Analysis (and geometry) of alternating projection algorithms

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Séminaire de l'équipe ACSIOM, Montpellier – March 2010

based on joint work with Adrian Lewis and Russell Luke

Outline

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Projection, distance and convexity

In a Euclidean space $(\mathbb{R}^n, \|\cdot\|)$

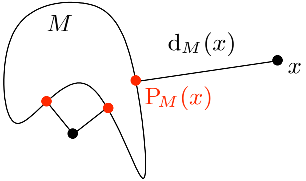
For closed $M \subset \mathbb{R}^n$, the distance of x from M

$$\mathrm{d}_M(x) = \min\{\|x - y\| : \; y \in M\}$$

and the projection of x onto M

$$\mathrm{P}_M(x) = \operatorname{argmin}\{\|x - y\| : \; y \in M\}$$

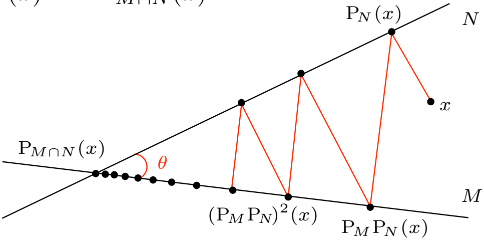
If M is convex, $\mathrm{P}_M(x)$ is singleton. Otherwise, it is not for some x for sure!



Alternating projections on subspaces

For affine subspaces M and N, von Neumann '33 studied:

$$(\mathrm{P}_M\,\mathrm{P}_M)^n(x) \longrightarrow \mathrm{P}_{M\cap N}(x)$$



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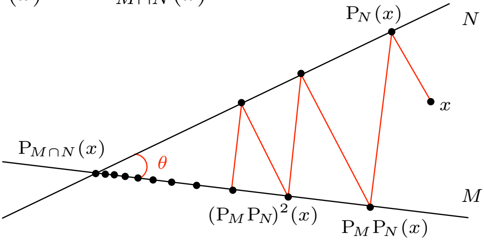


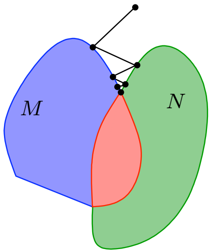
Convergence is linear at rate $(\cos\theta)^2$, indeed:

$$\|(\mathrm{P}_M\,\mathrm{P}_M)^n(x) - \mathrm{P}_{M\cap N}(x)\| \leqslant (\cos\theta)^{2n-1}\|x\|$$

where θ is the angle between M and N (Aronszajn '50)

Alternating convex projections

Alternating projections naturally extends to closed convex sets M and N



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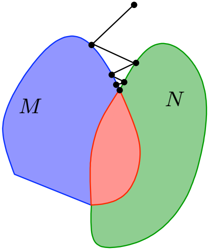

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Convergence is linear providing

$$M \cap \operatorname{int} N \neq \varnothing \qquad \text{(more generally } \operatorname{ri} M \cap \operatorname{ri} N \neq \varnothing\text{)}$$

(Polyak *et al* '67, Bauschke-Borwein '93)

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- in Hilbert, complex spaces...
- several sets, averaged, cyclic projections...
- relaxations, regularization (under/over-relaxed, AAR, Dykstra,...)

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Ex3: Combettes et al. '10 "on the effectiveness of projection methods for convex feasibility problems with linear inequality constraints"

Example in Finance

For symmetric matrix C, computing the nearest correlation matrix: computing the projection of C onto the intersection of $\mathcal{S}_n^+$ the semidefinite positive matrices, and the matrices with ones the diagonal

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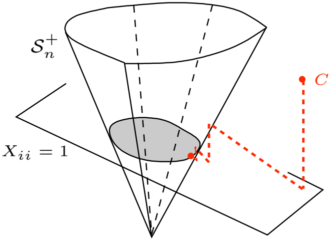

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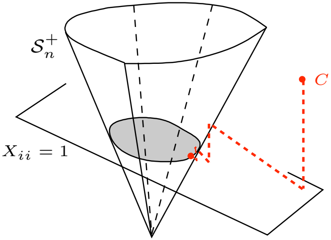

1. alternating projection (Higham '02) (+ Dykstra correction)
2. Lagrangian duality (Malick '04) $\longrightarrow$ efficient algorithm

Nonconvex heuristic

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- Optics : phase retrieval of images (Combettes *et al* '02)
  Simple version : given $a_j \in \mathbb{C}^k$, find $x \in \mathbb{C}^k$, so

  $$|\langle a_j, x\rangle| = b_j \quad (j = 1, \ldots, m)$$

  with alternative projections onto

  $$\begin{aligned} M &= \{(x,z) \in \mathbb{C}^n \times \mathbb{C}^m : \; Ax = z\} \\ N &= \{(x,z) : \; |z_j| = b_j, \; (j = 1, \ldots, m)\} \end{aligned}$$

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- Control : low-order control design (eg Grigoriadis-Beran '00)

  $$\begin{aligned} \text{affine } M &\subset \{n\text{-by-}n \text{ symmetric matrices}\} \\ N &= \{\text{positive semidefinite matrices of rank } r\} \end{aligned}$$

Numerical illustration

Find a 100-by-110 matrix X of rank 4, satisfying 450 equations

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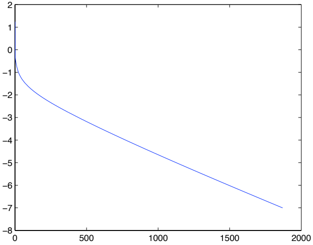

Alternating nonconvex projections

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Few answers:

- 1st results in Combettes-Trussel '90 (with convex-like techniques)
- linear cvg in special cases (Orsi '06 for a matrix analysis pb),...
- or of special algos (Attouch-Bolte-Redont-Soubeyrand '08, Luke '09)
- Lewis-Malick '07, Lewis-Luke-Malick '08 – whose ingredients are:
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In this talk:

1. easy-to-compute nonconvex projections
2. convergence of the algorithm through nice geometry

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For closed nonconvex $M \subset \mathbb{R}^n$, the projection $\mathrm{P}_M(x)$ is somewhere nonsingleton. But projection may still be easy...

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- Single quadratic constraint

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- Rank constraint

$$M = \left\{ X \in \mathbb{R}^{n \times m} : \quad \mathrm{rank}(X) = r \right\}$$

  To project, find a singular value decomposition $X = UDV$ and zero all but the first r largest singular values in D

Spectral sets

For permutation-invariant $K \subset \mathbb{R}^n$, the spectral set of symmetric matrices

$$\lambda^{-1}(K) = \{X \in \mathcal{S}_n : \quad (\lambda_1(X), \ldots, \lambda_n(X)) \in K\}$$

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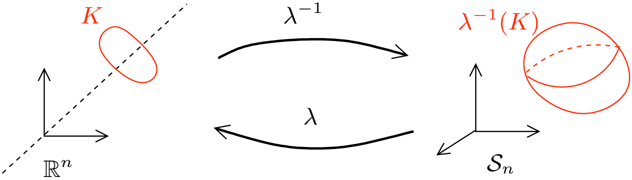

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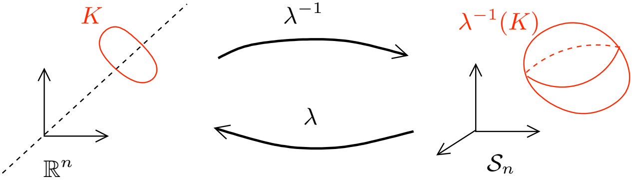


Examples

- $K = \mathbb{R}^n_+$ gives the positive semidefinite cone $\mathcal{S}^+_n$
- $K = \Sigma_n \cdot x$ gives an isospectral set (given x)
- $K = \{x : \ \mathrm{Card}\,(\mathrm{argmax}\{x_i\}) = r\}$ gives $\{X : \ \lambda_{\max}(X)$ with $r\}$

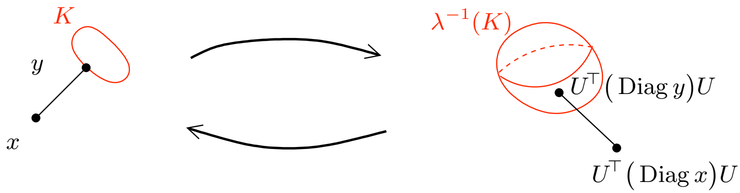
Easy spectral projections

The following result (Lewis-Malick '07) generalizes previous (partial) results about projections onto some spectral sets (eg Higham '88, Oustry '02)

Theorem (projection onto spectral sets)

If $y \in \mathrm{P}_K(x)$ *and* U *orthogonal, then*

$$U^\top(\mathrm{Diag}\, y)U \;\in\; \mathrm{P}_{\lambda^{-1}(K)}(U^\top(\mathrm{Diag}\, x)U)$$



Prox-regular spectral sets

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$$K \text{ prox-regular } \implies \lambda^{-1}(K) \text{ prox-regular}$$

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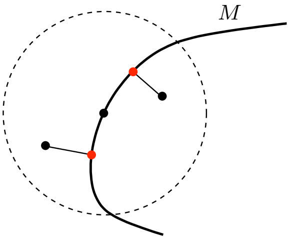
General notion of prox-regularity
(eg Rock.-Poliquin-Thibault '00):

P_M is locally unique



⟶ prox-regular spectral sets have locally all the good properties !

(Ex: manifolds...)

Prox-regular spectral sets in practice

$\longrightarrow$ Many spectral sets in alternative nonconvex projections

1. Numerical algebra: nonnegative inverse eigenvalue pb (Orsi '06)
For $\bar{\lambda}$ given, find $X \in M \cap N$

$$\begin{aligned} M &= \{X \in \mathbb{R}^{n\times n} : \ \lambda(X) = \bar{\lambda}\,\} \\ N &= \{X \in \mathbb{R}^{n\times n} : \ X_{ij} \geqslant 0\,\} \end{aligned}$$

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2. Image processing: design of tight frames (Tropp *et al* '05)
   Find the associated Gram matrix $X \in M \cap N$

$$\begin{aligned} M &= \{X \in \mathbb{C}^{n \times n} : \ \lambda(X) = (n, \ldots, n, 0, \ldots, 0)/d \,\} \\ N &= \{X \in \mathbb{C}^{n \times n} : \ X_{ii} = 1, \ \|X\|_\infty \leqslant \mu \,\} \end{aligned}$$

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⟶ Many other types of sets (ex: Phase retrieval)...

Last word on spectral sets

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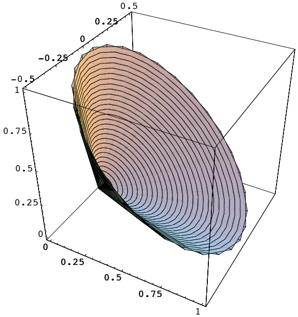
- No, in general

  Ex: $M =]-1,1[\times\{0\} \subset \mathbb{R}^2$,
  and $\lambda^{-1}(M)$ has a kink

- Yes, if M is loccally symmetric !

  a neighbd of M around $x \in M$
  is invariant under permutations
  σ such that $\sigma x = x$



Moreover, we know the dimension of $\lambda^{-1}(M)$

Not straighforward... 43 pages of sheer joy (Daniilidis-Malick-Sendov '09)

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Theorem (local linear convergence)

For closed sets $M, N \subset \mathbb{R}^n$. Assume

- *strong regularity holds at $\bar{x} \in M \cap N$*
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- Inexact versions still work
- More in Lewis-Luke-Malick '08...

Strong regularity

Simple definition:

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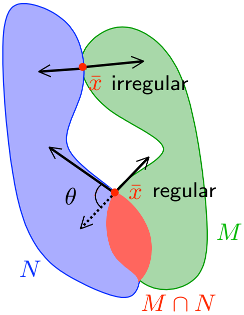

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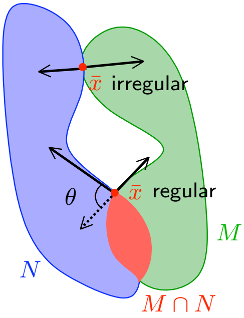

Examples

1. The intersection of two smooth manifolds is strongly regular
   $\iff$ the manifolds are transverse
2. The intersection of two convex sets is strongly regular
   $\iff$ no separating hyperplane

Super-regularity

Notion (not standard!) introduced in
Lewis-Luke-Malick '08

Examples of super-regular sets

1. convex sets
2. smooth manifolds

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Examples of super-regular sets

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2. smooth manifolds
3. prox-regular sets

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Examples of super-regular sets

1. convex sets
2. smooth manifolds
3. prox-regular sets
4. constraint sets with Mangasarian-Fromovitz

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Notion (not standard!) introduced in Lewis-Luke-Malick '08

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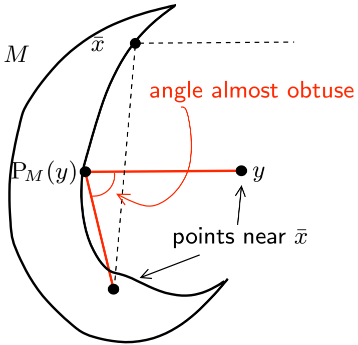

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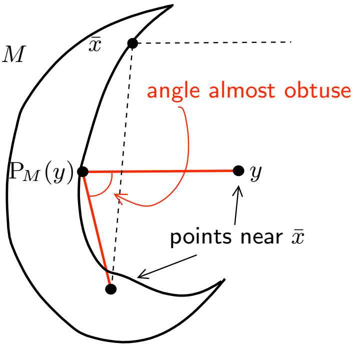
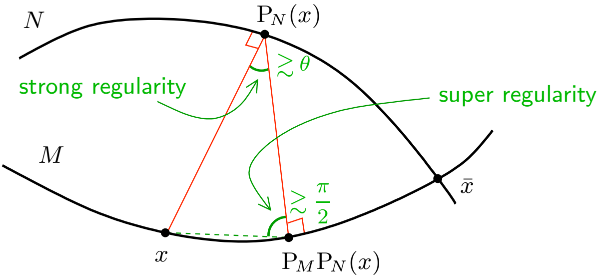


prox-regular $\subset$ super-regular $\subset$ (Clarke) regular

Sketch proof

The geometry controls the asymptotical improvement:



For $x \in M$ near $\bar{x}$,

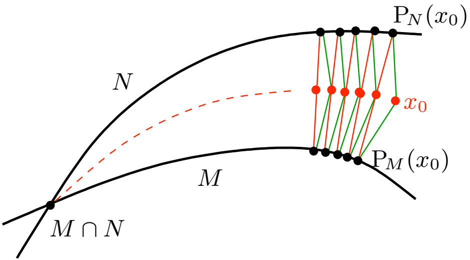
$$\frac{\| P_M P_N(x) - P_N(x)\|}{\| P_N(x) - x\|}$$

is not much larger than $\cos\theta$

Consequence for averaged projections

Method of averaged projections

$$
\begin{aligned}
z_M &\in P_M(x) \\
z_N &\in P_N(x) \\
x &\leftarrow \frac{1}{2}(z_N + z_M)
\end{aligned}
$$



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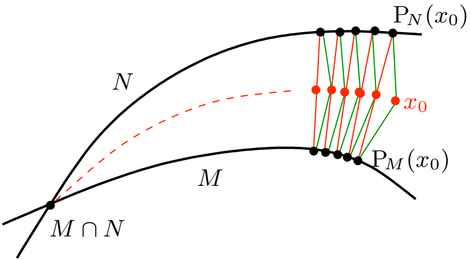

Corollary (linear convergence of averaged projections)

For any closed $M, N \subset \mathbb{R}^n$, if strong regularity holds at $\bar{x} \in M \cap N$, then starting with x_0 near $\bar{x}$, averaged projections converges linearly to $M \cap N$

More on averaged projections

1. **Proof:** Following Auslender '69 in the convex case, just consider alternating projections in $\mathbb{R}^n \times \mathbb{R}^n$ between

$$M \times N \qquad \{(x,x) : \ x \in \mathbb{R}^n\} \text{ (super-regular)}$$

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2. **Interpretation as minimization** (Lewis-Luke-Malick '08)
If M and N both prox-regular, averaged projections is just the steepest descent (with unit step size) applied to

$$f(x) = \frac{1}{4}\left(\mathrm{d}^2_M(x) + \mathrm{d}^2_N(x) \right)$$

$\longrightarrow$ Q-linear convergence: improvements at each iteration

$$\frac{f(x_{k+1})}{f(x_k)} < 1 - \frac{1}{2\kappa^2}$$

Numerical illustration

Averaged projections to find d-by-m matrix $U \in L \cap M \cap C$

$$
\begin{array}{rrcl}
\text{(linear)} & L & = & \{U \in \mathbb{R}^{d\times m} :\ U = PW\} \\
\text{(smooth)} & M & = & \{U \in \mathbb{R}^{d\times m} :\ UU^\top = I\} \\
\text{(convex)} & C & = & \{U \in \mathbb{R}^{d\times m} :\ \|U\|_\infty \leqslant \alpha\}
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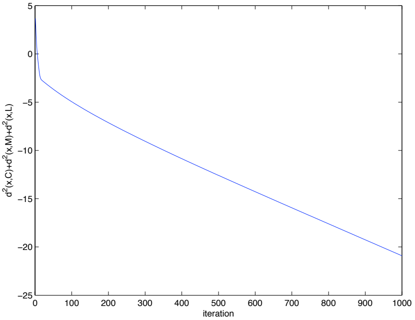


$$\frac{f(U_{k+1})}{f(U_k)} \leqslant 0.96 < 1$$

where f is the sum of the squared distances

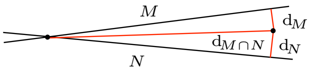
Outline

1. Alternating convex projections
2. Nonconvex projections
3. Alternating nonconvex projections
4. **Regularity and conditionning**

alternating projections + small angle

Weak error bound:

$$\mathrm{d}^2_{M\cap N}(\cdot) \leqslant \rho\big(\,\mathrm{d}^2_M(\cdot) + \mathrm{d}^2_N(\cdot)\big)$$



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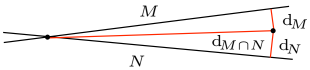

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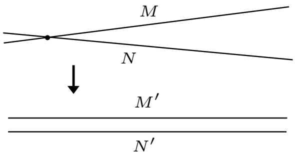
Small perturbations render the problem ill-posed



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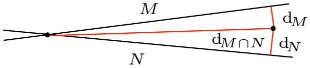

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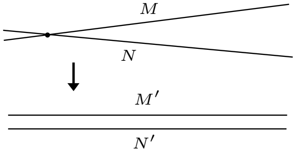
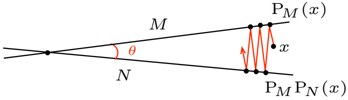


Alternating method converges with slow linear rate $\cos\theta$



Conditionning

For linear subspaces for example, it is obvious on picture that instances are not well-conditionning, when the angle θ is small...

... moreover $\cos\theta$ controls so the speed of alternating (and averaged) projection algorithms.

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- Demmel paradigm (Demmel '87)
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For many computational problems, three equivalent properties characterize "hard" instances:

1. a posteriori error bounds are weak
2. the distance to an ill-posed instance is small
3. basic algorithms are slow

Simple example: solving linear system

Consider the positive-definite linear system

$$A\,x = y$$

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$$\|x - A^{-1}y\| \leqslant \frac{1}{\lambda_{\min}(A)}\|Ax - y\|$$

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3. Basic algorithms converge with slow linear rate. Eg:

$$\left(\frac{\kappa - 1}{\kappa + 1}\right)^2 \quad \text{and} \quad \left(\frac{\sqrt{\kappa} - 1}{\sqrt{\kappa} + 1}\right)^2$$

for steepest descent and conjuguate gradients, where $\kappa = \lambda_{\max}(A)/\lambda_{\min}(A)$

General framework

A very general framework: invertion
Given set-valued $F\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^p$, suppose

- $F(x)$ is easy to compute
- $F^{-1}(x)$ is hard to compute

Problem: Given a point $\bar{y}$, find some point $\bar{x} \in F^{-1}(\bar{y})$

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Example: (Lewis-Luke-Malick '08)
Given $M, N \subset \mathbb{R}^n$, find $\bar{x} \in M \cap N$
Define $G\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^n \times \mathbb{R}^n$ by

$$G(x) = (M - x) \times (N - x)$$

Finding $\bar{x} \in G^{-1}(0,0) = M \cap N$ is solving the generalized equation

$$(0,0) \in G(x)$$

To quantify local error bounds

Definition (Metric regularity)

Suppose $\bar{y} \in F(\bar{x})$. We say that F is metrically regular at $(\bar{x}, \bar{y})$ if the local error bound

$$\mathrm{d}_{F^{-1}(y)}(x) \leqslant \rho \, \mathrm{d}_{F(x)}(y) \qquad \text{for all } (x, y) \text{ near } (\bar{x}, \bar{y})$$

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  $\iff$ closed M and N have strongly regular intersection
- With θ the minimum angle between $\mathrm{N}_M(\bar{x})$ and $-\mathrm{N}_N(\bar{x})$
  $$\operatorname{reg} G(\bar{x}, (0,0)) = \frac{1}{\sqrt{1 - \cos\theta}}$$
  hence the modulus controls local linear convergence rates

Condition number

Modulus of metric regularity quantifies hardness of instances:

1. the a posteriori error bounds (definition of reg)
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- linear systems
- alternating projections (Lewis-Malick '07, Lewis-Luke-Malick '08)
- proximal point methods (Aragon-Artacho-Dontchev-Geoffroy '05)
- several conceptual algorithms (Klatte-Kummer '07)
- errors bounds and descent methods (Luo-Tseng '93)
- more ?...

Summary

- $\longrightarrow$ Nonconvex projections are tractable in some usual situations
- $\longrightarrow$ Alternating nonconvex projections is a tempting natural heuristic, often converges linearly, and is thus popular !
- $\longrightarrow$ The linear rate reflects the "condition number"

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