic function.

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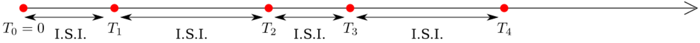

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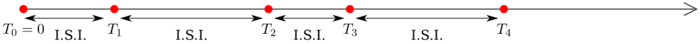

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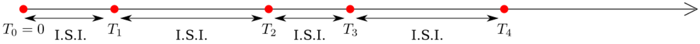

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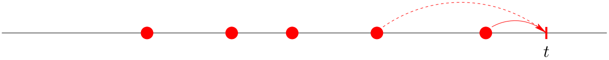
$$= \mu + \sum_{\substack{V \in N \\ V < t}} h(t - V).$$



$$\int_{-\infty}^{t-} h(t-v) N(dv) \quad \longleftrightarrow \quad \int_0^t d(v) m(t-v) dv = X(t).$$

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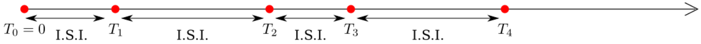

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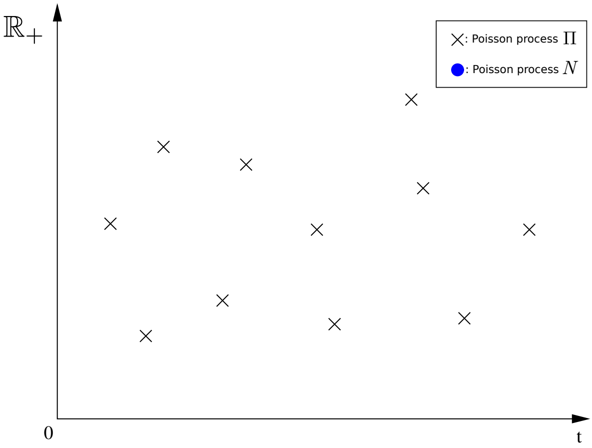
We use the SDE representation of these processes induced by Thinning.

Lewis and Shedler's Thinning, 1979



- Π is a Poisson process with intensity 1.
- $\Pi(dt, dx) = \sum \delta_X$.
- $\mathbb{E}[\Pi(dt, dx)] = dtdx$.
- Spatial independence.

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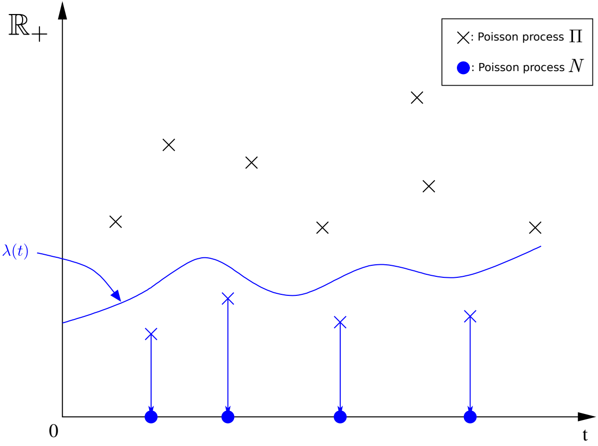

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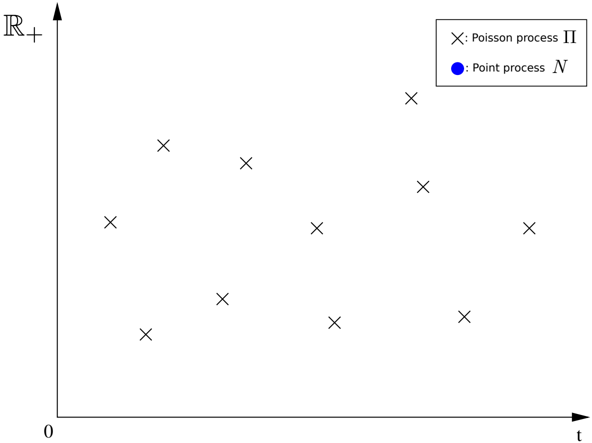
- λ is deterministic.
- N admits λ as an intensity.

Ogata's Thinning, 1981



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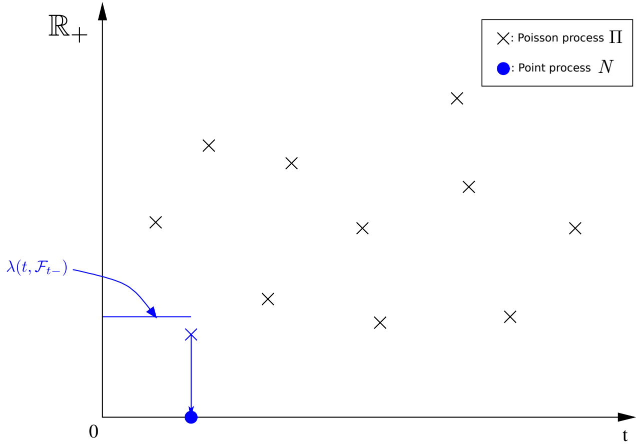

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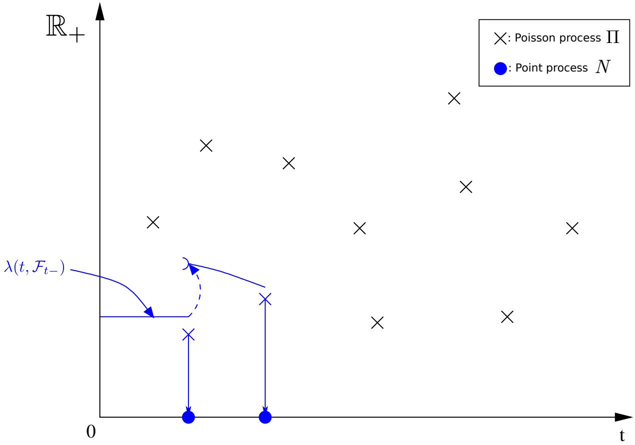

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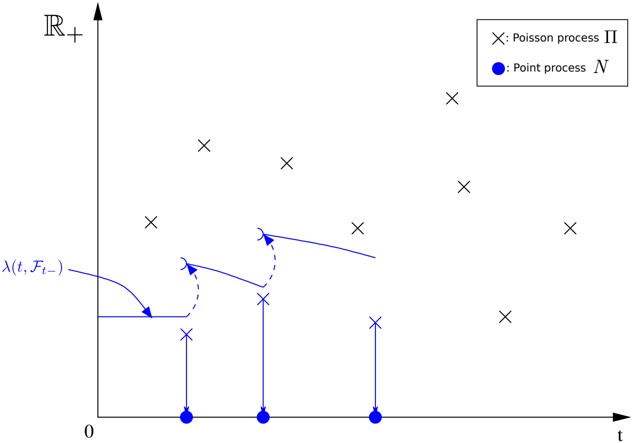

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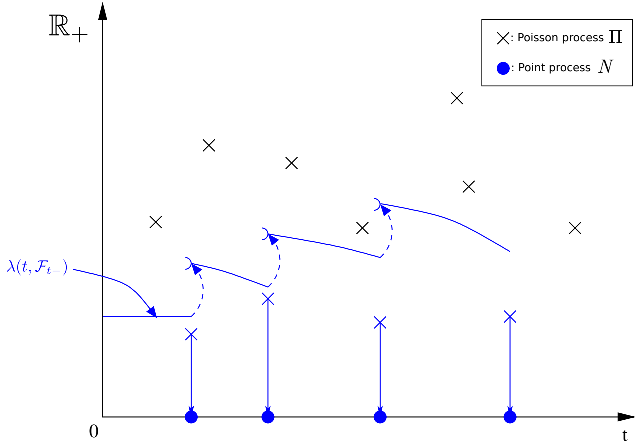

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Thinning

Theorem

Let Π *be a* $(\mathscr{F}_t)$*-Poisson process with intensity* 1 *on* $\mathbb{R}_+^2$*. Let* $\lambda(t, \mathscr{F}_{t-})$ *be a non-negative* $(\mathscr{F}_t)$*-predictable process which is* L^1_{loc} *a.s. and define the point process* N_+ *(on* $(0, \infty)$*) by*

$$N_+(C) = \int_{C \times \mathbb{R}_+} \mathbf{1}_{[0, \lambda(t, \mathscr{F}_{t-})]}(x) \, \Pi(dt, dx),$$

for all $C \in \mathscr{B}(\mathbb{R}_+)$*. Then* N_+ *admits* $\lambda(t, \mathscr{F}_{t-})$ *as a* $(\mathscr{F}_t)$*-predictable intensity.*

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What you should remind

$$N_+(dt) = \int_{x=0}^{\lambda(t, \mathscr{F}_{t-}^N)} \Pi(dt, dx).$$

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What we need

- Random measure U on $\mathbb{R}^2$.
- Action over test functions: $\forall \varphi \in C^\infty_{c,b}(\mathbb{R}^2_+)$,

$$\int \varphi(t,s)U(dt,ds) = \int \varphi(t,S_{t-})dt.$$

What we define

- We construct an ad hoc random measure U which satisfies a system of stochastic differential equations similar to (PPS).

Random system

Theorem

Let Π be a Poisson measure. Let $\left(\lambda(t,\mathscr{F}^N_{t-})\right)_{t>0}$ be some non negative predictable process which is L^1_{loc} a.s.
The measure U satisfies the following system a.s.

$$\begin{cases} (\partial_t+\partial_s)\{U(dt,ds)\}+\left(\int_{x=0}^{\lambda(t,\mathscr{F}^N_{t-})}\Pi(dt,dx)\right)U(t,ds)=0, \\ U(dt,0)=\int_{s\in\mathbb{R}}\left(\int_{x=0}^{\lambda(t,\mathscr{F}^N_{t-})}\Pi(dt,dx)\right)U(t,ds), \end{cases}$$

in the weak sense with initial condition $\lim_{t\to 0^+}U(t,\cdot)=\delta_{-T_0}$. ($-T_0$ is the age at time 0)

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$$\mathbb{E}\left[\int_{x=0}^{\lambda(t,\mathscr{F}_{t-}^N)} \Pi(dt,dx) \middle| \mathscr{F}_{t-}^N\right] = \lambda\left(t,\mathscr{F}_{t-}^N\right) dt.$$

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Technical difficulty

Product of measures

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Product of measures

- Parametrized families of measures $U(t,ds)$ and $U(dt,s)$, e.g.

$$U(t,ds) = \delta_{S_{t-}}(ds)$$

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Theorem

The measure U satisfies the following system a.s.

$$\begin{cases} (\partial_t + \partial_s)\{U(dt,ds)\} + \left(\int_{x=0}^{\lambda(t,\mathscr{F}_{t-}^N)} \Pi(dt,dx)\right) U(t,ds) = 0, \\ U(dt,0) = \int_{s\in\mathbb{R}} \left(\int_{x=0}^{\lambda(t,\mathscr{F}_{t-}^N)} \Pi(dt,dx)\right) U(t,ds), \end{cases}$$

in the weak sense with initial condition $\lim_{t\to 0^+} U(t,\cdot) = \delta_{-T_0}$. ($-T_0$ is the age at time 0)

Technical difficulty

Product of measures

- Parametrized families of measures $U(t,ds)$ and $U(dt,s)$, e.g.

$$U(t,ds) = \delta_{S_{t-}}(ds)$$

- Fubini property: $U(t,ds)dt = U(dt,s)ds = U(dt,ds)$.

Outline

Taking the expectation

- Can we consider the expectation measure $u(dt, ds) = \mathbb{E}[U(dt, ds)]$?

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We need the intensity to be L^1_{loc} in expectation.

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Property

$u(t,.)$ is the distribution of S_{t-}.

System in expectation

Theorem

Let $\left(\lambda(t, \mathscr{F}^N_{t-})\right)_{t>0}$ *be some non negative predictable process which is* L^1_{loc} *a.s.*
The measure U *satisfies the following system,*

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\begin{cases}
(\partial_t + \partial_s)\{U(dt, ds)\} + \left(\int_{x=0}^{\lambda(t,\mathscr{F}^N_{t-})} \Pi(dt, dx)\right) U(t, ds) = 0, \\
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\end{cases}
$$

in the weak sense with initial condition $\lim_{t\to 0^+} U(t, \cdot) = \delta_{-T_0}$.

- There are two (highly correlated) random measures: U and Π.

System in expectation

Theorem

Let $\left(\lambda(t,\mathscr{F}^N_{t-})\right)_{t>0}$ *be some non negative predictable process which is* L^1_{loc} *in expectation, and which admits a finite mean.*
The measure u *satisfies the following system,*

$$\begin{cases}(\partial_t+\partial_s)u(dt,ds)+\rho_{\lambda,\mathbb{P}_0}(t,s)u(dt,ds)=0,\\ u(dt,0)=\int_{s\in\mathbb{R}}\rho_{\lambda,\mathbb{P}_0}(t,s)u(t,ds)\,dt,\end{cases}$$

in the weak sense where $\rho_{\lambda,\mathbb{P}_0}(t,s)=\mathbb{E}\left[\lambda\left(t,\mathscr{F}^N_{t-}\right)\middle|S_{t-}=s\right]$ *for almost every* t*. The initial condition* $\lim_{t\to0^+}u(t,\cdot)$ *is given by the distribution of* $-T_0$*.*

Idea of Proof

- We deal with the equations in the weak sense.
- The terms that do not involve the spiking measure $\int_{x=0}^{\lambda(t,\mathscr{F}_{t-}^{N})} \Pi(dt, dx)$ are easy to deal with.

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$$\begin{aligned}
\mathbb{E}\left[\int_t \varphi(t,S_{t-})\lambda\left(t,\mathscr{F}_{t-}^N\right)dt\right] &= \mathbb{E}\left[\int_t \varphi(t,S_{t-}) \quad \rho_{\lambda,\mathbb{P}_0}(t,S_{t-}) \quad dt\right] \\
&= \int_t\int_s \varphi(t,s)\rho_{\lambda,\mathbb{P}_0}(t,s)\,u(t,ds)dt.
\end{aligned}$$

Law of large numbers

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Let $(N^i)_{i\geq 1}$ be some i.i.d. point processes on $\mathbb{R}$ with L^1_{loc} intensity in expectation. For each i, let $(S^i_{t-})_{t>0}$ denote the age process associated to N^i. Then, for every test function φ,

$$\int \varphi(t,s) \left(\frac{1}{n} \sum_{i=1}^{n} \delta_{S^i_{t-}}(ds) \right) dt \xrightarrow[n\to\infty]{a.s.} \int \varphi(t,s) u(dt,ds),$$

with u satisfying the deterministic system.

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Idea of proof

Thinning inversion Theorem $\Rightarrow$ recover some Poisson measures $(\Pi^i)_{i\geq 1}$ and microscopic measures $(U^i)_{i\geq 1}$.

Outline

Review of the examples

The system in expectation

$$\begin{cases} (\partial_t + \partial_s) u(dt, ds) + \rho_{\lambda,\mathbb{P}_0}(t,s)\, u(dt, ds) = 0, \\ u(dt, 0) = \int_{s\in\mathbb{R}} \rho_{\lambda,\mathbb{P}_0}(t,s)\, u(t, ds)\, dt. \end{cases}$$

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- Renewal process. $\rightarrow \rho_{\lambda,\mathbb{P}_0}(t,s) = f(s)$.
- Hawkes process. $\rightarrow \rho_{\lambda,\mathbb{P}_0}$ is much more complex.

Overview of the results

Recall that

$$\int_{-\infty}^{t-} h(t-x)N(dx) \quad \longleftrightarrow \quad \int_0^t d(x)m(t-x)dx = X(t).$$

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Technical difficulty

$\rho_{\lambda, \mathbb{P}_0}(t, s) = \mathbb{E}\left[\lambda\left(t, \mathscr{F}_{t-}^N\right) \middle| S_{t-} = s\right]$ is not so easy to compute.

Cluster process

$$\lambda(t,\mathscr{F}_{t-}^{N}) = \mu + \int_0^{t-} h(t-v)N(dv) = \mu + \sum_{\substack{V\in N \\ V<t}} h(t-V)$$

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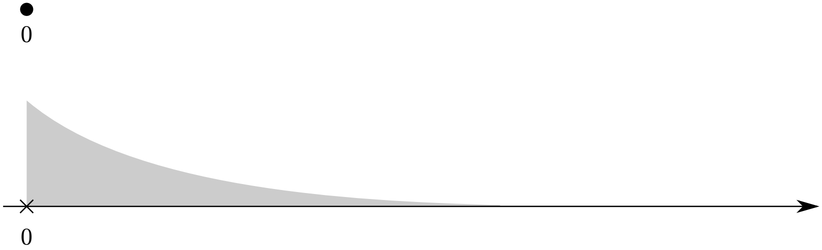

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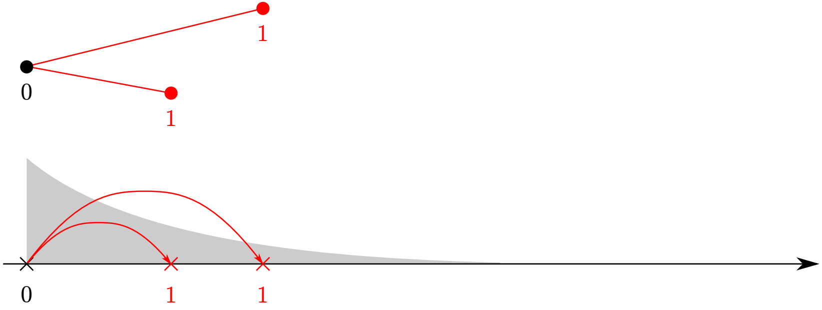

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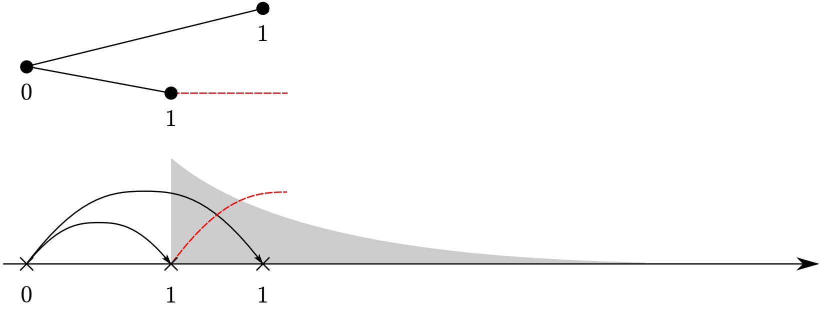

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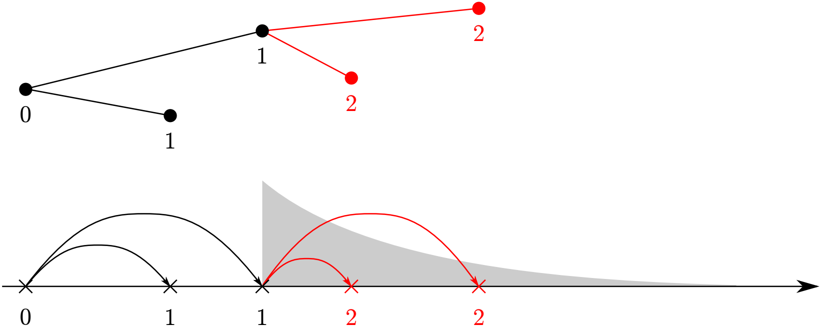

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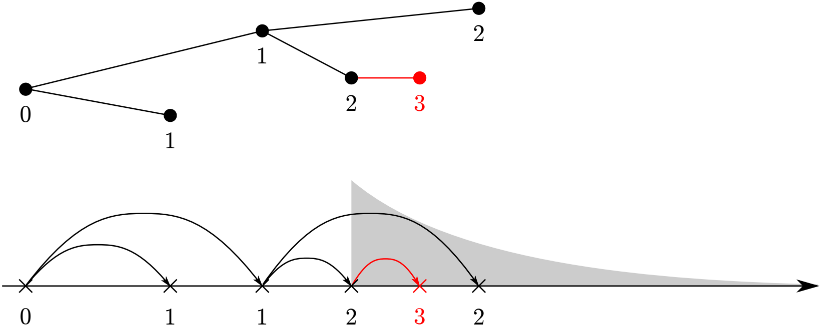

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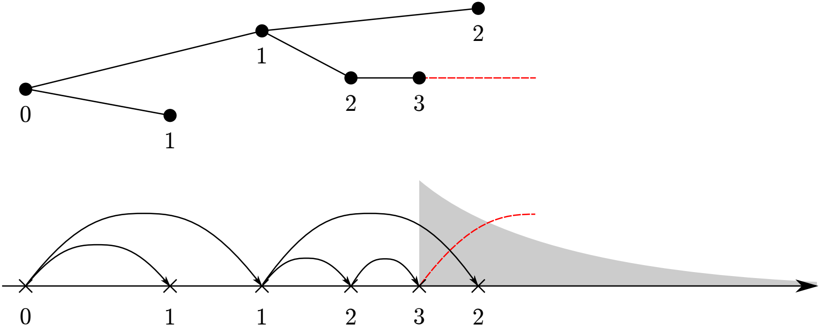

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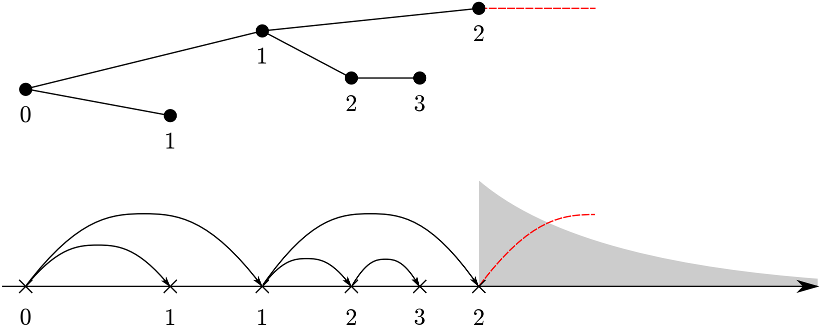

- Cluster process N_c associated to h: The set of points of generation greater than 1.
- number of children $\sim \mathscr{P}(||h||_1)$: $||h||_1 < 1 \Rightarrow N_c$ is finite a.s.

Cluster decomposition of the linear Hawkes process

- Recall that: $N_- = N \cap (-\infty, 0]$ and $N_+ = N \cap (0, +\infty)$.
- N_- is a point process on $\mathbb{R}_-$ distributed according to $\mathbb{P}_0$.

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$$N_{\leq 0} = N_- \cup \left(\bigcup_{T \in N_-} N_1^T \cup \left(\bigcup_{V \in N_1^T} V + N_c^{T,V} \right) \right). \tag{1}$$

The process $N_{\leq 0}$ admits $t \mapsto \int_{-\infty}^{t-} h(t-x) N_{\leq 0}(dx)$ as an intensity on $(0, \infty)$.

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$$N_{>0} = N_{anc} \cup \left(\bigcup_{X \in N_{anc}} X + N_c^X \right). \quad (2)$$

The process $N_{>0}$ admits $t \mapsto \mu + \int_0^{t-} h(t-x) N_{>0}(dx)$ as an intensity on $(0, \infty)$.

Cluster decomposition of the linear Hawkes process

$$N_{\leq 0} = N_{-} \cup \left(\bigcup_{T \in N_{-}} N_1^T \cup \left(\bigcup_{V \in N_1^T} V + N_c^{T,V} \right) \right). \tag{1}$$

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Proposition (Hawkes, 1974)

The processes $N_{\leq 0}$ and $N_{>0}$ are independent and

$$N = N_{\leq 0} \cup N_{>0}$$

has intensity on $(0,\infty)$ given by

$$\lambda(t, \mathscr{F}_{t-}^N) = \mu + \int_{-\infty}^{t-} h(t-x) N(dx).$$

Coming back to the conditional expectation

$\rho_{\lambda,\mathbb{P}_0}(t,s) = \rho^{\mu,h}_{\mathbb{P}_0}(t,s)$ (in this case) is hard to compute directly. We prefer

$$\Phi^{\mu,h}_{\mathbb{P}_0}(t,s) = \mathbb{E}\left[\lambda(t,\mathscr{F}^N_{t-}) \middle| S_{t-} \geq s\right]$$

For any point process N and any real numbers $s < t$, let

$$\mathscr{E}_{t,s}(N) = \{N \cap (t-s,t) = \emptyset\} = \{S_{t-} \geq s\}.$$

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No general formula available for $\Phi_{-,\mathbb{P}_0}^{\mu,h}$. Two cases are studied in the article:

- N_- is a Poisson process.
- N_- is a one point process ($N_- = \{T_0\}$).

$$\Phi_{+}^{\mu,h}(t,s) = \mathbb{E}\left[\underbrace{\mu + \int_0^{t-} h(t-x)N_{>0}(dx)}_{\text{intensity of } N_{>0}} \middle| \mathscr{E}_{t,s}(N_{>0})\right].$$

Linear Hawkes process

$$\Phi_{+}^{\mu,h}(t,s) = \mathbb{E}\left[\underbrace{\mu + \int_0^{t-} h(t-x) N_{>0}(dx)}_{\text{intensity of } N_{>\mathbf{0}}} \middle| \mathscr{E}_{t,s}(N_{>0}) \right] = \mu + L_s^{\mu,h}(t).$$

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Lemma

Let N be a linear Hawkes process with

$$\lambda(t,\mathscr{F}_{t-}^N) = g(t) + \int_0^{t-} h(t-x)N(dx),$$

and $||h||_1 < 1$. *Let*

$$\begin{cases} L_s^{g,h}(x) = \mathbb{E}\left[\int_0^x h(x-z)N(dz) \middle| \mathscr{E}_{x,s}(N)\right] \\ G_s^{g,h}(x) = \mathbb{P}(\mathscr{E}_{x,s}(N)), \end{cases}$$

for any $x,s \geq 0$. *Then, for any* $x,s \geq 0$,

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- Remind that $\Phi_{\mathbb{P}_0}^{\mu,h}(t,s) = \mathbb{E}\left[\lambda\left(t, \mathscr{F}_{t-}^{N}\right) \middle| S_{t-} \geq s\right]$,
- Hence $\rho_{\mathbb{P}_0}^{\mu,h}(t,s) = \mathbb{E}\left[\lambda\left(t, \mathscr{F}_{t-}^{N}\right) \middle| S_{t-} = s\right]$ can be recovered as the derivative of $\Phi_{\mathbb{P}_0}^{\mu,h}$.

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