OR Complementary – Final Exam 2025

- This is a 2-hour and 2-page exam with 3 independent exercises.
- Approximate grading: 6 – 5 – 10.
- The quality of the presentation and argumentation will be an important element of the evaluation.
- Document allowed: one page front/back.

Exercise 1 – Random questions taken from the course. Provide short and accurate answers to the following technical questions, taken from the course or left as exercices in the course.

a) Adjacency matrix. Let $G = (V, E)$ a graph and A its adjacency matrix. What appears on the diagonal of A^2 ?

b) Spectral radius. Consider a matrix $A \in \mathbb{R}^{n\times n}$, its spectral radius $\rho(A)$, and the induced norm $|||A||| = \max_{x\neq 0} \frac{\|Ax\|}{\|x\|}$. Show that $\rho(A) \leqslant |||A|||$.

c) Decomposition. Let X a symmetric $n \times n$ matrix. Show the equivalence: X is positive semidefinite $\iff$ $X = L^\top L$ with $L \in \mathbb{R}^{n\times n}$.

d) Positive semidefinite cone. Show that the set $\mathbb{S}_n^+$ of positive semidefinite matrices is a convex cone in the space $\mathbb{S}_n$ of $n \times n$ symmetric matrices. [Bonus question: show that $\mathbb{S}_n^+$ is closed and that its interior corresponds to the set of positive *definite* matrices.]

e) Convex duality. Using the result of the previous question, explain why there is no duality gap in the pair of primal-dual problems, with $u \in \mathbb{R}^n$ and $X \in \mathbb{S}_n$

$$\left\{\begin{array}{ll} \max_u & c^\top u \\ & W + \mathrm{Diag}(u) \in \mathbb{S}_n^+ \end{array}\right. \qquad \left\{\begin{array}{ll} \min_X & \mathrm{trace}(WX) \\ & \mathrm{diag}(X) = c,\ X \in \mathbb{S}_n^+ \end{array}\right.$$

where $c \in \mathbb{R}^n$, $W \in \mathbb{S}_n$, and Diag/diag are the two (adjoint) operators of the course[1].

Exercise 2 – Small parametric game. Consider this game depending on the parameter $x \in \mathbb{R}$:

		Player 2: A	Player 2: B
Player 1	A	(0.5,0.5)	$(x, 1-x)$
	B	$(1-x, x)$	(0.5, 0.5)

a) What are the pure Nash equilibrium of this game, depending on x ?

b) Given $(q, 1-q)$ a mixed strategy for Player 2, what is the expected payoff for Player 1 if he plays A? Same question if Player 1 plays B.

c) Following the notation of the course, let a mixed Nash equilibrium $\big((p^*, 1-p^*), (q^*, 1-q^*)\big)$ (not a pure one, so $p^* \notin \{0, 1\}$). Show that we have: $0.5q^\star + (1-q^\star)x - (1-x)q^\star - 0.5(1-q^\star) = 0$. Explain briefly why this makes sense and why this property is called "indifference".

d) What are the mixed Nash equilibrium of this game, depending on x ?

[1] Diag: $\mathbb{R}^n \to \mathbb{S}^n$ associates, to a vector $u \in \mathbb{R}^n$, the diagonal matrix with u on the diagonal; Diag: $\mathbb{S}^n \to \mathbb{R}^n$ associates, to a symmetric matrix X, the vector of its diagonal entries $u = (X_{11}, \ldots, X_{nn})$.

Exercise 3 – Augmented Lagrangian relaxation. We start this exercice with studying the following simple optimization problem in $\mathbb{R}^2$

$$\begin{cases} \max & -x_1 - 2x_2 \\ & x_1 + x_2 = 3 \\ & x_1 \in [0,2],\ x_2 \in \{0,2\}. \end{cases} \tag{P}$$

a) By observing that (P) reduces to the trivial problem

$$\begin{cases} \max & -x_1 - 4 \\ & x_1 = 1 \\ & x_1 \in [0,2], \end{cases}$$

give the optimal solution and the optimal value of (P).

b) What is the optimal solution and the optimal value of the convexified problem ? (where the constraint $x_2 \in \{0,2\}$ is replaced by $x_2 \in [0,2]$).

c) Write the Lagrangian and the dual function θ associated to the dualization in (P) of the constraint $x_1 + x_2 - 3 = 0$.

d) Draw the graph of θ. Give the dual optimal solution, the dual optimal value, and the duality gap.

Let's now turn to the general framework of the course

$$\begin{cases} \max & \varphi(x) \\ & c(x) = 0,\ x \in X. \end{cases}$$

For a parameter $\rho > 0$, we define the augmented Lagrangian function by

$$L^\rho(x;u) := \varphi(x) - u^\top c(x) - \rho\|c(x)\|^2$$

and the associated augmented dual function by

$$\theta^\rho(u) := \max_{x \in X} L^\rho(x;u).$$

d) Show that $\theta^\rho \colon \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is convex. Show that for any dual variable u and any primal feasible variable $x \in X$ such that $c(x) = 0$, we have $\theta^\rho(u) \geqslant \varphi(x)$.

e) Fix $\bar{u}$ and $x(\bar{u}) \in X$ such that $\theta^\rho(\bar{u}) = L^\rho(x(\bar{u});\bar{u})$. Prove that, if $c(x(\bar{u})) = 0$, then $\bar{u}$ minimizes θ^ρ, $x(\bar{u})$ is a primal optimal solution, and that there is no duality gap.

Augmented Lagrangians have the following nice property. Contrary to *standard* Lagrangian duality, *augmented* Lagrangian duality always zeroes the duality gap and recovers primal solutions (when ρ is large enough). The aim of this exercise is to prove this property for (P) and $\rho = 3$.

f) Write the augmented Lagrangian and the augmented dual function θ^3 (that is, θ^ρ for $\rho = 3$) associated to the dualization of $x_1 + x_2 - 3 = 0$ in problem (P). Show that θ^3 can be cast as

$$\theta^3(u) = \max\{\theta_0^3(u), \theta_2^3(u)\}$$

with two concave functions that we denote by θ_0^3 and θ_2^3 (no need to develop them explicitly).

g) Show that $\theta^3(-1) = -5$.

h) Conclude that $\bar{u} = -1$ minimizes θ^3 and that there is no duality gap.

i) Thus solving the augmented Lagrangian dual allows us to solve the primal problem! But there is no free lunch: what is the big disadvantage of augmented Lagrangian (versus the usual Lagrangian)?