Convex optimization: applications, formulations, relaxations

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Convex optimization, useful applied maths

Optimization in two words:

- "the maths of doing-better" or "the maths of decision-making"
- Mature disciplin of applied maths (theory, algorithms, software)
- Recent explosion of applications in engineering sciences

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And why convex ?... because it's the favorable case !

"the great watershed in optimization isn't between linearity and nonlinearity, but convexity and nonconvexity" T. Rockafellar

- Geometrical properties : globality, guarantees,...
- Useful tools: duality, sensitivity analysis... and algorithms !

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Some current trends and challenges in (convex) optimization

1. more complex models ("nonlinear mixed-integer programming")
2. very large-scale data ("huge-scale optimization")
3. noisy or unknown data ("robust optimization")
4. "applied optimization" ("incremental optimization")

Optimization: BiPoP... in fact, me :-)

Projects that I have been leading (since I arrived in nov. 2006)

- Algorithms for optimisation convex and...
  - ... differentiable (with M. Fuentes, ancien post-doc)
  - ... differentiable on submanifold (with P.-A. Absil, Belgium)
  - ... nonsmooth (with A. Daniilidis, Barcelona)
  - ... semidefinite (with F. Rendl, Austria)
- Industrial applications:
  - electrical production (with C. Lemaréchal and EDF)
  - finance (with RaisePartner)
- Applications in other disciplins :
  - combinatorial optimisation (with F. Roupin, Paris)
  - polynomial optimisation (with D. Henrion, LAAS)
  - computational mechanics (with V. Acary and Schneider)
  - statistical learning (with Lear and Yahoo!)
- and some related theoretical questions... in particular:
  - properties of spectral functions (with H. Sendov, Canada)
  - convergence of projection algorithms (with A. Lewis, US)

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Today's presentation

Roadmap

1. Optimization of the French electricity production (20 years)
   Bipop (C. Lemaréchal, J. Malick, S. Zaourar) and EDF
2. Low-rank algorithm for multiclass classification (1 year)
   Lear (Z. Harchaoui), Bipop (J. Malick) and Yahoo! (M. Dudik)
3. New SDP/LMI relaxations in combinatorial optimization (3 years)
   Bipop (N. Krislock, J. Malick) and Paris 13 (F. Roupin)

"Goals" (modest)

- show problems, issues, ideas - with no details
- for you : give a (partial) overview (of what happens upstairs)
- for me : insisting on my hot topics... and do local advertising

Examples of applications of convex optimization

1. **Optimization of electricity production**
2. Low-rank penalization for multiclass classification
3. New SDP relaxations in combinatorial optimization

Short-term electricity production managment

In France: electricity produced by $n \simeq 200$ production units

nuclear 80%

oil + coal 3%

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Hard optimization problem: large-scale, heterogeneous, short time

$$\begin{cases} \min \quad \sum_i c_i(p_i) & \text{(sum of costs)} \\ \quad p_i \in P_i \quad i = 1, \ldots, n & \text{(technical constraints)} \\ \quad \sum_i p_i^t = d^t \quad t = 1, \ldots, T & \text{(answer to demand)} \end{cases}$$

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Decomposable by duality !

$$L(p, \lambda) \;:=\; \sum_i c_i(p_i) + \sum_t \lambda^t \Big(\sum_i p_i^t - d^t\Big) = \sum_i \Big(c_i(p_i) + \sum_t \lambda^t (p_i^t - d^t)\Big)$$

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$$\theta(\lambda) := \min_{p \in P} L(p, \lambda) = \sum_i \min_{p_i \in P_i} \Big(c_i(p_i) + \sum_t \lambda^t (p_i^t - d^t) \Big)$$

Centralized resolution by convex optimization

- Computing $\theta(\lambda) =$ solving the n independent sub-problems
  - thermic units: dynamic optimization
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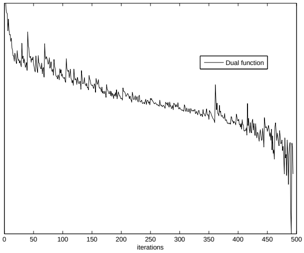
  - gives optimal prices $\lambda^\star \in \mathbb{R}^T$
  - initializes a 2nd phase heuristic that computes feasible plannings p_i
- E.g.: production 35000 - 70000 MW, mismatch at most 30MW



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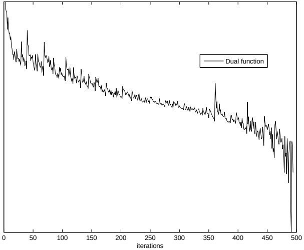

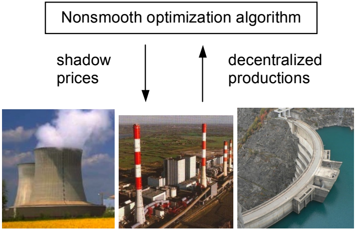

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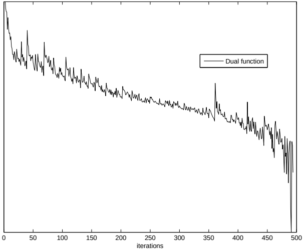

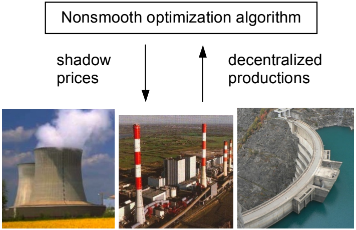

- On-going research: **1)** handle noise, stabilisation of prices
  2) interpretation of prices, new modeling

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Supervized multiclass classification

- Goal : classify objets (associated features)
  - data : couples feature/class $(x_i, y_i) \in \mathbb{R}^p \times \{0,1\}^K$
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- E.g. : biostats - computer vision

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- Optimization: learning a classifier from data
  - compute a "weight" matrix $W \in \mathbb{R}^{p \times K}$
  - minimize an error function (+ regularization)

$$\min_{W \in \mathbb{R}^{p \times K}} \frac{1}{n} \sum_{i=1}^{n} L(y_i, W^\top x_i) + \alpha \operatorname{Reg}(W)$$

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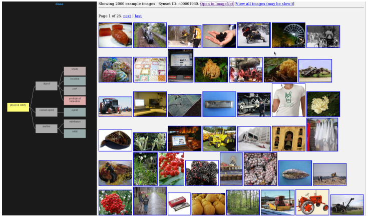
- State of the art: OK for $K = 2$ (and $K \approx 10$)
- All this is well-known, but the challenge is : K large
- Eg: Pascal Challenge '10 "imagenet" : $K \approx 1000$ (but not reliable)

Multiclass classification with sub-structure

- If unknown underlying structure...
- $\rightarrow$ Low rank W
- $W = UV^\top,\ U \in \mathbb{R}^{p \times r}, V \in \mathbb{R}^{K \times r}$
- Factorization of information
  (and cheaper computations)



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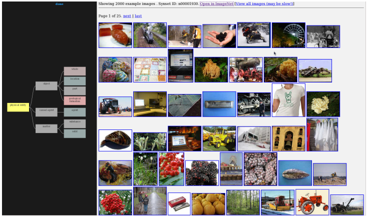

- Optimization: large-scale non-convex problem

$$\min_{W \in \mathbb{R}^{p \times K}} \; \alpha \, \mathrm{rank}(W) \; + \frac{1}{n} \sum_{i=1}^{n} L(y_i, W^{\top} x_i)$$

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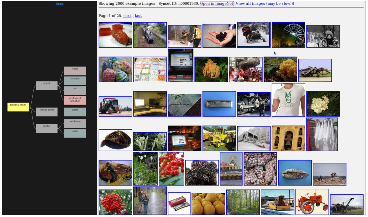

- Optimization: nonsmooth convex relaxation

$$\min_{W \in \mathbb{R}^{p \times K}} \ \alpha \, \|\sigma(W)\|_1 \ + \frac{1}{n} \sum_{i=1}^{n} L(y_i, W^\top x_i)$$

where $\sigma(W)$ is the vector of singular values

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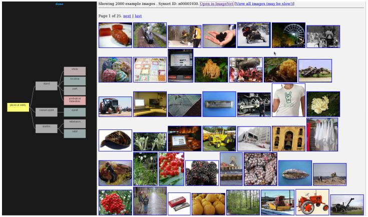

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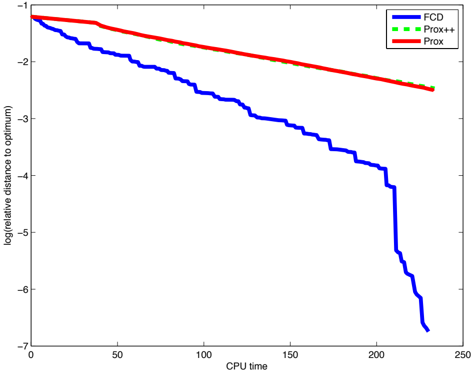
  where $\sigma(W)$ is the vector of singular values
- Our approach: constructive, rang control, no SVD
  (handle only rank-one matrices)
- Cheap algorithm of type "greedy coordinate descent"

Numerical illustration

- 2010 Pascal Visual Object Classes Challenge ("imagenet")
- $K = 1000$ $(p = 600)$
- Plot: relative error $\log\left((f(x_k) - f^*)/f^*\right)$ vs time
- Our algorithm outperforms the state-of-the-art



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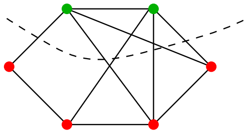
Combinatorial optimization

- Many combinatorial optimization problems can be written as quadratic problems under quadratic constraints

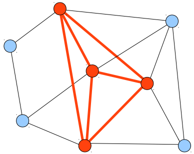
$$(\mathrm{P}) \quad \begin{cases} \min \quad x^\top Q_0\, x + {b_0}^\top x \\ \quad x^\top Q_i\, x + {b_i}^\top x + c_i = (\text{or} \leqslant)\, 0 \\ \quad x \in \{0,1\}^n \end{cases}$$

- Examples: (in graphs, models for real-life applications)

find a max cut



find a densest k-subgraph



- Problems: NP-hard... and, in fact, hard to solve (eg: $n = 100$)
- Tool: smart enumeration (branch-and-bound) using lower bounds of the optimal value of (P) (and sub-problems of (P))

Convex relaxations

- "Relaxation" : relax some constraints of (P) to get a convex problem (R) that we solve "easily"

$$\mathrm{val(P)} \geqslant \mathrm{val(R)}$$

- Eg: replace $\{0, 1\}$ by $[0, 1]$ $\rightarrow$ linear problems (CPLEX)

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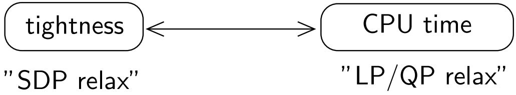

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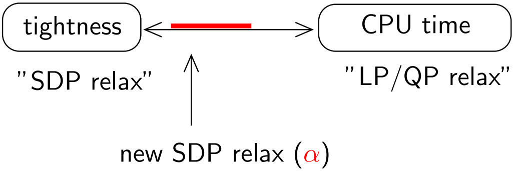
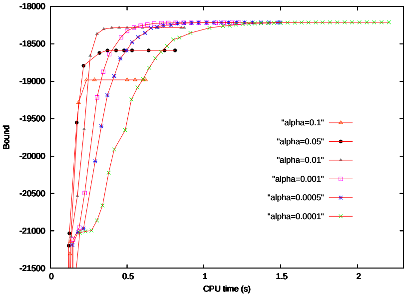
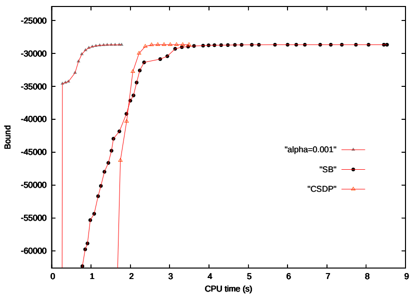


- New relaxations of type SDP (ou LMI)
  - goal: have (R) solved quickly
  - strategy: have SDP quality without its price (good pruning in B&B)
  - technique: original reformulation + convex duality

Numerical illustrations





- New bounds and solver allow us
  - to attain LP/QP bounds quicker than with dedicated solvers
  - to (almost) attain SDP bounds quicker than with dedicated solvers

- Densest k-subgraph: a simple exact resolution scheme using these bounds outperforms the best existing approach (using CPLEX) (due to a better prunning in the B&B search tree...)