Longstaff Schwartz algorithm and Neural Network regression

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Introduction

- Computing an American option involving a large number of assets remains numerically challenging.
- A hope: Neural Network (NN) can (may) help to reduce the computational burden.
- Some previous works using NN for optimal stopping (not LS algorithm though)
  - Michael Kohler, Adam Krzyżak, and Nebojsa Todorovic. Pricing of high-dimensional american options by neural networks. *Mathematical Finance: An International Journal of Mathematics, Statistics and Financial Economics*, 20(3):383–410, 2010
  - S. Becker, P. Cheridito, and A. Jentzen. Deep optimal stopping. *Journal of Machine Learning Research*, 20(74):1–25, 2019

Computing Bermudan options prices

- A discrete time (discounted) payoff process $(Z_{T_k})_{0\leq k\leq N}$ adapted to $(\mathcal{F}_{T_k})_{0\leq k\leq N}$. $\max_{0\leq k\leq N}|Z_{T_k}|\in L^2$.
- The time-T_k discounted value of the Bermudan option is given by

$$U_{T_k} = \text{esssup}_{\tau\in\mathcal{T}_{T_k}}\mathbb{E}[Z_\tau|\mathcal{F}_{T_k}]$$

  where $\mathcal{T}_t$ is the set of all $\mathcal{F}-$ stopping times with values in $\{T_k, T_{k+1}, ..., T\}$.
- From the Snell enveloppe theory, we derive the standard dynamic programming algorithm ($\rightarrow$ "Tsistsiklis-Van Roy" type algorithms).

$$\begin{cases} U_{T_N} = Z_{T_N} \\ U_{T_k} = \max\left(Z_{T_k}, \mathbb{E}[U_{T_{k+1}}|\mathcal{F}_{T_k}]\right) \end{cases} \tag{1}$$

The policy iteration approach...

Let τ_k be the smallest optimal stopping time after T_k.

$$\begin{cases} \tau_N = T_N \\ \tau_k = T_k \mathbf{1}_{\{Z_{T_k} \geq \mathbb{E}[Z_{\tau_{k+1}} | \mathcal{F}_{T_k}]\}} + \tau_{k+1} \mathbf{1}_{\{Z_{T_k} < \mathbb{E}[Z_{\tau_{k+1}} | \mathcal{F}_{T_k}]\}}. \end{cases} \tag{2}$$

This is a dynamic programming principle on the policy not on the value function $\rightarrow$ "Longstaff-Schwartz" algorithm.
This approach has the practitioners' favour for its robustness.

Difficulty: how to compute the conditional expectations?

...in a Markovian context

- Markovian context: $(X_t)_{0\leq t\leq T}$ is a Markov process and $Z_{T_k} = \phi_k(X_{T_k})$.

$$\mathbb{E}[Z_{\tau_{k+1}}|\mathcal{F}_{T_k}] = \mathbb{E}[Z_{\tau_{k+1}}|X_{T_k}] = \psi_k(X_{T_k})$$

  where ψ_k is a measurable function.
- Because of the L^2 assumption, ψ_k can be computed by a least-square problem

$$\inf_{\psi\in L^2(\mathcal{L}(X_{T_k}))} \mathbb{E}\left[\left|Z_{\tau_{k+1}} - \psi(X_{T_k})\right|^2\right]$$

Different numerical strategies

- The standard numerical (LS) approach: approximate the space L^2 by a finite dimensional vector space (polynomials, ...)
- We investigate the use of Neural Networks to approximate ψ_k.
- Kohler et al. [2010]: neural networks but in a different context (approximation of the value function Tsitsiklis and Roy [2001], equation (1)) and re-simulation of the paths at each time steps.

LS: truncation step

Longstaff-Schwartz type algorithms rely on direct approximation of *stopping times* and use of *the same simulated paths* for all time steps (obvious and large computational gains).

- $(g_k, k \geq 1)$ is an $L^2(\mathcal{L}(X))$ basis and $\Phi_p(X,\theta) = \sum_{k=1}^{p} \theta_k \, g_k(X)$.
- Backward approximation of iteration policy using (2),

$$\begin{cases} \widehat{\tau}_N^{p,} = T_N \\ \widehat{\tau}_n^p = T_n \, \mathbf{1}_{\left\{Z_{T_n} \geq \Phi_p(X_{T_n};\widehat{\theta}_n^p)\right\}} + \widehat{\tau}_{n+1}^p \, \mathbf{1}_{\left\{Z_{T_n} < \Phi_p(X_{T_n};\widehat{\theta}_n^p)\right\}} \end{cases}$$

- with conditional expectation computed using a Monte Carlo minimization problem: $\widehat{\theta}_n^p$ is a minimizer of

$$\inf_{\theta} \mathbb{E}\left(\left|\Phi_p(X_{T_n};\theta) - Z_{\widehat{\tau}_{n+1}^p}\right|^2\right).$$

- Price approximation: $U_0^p = \max\left(Z_0, \mathbb{E}\left(Z_{\widehat{\tau}_1^p}\right)\right)$.

The LS algorithm

- $(g_k, k \geq 1)$ is an $L^2(\mathcal{L}(X))$ basis and $\Phi_p(X,\theta) = \sum_{k=1}^{p} \theta_k \, g_k(X)$.
- Paths $X_{T_0}^{(m)}, X_{T_1}^{(m)}, \ldots, X_{T_N}^{(m)}$ and payoff paths $Z_{T_0}^{(m)}, Z_{T_1}^{(m)}, \ldots, Z_{T_N}^{(m)}$, $m = 1, \ldots, M$.
- Backward approximation of iteration policy,

$$\begin{cases} \widehat{\tau}_N^{p,(m)} &= T_N \\ \widehat{\tau}_n^{p,(m)} &= T_n \mathbf{1}_{\left\{Z_{T_n}^{(m)} \geq \Phi_p(X_{T_n}^{(m)}; \widehat{\theta}_n^{p,M})\right\}} + \widehat{\tau}_{n+1}^{p,(m)} \mathbf{1}_{\left\{Z_{T_n}^{(m)} < \Phi_p(X_{T_n}^{(m)}; \widehat{\theta}_n^{p,M})\right\}} \end{cases}$$

- with conditional expectation computed using a Monte Carlo minimization problem: $\widehat{\theta}_n^{p,M}$ is a minimizer of

$$\inf_{\theta} \frac{1}{M} \sum_{m=1}^{M} \left| \Phi_p(X_{T_n}^{(m)}; \theta) - Z_{\tau_{n+1}^{p,(m)}}^{(m)} \right|^2 .$$

- Price approximation: $U_0^{p,M} = \max\left(Z_0, \frac{1}{M} \sum_{m=1}^{M} Z_{\widehat{\tau}_1^{p,(m)}}^{(m)}\right)$.

Reference papers

- Description of the algorithm:

  F.A. Longstaff and R.S. Schwartz. Valuing American options by simulation : A simple least-square approach.
  Review of Financial Studies, 14:113–147, 2001.

- Rigorous approach:

  Emmanuelle Clément, Damien Lamberton, and Philip Protter. An analysis of a least squares regression method for american option pricing.
  Finance and Stochastics, 6(4):449–471, 2002.

  - U_0^p converge to U_0, $p \to +\infty$
  - $U_0^{p,M}$ converge to U_0^p, $M \to +\infty$ a.s.
  - "almost" a central limit theorem

The modified algorithm

- In LS algorithm replace the approximation on a Hilbert basis $\Phi_p(.;\theta)$ by a Neural Network. This is not a vector space approximation (non linear).
- The optimization problem is non linear, non convex, ...
- Aim: extending the proof of (a.s.) convergence results

A quick view of Neural Networks

- In short, a NN: $x \to \Phi_p(x, \theta) \in \mathbb{R}$, with $\theta \in \mathbb{R}^d$, d large
- $\Phi_p = A_L \circ \sigma_a \circ A_{L-1} \circ \cdots \circ \sigma_a \circ A_1$, $L \geq 2$
- $A_l(x_l) = w_l x_l + \beta_l$ (affine functions)
- $L - 2$ "number of hidden layers"
- p "maximum number of neurons per layer" (i.e. sizes of the w_l matrix)
- σ_a a fixed non linear (called *activation function*) applied component wise
- $\theta := (w_l, \beta_l)_{l=1,\ldots,L}$ parameters of all the layers
- Restriction to a compact set $\Theta_p = \{\theta : |\theta| \leq \gamma_p\}$ and assume $\lim_{p\to\infty} \gamma_p = \infty$. $\to$ use the USLLN.
- $\mathcal{NN}_p = \{\Phi_p(\cdot, \theta) : \theta \in \Theta_p\}$ and $\mathcal{NN}_\infty = \cup_{p\in\mathbb{N}} \mathcal{NN}_p$

Hypothesis **H**

- For every p, there exists $q \geq 1$

$$\forall \theta \in \Theta_p, \quad |\Phi_p(x,\theta)| \leq \kappa_q(1+|x|^q)$$

  a.s. the random function $\theta \in \Theta_p \longmapsto \Phi_p(X_{T_n},\theta)$ are continuous.
- $\mathbb{E}[|X_{T_n}|^{2q}] < \infty$ for all $0 \leq n \leq N$.
- For all p, $n < N$, $\mathbb{P}\left(Z_{T_n} = \Phi_p(X_{T_n};\theta_n^p)\right) = 0$.
- If θ_1 and θ_2 solve

$$\inf_{\theta \in \Theta_p} \mathbb{E}\left(\left|\Phi_p(X_{T_n};\theta) - Z_{\widehat{\tau}_{n+1}^p}\right|^2\right),$$

  then $\Phi_p(x,\theta_1) = \Phi_p(x,\theta_2)$ for almost all x
  No need of a unique minimizer but only of the represented function.

The result

Theorem 1

Under hypothesis **H**

- *Convergence of the Neural network approximation*

$$\lim_{p\to\infty} \mathbb{E}[Z_{\tau_n^p}|\mathcal{F}_{T_n}] = \mathbb{E}[Z_{\tau_n}|\mathcal{F}_{T_n}] \text{ in } L^2(\Omega) \quad (i.e.\ U_0^p \to U_0).$$

- *SLLN: for every* $k = 1, \dots, N$,

$$\lim_{M\to\infty} \frac{1}{M} \sum_{m=1}^{M} Z^{(m)}_{\widehat{\tau}_k^{p,(m)}} = \mathbb{E}\left[Z_{\tau_k^p}\right] \quad a.s. \quad (i.e.\ U_0^{p,M} \to U_0^p)$$

Convergence of the NN approximation

A simple consequence of Hornik [1991].

- Also known as the "Universal Approximation Theorem".

Theorem 2 (Hornik)

Assume that the function σ_a is non constant and bounded. Let μ denote a probability measure on $\mathbb{R}^r$, then $\mathcal{NN}_\infty$ is dense in $L^2(\mathbb{R}^r, \mu)$.

- Corollary: If for every p, $\alpha_p \in \Theta_p$ is a minimizer of

$$\inf_{\theta \in \Theta_p} \mathbb{E}[|\Phi_p(X;\theta) - Y|^2],$$

  $(\Phi_p(X;\alpha_p))_p$ converges to $\mathbb{E}[Y|X]$ in $L^2(\Omega)$ when $p \to \infty$.
- proof of the convergence of the "non-linear approximation" $\Phi_p(X;\theta)$.

Convergence of Monte-Carlo approximation

- p is fixed, $N \to +\infty$
- Now, minimisation problems are non linear, need more abstract arguments to prove convergence
- Two ingredients (quite "old" results)
- First result: approximation of minimization problems

Lemma 3 (Rubinstein and Shapiro [1993])

- *$(f_n)_n$ defined on a compact set $K \subset \mathbb{R}^d$. $v_n = \inf_{x \in K} f_n(x)$*
- *x_n a sequence of minimizers $f_n(x_n) = \inf_{x \in K} f_n(x)$.*
- *$v^\star = \inf_{x \in K} f(x)$ and $\mathcal{S}^\star = \{x \in K : f(x) = v^\star\}$.*

If $(f_n)_n$ converges uniformly on K to a continuous function f, then $v_n \to v^\star$ and $d(x_n, \mathcal{S}^\star) \to 0$ a.s.

Convergence of Monte-Carlo approximation

- Second result: SLLN in Banach spaces (Ledoux and Talagrand [1991], goes back to Mourier [1953]).

Lemma 4

Let $(\xi_i)_{i\geq 1}$ *i.i.d.* $\mathbb{R}^m$*-valued,* $h : \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}$*. If*

- *a.s.,* $\theta \in \mathbb{R}^d \mapsto h(\theta, \xi_1)$ *is continuous,*
- $\forall K > 0,\ \mathbb{E}\left[\sup_{|\theta|\leq K} |h(\theta, \xi_1)|\right] < +\infty.$

Then

$$\lim_{n\to\infty} \sup_{|\theta|\leq K} \left| \frac{1}{n} \sum_{i=1}^{n} h(\theta, \xi_i) - \mathbb{E}[h(\theta, \xi_1)] \right| = 0 \quad a.s.$$

Convergence of Monte-Carlo approximation

Combining these two results with the backward iteration introduced by Clément et al. [2002], we get

Proposition

Under hypothesis **H**, *for every* $n = 1, \dots, N$, $\Phi(X_{T_n}^{(1)}; \widehat{\theta}_n^{p,M})$ *converges to* $\Phi_p(X_{T_n}^{(1)}; \theta_n^p)$ *a.s. when* $M \to \infty$.

Implementation details

- Python code with TensorFlow.
- We use ADAM algorithm to fit the neural network at each time step.
- We use the same NN through all time steps: take $\widehat{\theta}^{p,M}_{n+1}$, as the starting point of the training algorithm at time time n.
- No use of setting *epochs*> 1 for $n < N-1$. This allows for huge computational time savings.
- We only use the in-the-money paths

$$\inf_{\theta \in \Theta_p} \mathbb{E}\left[\left|\Phi_p(X_{T_n};\theta) - Z_{\tau^p_{n+1}}\right|^2 \mathbf{1}_{\{Z_{T_n}\}} > 0\right].$$

Put Basket option in the Black Scholes model

L	d_l	epochs=1	epochs=5	epochs=10
2	32	4.08 ($\pm$ 0.031)	4.1 ($\pm$ 0.034)	4.11 ($\pm$ 0.029)
2	128	4.08 ($\pm$ 0.036)	4.09 ($\pm$ 0.034)	4.1 ($\pm$ 0.032)
2	512	4.07 ($\pm$ 0.034)	4.09 ($\pm$ 0.036)	4.1 ($\pm$ 0.033)
4	32	4.07 ($\pm$ 0.034)	4.09 ($\pm$ 0.033)	4.1 ($\pm$ 0.032)
4	128	4.06 ($\pm$ 0.039)	4.09 ($\pm$ 0.04)	4.1 ($\pm$ 0.037)
4	512	4.05 ($\pm$ 0.037)	4.08 ($\pm$ 0.034)	4.09 ($\pm$ 0.031)
8	32	4.07 ($\pm$ 0.034)	4.09 ($\pm$ 0.037)	4.1 ($\pm$ 0.035)
8	128	4.06 ($\pm$ 0.039)	4.09 ($\pm$ 0.032)	4.1 ($\pm$ 0.035)
8	512	4.04 ($\pm$ 0.066)	4.07 ($\pm$ 0.069)	4.08 ($\pm$ 0.063)

Table: Basket option with $r = 0.05$, $d = 5$, $\sigma^i = 0.2$, $\omega^i = 1/d$, $S_0^i = 100$, $\rho = 0.2$, $K = 100$, $N = 20$ and $M = 100,000$. The standard Longstaff Schwartz algorithm yields 4.11 ± 0.03 (resp. 4.04 ± 0.034) for an order 3 (resp. 1) polynomial regression.

Put option in the Heston model

L	d_l	epochs=1	epochs=5	epochs=10
2	32	1.69 ($\pm$ 0.017)	1.7 ($\pm$ 0.017)	1.7 ($\pm$ 0.016)
2	128	1.69 ($\pm$ 0.017)	1.7 ($\pm$ 0.019)	1.7 ($\pm$ 0.019)
2	512	1.69 ($\pm$ 0.019)	1.69 ($\pm$ 0.019)	1.69 ($\pm$ 0.018)
4	32	1.69 ($\pm$ 0.022)	1.69 ($\pm$ 0.017)	1.7 ($\pm$ 0.018)
4	128	1.69 ($\pm$ 0.024)	1.69 ($\pm$ 0.02)	1.7 ($\pm$ 0.016)
4	512	1.68 ($\pm$ 0.025)	1.69 ($\pm$ 0.022)	1.69 ($\pm$ 0.022)
8	32	1.69 ($\pm$ 0.023)	1.69 ($\pm$ 0.02)	1.69 ($\pm$ 0.019)
8	128	1.68 ($\pm$ 0.03)	1.69 ($\pm$ 0.022)	1.69 ($\pm$ 0.02)
8	512	1.68 ($\pm$ 0.03)	1.68 ($\pm$ 0.041)	1.68 ($\pm$ 0.053)

Table: Prices for put option in the Heston model with parameters the geometric basket put option with parameters with $S_0 = K = 100$, $T = 1$, $\sigma_0 = 0.01$, $\xi = 0.2$, $\theta = 0.01$, $\kappa = 2$, $\rho = -0.3$, $r = 0.1$, $N = 10$ and $M = 100,000$. The standard Longstaff Schwartz algorithm yields 1.70 ± 0.008 (resp. 1.675 ± 0.005) for an order 6 (resp. 1) polynomial regression.

Conclusion

- Learn the continuation value using a NN instead of a polynomial regression
- NN do not help much for low dimensional problems but do scale far better
- Relatively small NN provide very accurate results (a few hundred neurons with 1 or 2 hidden layers)
- Setting $epochs = 1$ is fine for all dates but the last one.
- NN have proved to be a very versatile and efficient tool to compute Bermudan option prices...
- ...but keep in mind that using large NN is not green!

Bibliography (1)

S. Becker, P. Cheridito, and A. Jentzen. Deep optimal stopping. *Journal of Machine Learning Research*, 20(74):1–25, 2019.

Emmanuelle Clément, Damien Lamberton, and Philip Protter. An analysis of a least squares regression method for american option pricing. *Finance and Stochastics*, 6(4):449–471, 2002.

Kurt Hornik. Approximation capabilities of multilayer feedforward networks. *Neural Networks*, 4(2):251 – 257, 1991.

Côme Huré, Huyên Pham, Achref Bachouch, and Nicolas Langrené. Deep neural networks algorithms for stochastic control problems on finite horizon, part i: convergence analysis. *arXiv preprint arXiv:1812.04300*, 2018.

Michael Kohler, Adam Krzyżak, and Nebojsa Todorovic. Pricing of high-dimensional american options by neural networks. *Mathematical Finance: An International Journal of Mathematics, Statistics and Financial Economics*, 20(3):383–410, 2010.

Michel Ledoux and Michel Talagrand. *Probability in Banach spaces*, volume 23 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1991. ISBN 3-540-52013-9. Isoperimetry and processes.

F.A. Longstaff and R.S. Schwartz. Valuing American options by simulation : A simple least-square approach. *Review of Financial Studies*, 14:113–147, 2001.

Bibliography (2)

Edith Mourier. Eléments aléatoires dans un espace de Banach. *Ann. Inst. H. Poincaré*, 13: 161–244, 1953.

Reuven Y. Rubinstein and Alexander Shapiro. *Discrete event systems*. Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics. John Wiley & Sons Ltd., Chichester, 1993. ISBN 0-471-93419-4. Sensitivity analysis and stochastic optimization by the score function method.

J.N. Tsitsiklis and B. Van Roy. Regression methods for pricing complex American-style options. *IEEE Trans. Neural Netw.*, 12(4):694–703, 2001.