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Robust Influence Maximization via Continuous Submodular Minimization

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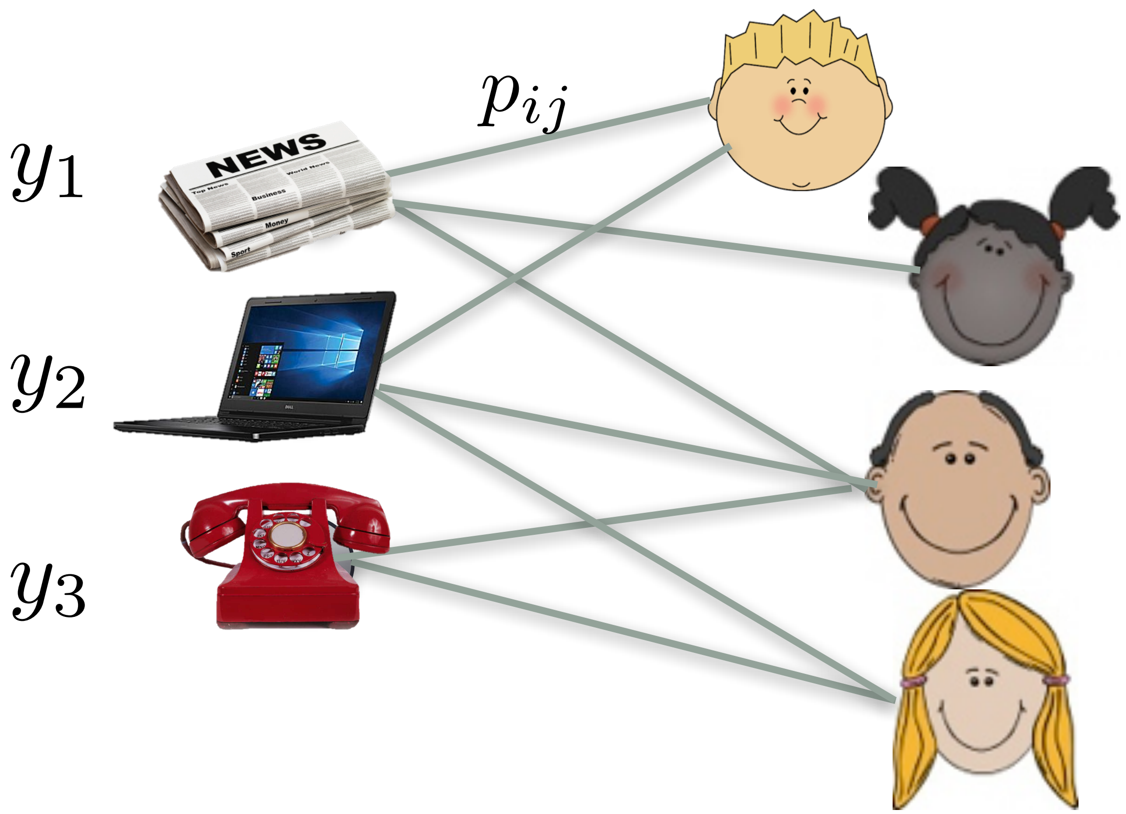
joint work with Matthew Staib

Les Houches, April 13 2017

Bipartite Influence Maximization

aka "Budget Allocation"



"Independent Cascade Model":

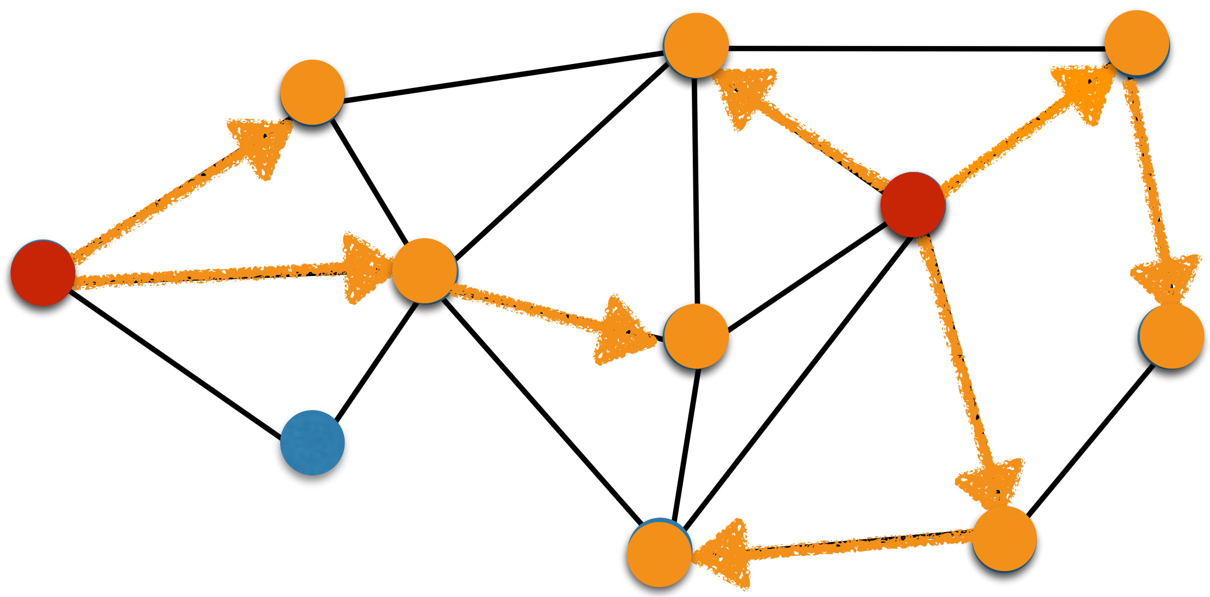
$$\mathbb{P}[j \text{ active}] = 1 - \prod_{i=1}^{n} (1 - p_{ij})^{y_i}$$

$$\mathcal{I}(y; p) = \sum_j \mathbb{P}[j \text{ active}]$$

$$\max_y \mathcal{I}(y; p) \qquad \text{s.t. } \sum_i y_i \leq k; \;\; 0 \leq y \leq u$$

(Alon-Gamzu-Tennenholtz 12; Soma et al 14; Soma & Yoshida 15; Hatano et al 15; Bian et al 17)

Influence Maximization



(Domingos & Richardson 01, Kempe et al 03, Mossel & Roch, Chen et al 09, Borgs et al 14, …)

Influence Maximization

$$\boxed{\max_{y} \; \mathcal{I}(y;p) \qquad \text{s.t.} \;\; \sum_i y_i \leq k; \;\; 0 \leq y \leq u}$$

- concave maximization over convex set
- y discrete: "monotone submodular maximization"
  $(1-\frac{1}{e})$-approximation via greedy algorithm

(Alon-Gamzu-Tennenholtz 12; Soma et al 14; Soma & Yoshida 15; Hatano et al 15; Bian et al 17)

Influence Maximization

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  $(1-\frac{1}{e})$ -approximation via greedy algorithm
- *What if we don't know p?*

Robust Influence Maximization

1. small collection of functions
   (He-Kempe 16, Chen et al 16, Krause et al)

$$\max_{y \in \mathcal{B}} \ \min_{\ell \in \mathcal{L}} \ \frac{\mathcal{I}_\ell(y)}{\mathcal{I}_\ell(y_\ell^*)}$$

$$\max_{y \in \mathcal{B}} \ \min_{\ell \in \mathcal{L}} \ \mathcal{I}_\ell$$

Robust Optimization

$$\max_{y \in \mathcal{B}} \min_{p \in \mathcal{P}} \mathcal{I}(y; p)$$

- *uncertainty sets:* e.g.

$$\mathcal{P}^Q = \{p \in [0,1]^d, (p - \hat{p})^\top \Sigma^{-1} (p - \hat{p}) \leq \gamma\}$$

$$\mathcal{P}^D = \left\{p \in [0,1]^d : \sum_{(i,j) \in E} \frac{p_{ij} - \hat{p}_{ij}}{u_{ij} - p_{ij}} \leq \gamma \right\}$$

- in y: min of concave functions -> *concave*
  use e.g. **subgradient method?**
- in p: ***not*** convex or quasiconvex …

But: continuous submodular in p…

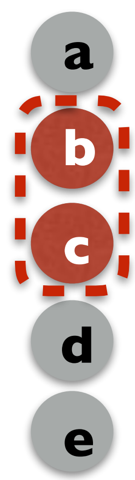
Corollary *(Staib-J17)*
Under a distinctness condition, we can solve the adversary's problem via Frank-Wolfe on a convex problem with dimension $O(n/\epsilon)$

Roadmap:

- Brief introduction to submodularity
- (Constrained) optimization and relaxations
- Implications

Submodular Set Functions

$S \subseteq V$



$$\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad x \in \{0,1\}^n$$

$F : 2^V \to \mathbb{R}$ ***submodular*** if for all *S,T*

$$F(S) + F(T) \geq F(S \cup T) + F(S \cap T)$$

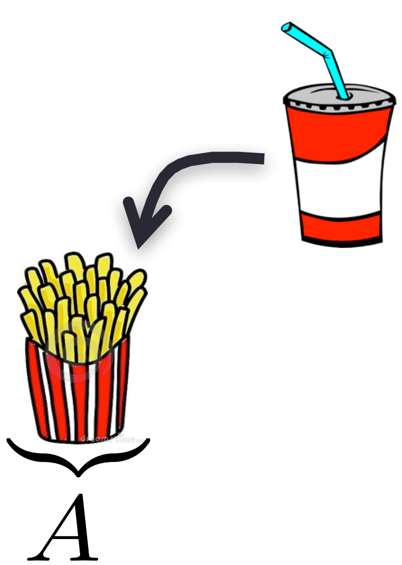
$$F(x) + F(y) \geq F(x \vee y) + F(x \wedge y)$$

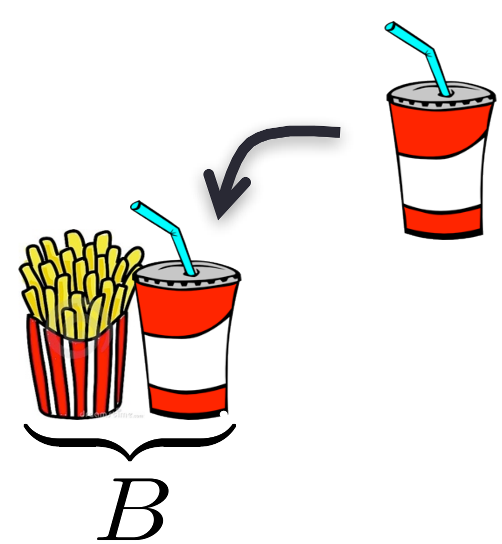
Diminishing Returns/Costs

$$F(S) \; + \; F(T) \quad \geq \quad F(S \cup T) \; + \; F(S \cap T)$$

equivalent condition: $\forall A \subseteq B \subset V, i \notin B :$

$$F(A \cup i) - F(A) \quad \geq \quad F(B \cup i) - F(B)$$



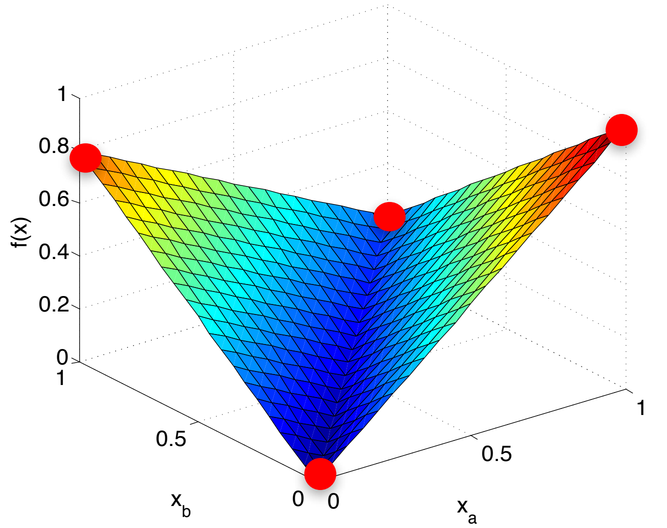


Why is submodularity useful?

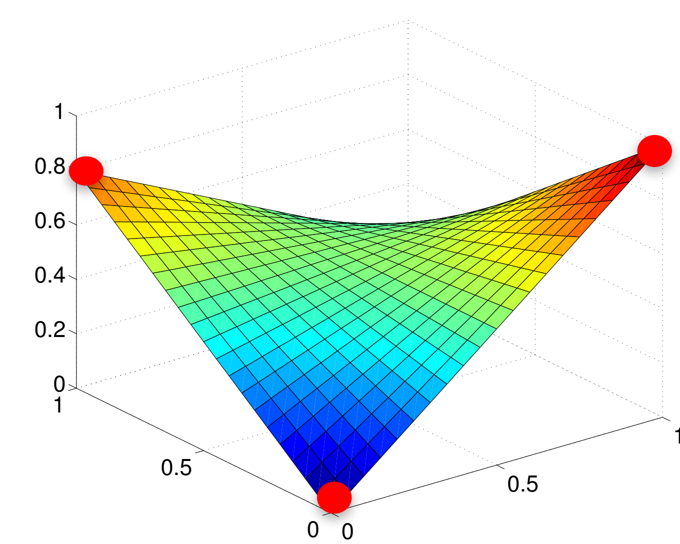
$$F(x), x \in \{0,1\}^n \text{ submodular}$$

- naturally occurs in many problems, and good optimization:

- **Minimization** over $\{0,1\}^n$ in ***polynomial time***
  "discrete convexity"
  (Groetschel-Lovasz-Schrijver 81, Schrijver 00, Iwata-Fleischer-Fujishige 01; Lee-Sidford-Wong 15; …)



- **Maximization** NP-hard, but constant-factor approximations
  "discrete concavity"
  (Nemhauser-Wolsey-Fisher 71; Calinescu-Chekuri-Pal-Vondrak 11; Vondrak ; Buchbinder-Naor-Feldman 12; …)



nonconvex optimization
lattice / continuous submodularity
many optimization & duality results generalize

probability measures
log-supermodular (→ positive assoc.)
log-submodular (← negative assoc.)
sampling, mode,
approx. partition function

submodular set functions

convexity:
minimization
max. coherence

dim. returns:
maximization
max. diversity

many examples:

- linear/modular functions
- entropy
- mutual information
- rank functions
- coverage
- diffusion in networks
- volume
- graph cut …

Submodularity beyond Sets

- straightforward generalization to (distributive) lattices:

$$F(x) \; + \; F(y) \quad \geq \quad F(x \vee y) \; + \; F(x \wedge y)$$

- for differentiable functions:

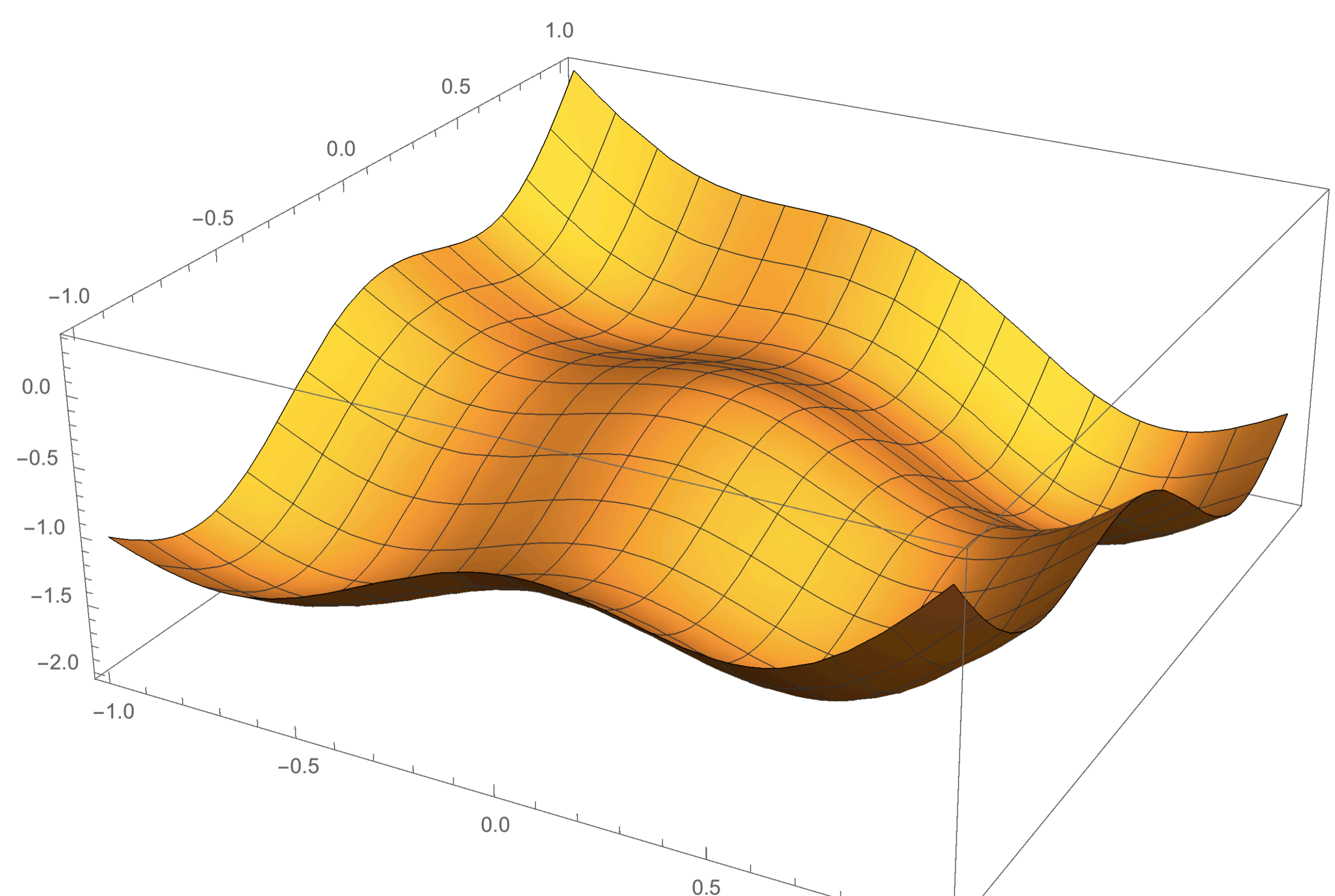
$$\frac{\partial^2 f}{\partial x_i \partial x_j} \; \leq \; 0 \qquad\qquad \forall i \neq j$$

- note: "diminishing returns" is stronger here!
  (DR-submodular)

Convex or Concave?

Examples:

- $F(x) = g(x_i - x_j)$ for convex g
- $F(x) = h(\sum_i x_i)$ for concave h
- $F(x) = x^\top Qx$ for Q with negative off-diagonals



Optimization

Minimization over Box: exact

- Reduction to sets *(Birkhoff 37 & Schrijver 00/Orlin 07)*
- Convex relaxation *(Bach 15)*

- Not all submodular functions are equal! Better for
  - *DR-submodular* functions
    (combining Ene & Nguyen 16 & e.g. Lee-Sidford-Wong 15)
  - $L^\natural$-convex functions *(Murota-Shioura)*

$$f(x) + f(y) \;\geq\; f\left(\left\lceil \frac{x+y}{2} \right\rceil\right) + f\left(\left\lfloor \frac{x+y}{2} \right\rfloor\right)$$

Maximization: greedy or non convex Frank-Wolfe

Back to Influence

$$\max_{y \in \mathcal{B}} \min_{p \in \mathcal{P}} \mathcal{I}(y; p)$$

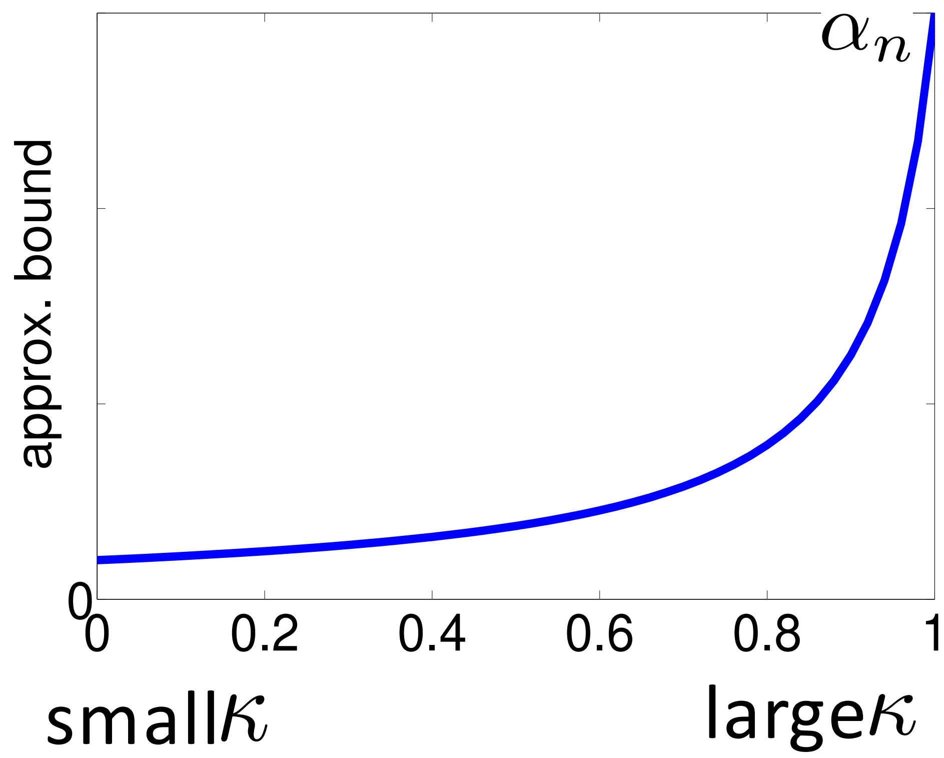
- submodular in 1-p
- DR-submodular if y integer

constrained submodular minimization?

Constrained Minimization?

- can be ***hard.***
  (Goel-Karande-Tripathi-Wang 09; Iwata-Nagano 09; Svitkina-Fleischer 11)

- ***But***, there is hope:
  for increasing functions,
  tight bounds via curvature
  (Iyer-Jegelka-Bilmes 13)



approximation

optimal

Constrained Minimization?

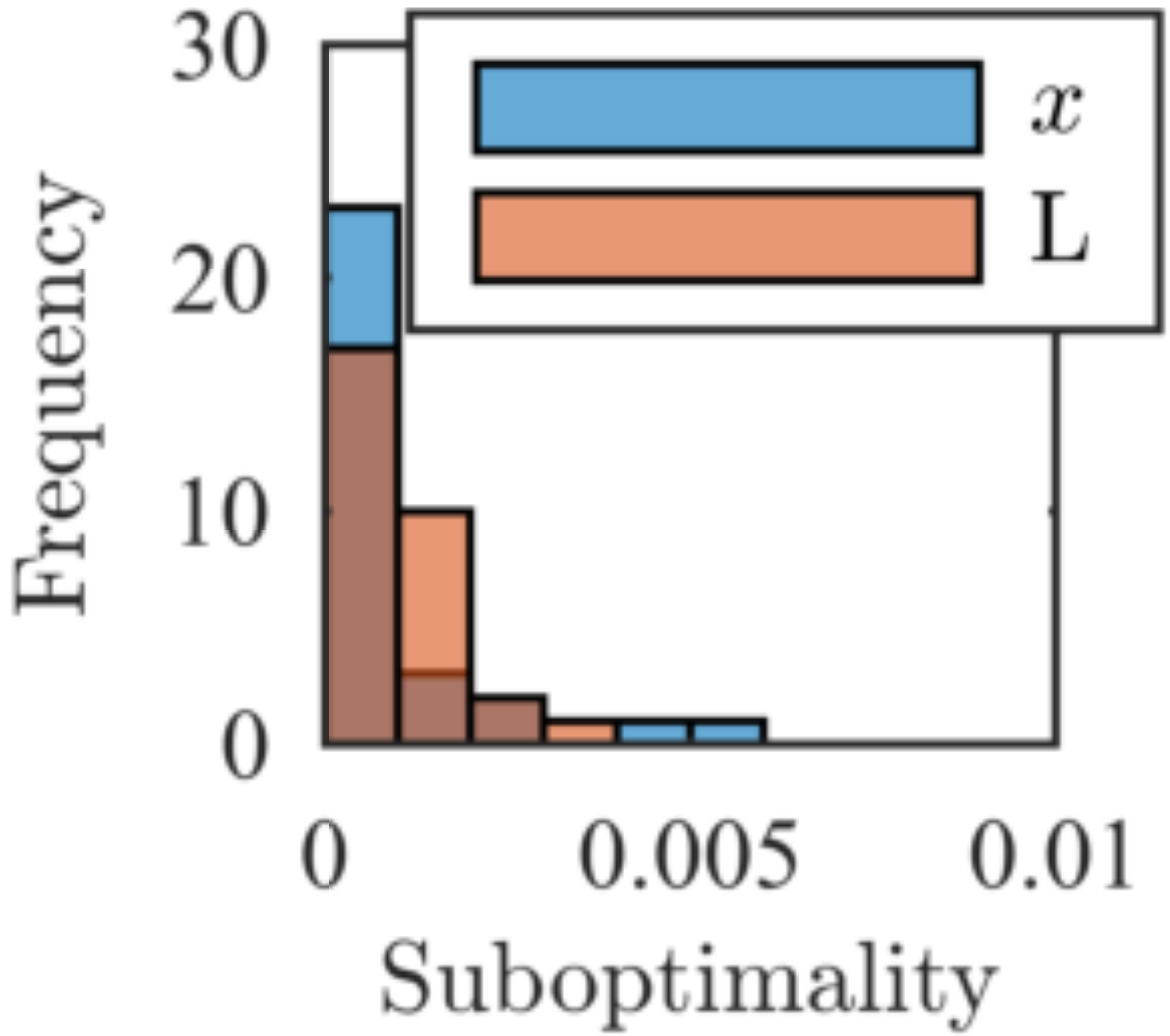
$$\begin{aligned} \min \quad & F(x) \\ \text{s.t.} \quad & R(x) \leq B \\ & \ell \leq x \leq u \end{aligned}$$

Theorem *(Staib-J 17)*

Under distinctness condition, for monotone non-increasing submodular F and separable, increasing R, we can this solve to accuracy ϵ via Frank-Wolfe for smooth convex problem with dimension $O(n/\epsilon)$.

Always: bounds on optimality

Optimality in Practice?



Yahoo Webscope data (>50k edges), 30 rep.

Relaxation

- Lagrangian relaxation

$$\begin{aligned} \min \quad & F(x) \\ \text{s.t.} \quad & R(x) \leq B \\ & \ell \leq x \leq u \end{aligned} \longrightarrow \begin{aligned} \min \quad & F(x) + \lambda R(x) \\ & \ell \leq x \leq u \end{aligned}$$

- Solve parametric problem (discretize)

$$\begin{aligned} \min \quad & F(z) + \lambda R(z) \\ & z_i \in [k_i] \end{aligned}$$

- Pick best feasible $x(\lambda)$

entire solution path
via a single optimization

Parametric Problem

$$\min \;\; F(z) + \lambda R(z)$$
$$z_i \in [k_i]$$

scalar smooth convex function

"decoding":

$$z(\lambda)_i = \rho_i^{-1}(\lambda)$$

$$\min_{\rho} \;\; f(\rho) + \sum_i \sum_{j=1}^{k_i - 1} a_{ij}(\rho_i(j))$$
$$\text{s.t.} \;\; \rho_i \in \mathbb{R}_{\downarrow}^{k_i - 1}$$

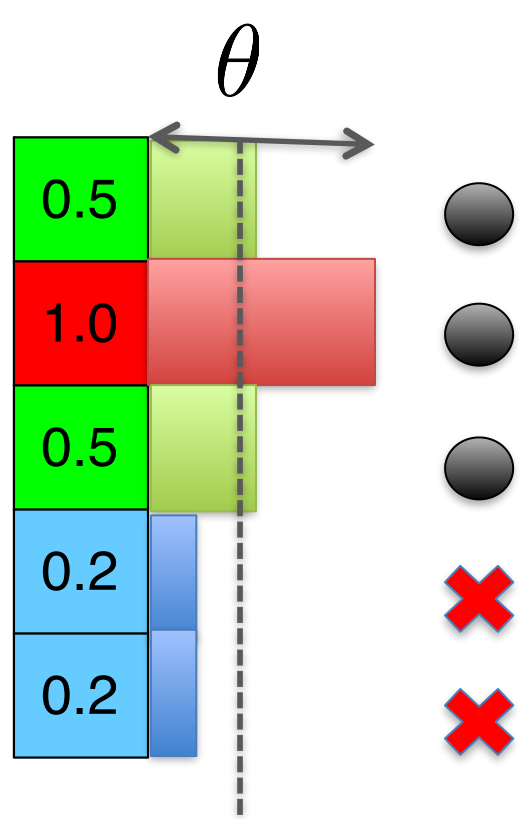
reverse CDF
(decreasing vector)

Set functions: Lovasz extension

$$F(S) \longrightarrow f(u), u \in [0,1]^n$$

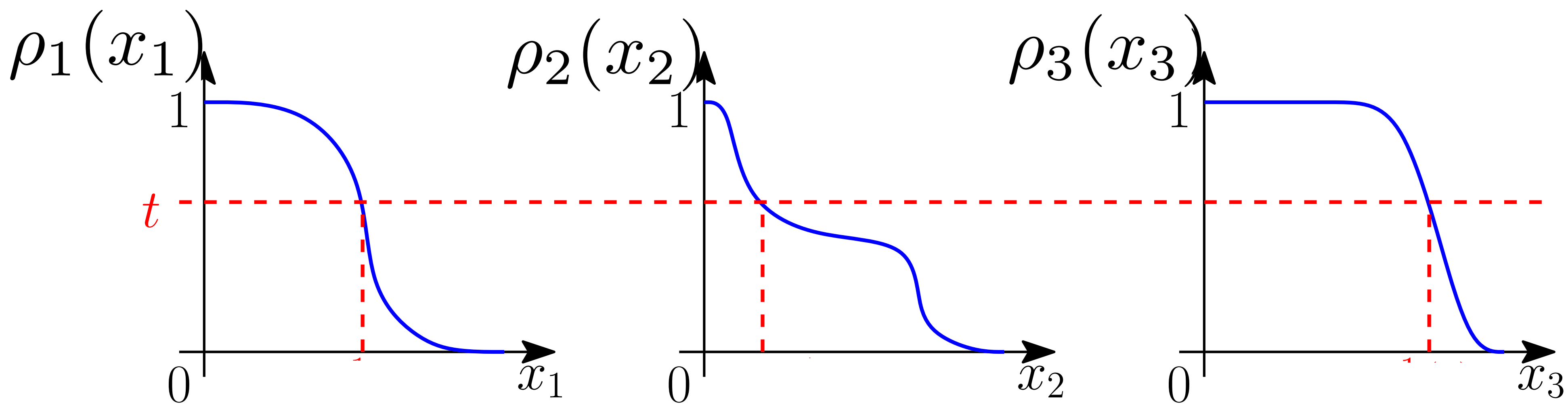
$$f(u) = \mathbb{E}[F(S_\theta)]$$

e.g. $\theta = 0.4$



- "couples/aligns" dimensions
- u_i "distribution" of coordinate i

Generalized extension



- distribution over values 1, … k
- reverse CDF:

$$\rho_i(x_i) = \mu_i\{y \in [k_i], y \geq x_i\}$$

- extension:

$$f(\mu_1, \ldots, \mu_n) = \int_0^1 F(\rho_1^{-1}(t), \ldots, \rho_n^{-1}(t)dt$$

Properties of Extensions I

2 ways to create extension:

- **"coupling/alignment":**
  computable but not always convex

- **convex envelope:**
  convex but not always computable

for submodular functions, these are the same!
(Lovasz 1982, Bach 15, Carlier 03)
via multi-marginal optimal transport

Properties of Extensions II

For

$$\min_S \quad F(S) + \lambda |S|$$

solve

$$\min_u \quad f(u) + \tfrac{1}{2}\|u\|^2$$

u^* gives optimal solutions for all λ

works also for integer functions

Insight: ***we can encode R(x) this way!***

Parametric Problem

$$\min \; F(z) + \lambda R(z)$$
$$z_i \in [k_i]$$

"decoding":

$$z(\lambda)_i = \rho_i^{-1}(\lambda)$$

scalar smooth
convex function

$$\min_{\rho} \; f(\rho) + \sum_i \sum_{j=1}^{k_i - 1} a_{ij}(\rho_i(j))$$

$$\text{s.t. } \; \rho_i \in \mathbb{R}_{\downarrow}^{k_i - 1}$$

reverse CDF
(decreasing vector)

solving relaxation:
pairwise Frank-Wolfe
(Lacoste-Julien & Jaggi 15)
on dual

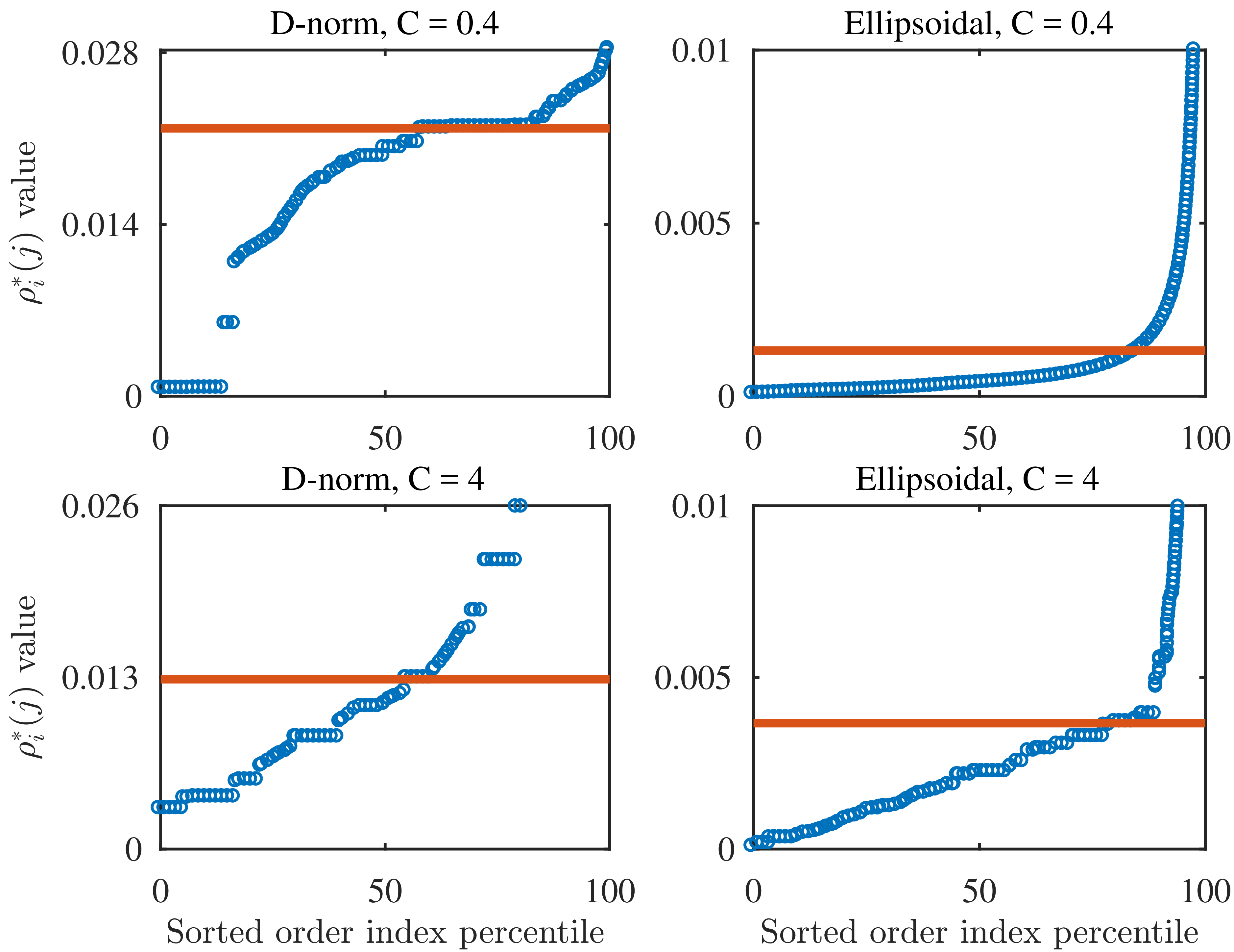
Optimality

"decoding":

$$z(\lambda)_i = \rho_i^{-1}(\lambda)$$

- **Distinctness condition:** all entries of ρ_i are distinct

Distinctness condition in practice



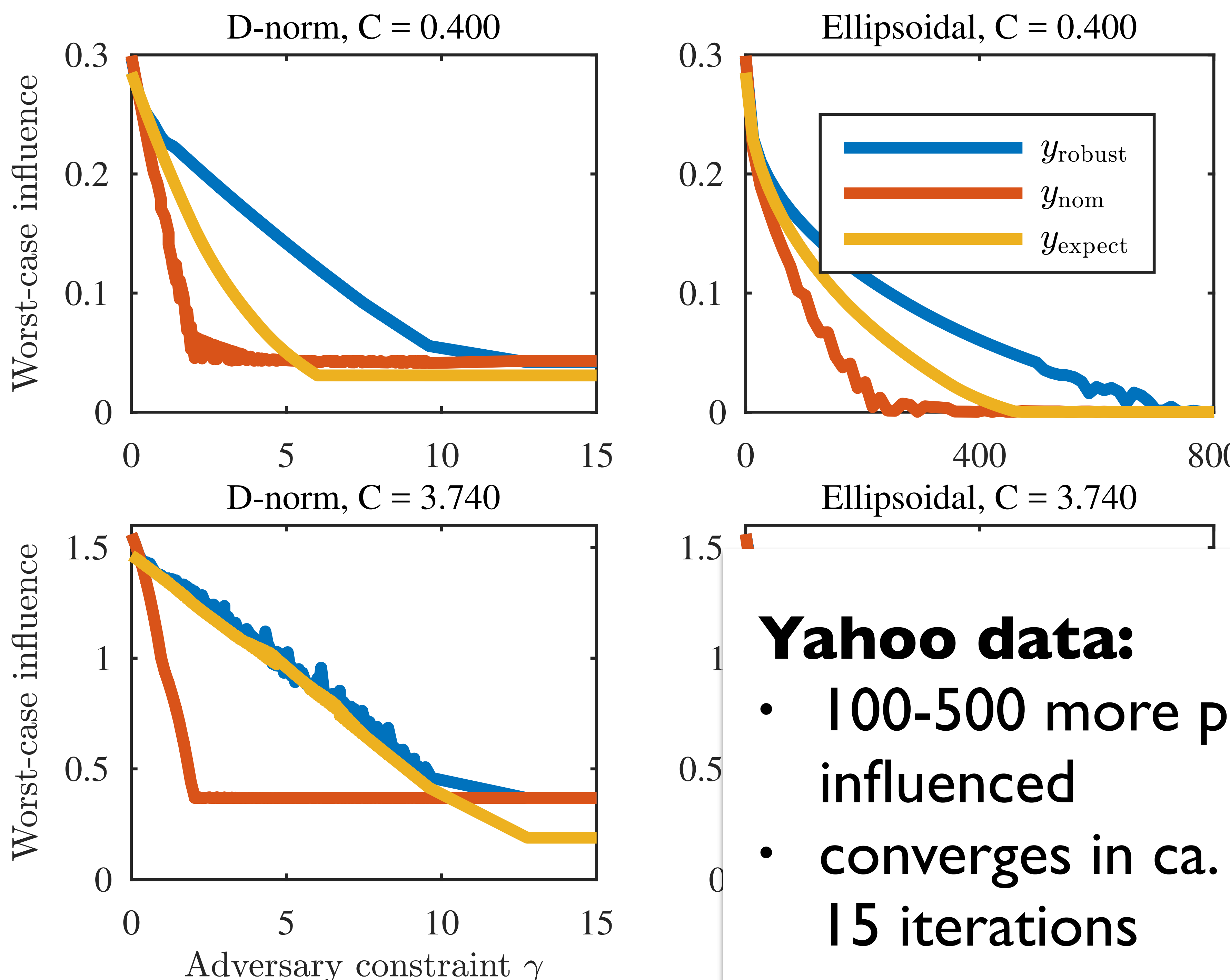
Putting things together

$$\max_{y \in \mathcal{B}} \min_{p \in \mathcal{P}} \mathcal{I}(y; p)$$

$$
\begin{aligned}
&\textbf{loop} \\
&\quad \boxed{p^{(k)} \leftarrow \operatorname{argmin}_{p \in \mathcal{P}} \mathcal{I}(y^{(k)}; p)} \\
&\quad g^{(k)} \leftarrow \nabla_y \mathcal{I}(y^{(k)}; p^{(k)}) \\
&\quad y^{(k+1)} \leftarrow \operatorname{proj}_{\mathcal{Y}}(y^{(k)} + \gamma^{(k)} g^{(k)}) \\
&\quad k \leftarrow k + 1 \\
&\textbf{end loop}
\end{aligned}
$$

continuous submodular minimization

Empirical Results



Yahoo data:

- 100-500 more people influenced
- converges in ca. 15 iterations

Summary

- Optimization for continuous constrained submodular minimization via parametric relaxation
- Optimality bounds
- Applications to robust influence maximization/budget allocation

M. Staib & S. Jegelka. Robust Budget Allocation via Continuous Submodular Functions. arXiv, 2017.