A pure dual approach for hedging Bermudan options

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Computing Bermudan options prices

- A discrete time (discounted) payoff process $(Z_k)_{0\le k\le N}$ adapted to $(\mathcal{F}_k)_{0\le k\le N}$. $\max_{0\le k\le N}|Z_k| \in L^p$, $p > 1$.
- The time-T_k discounted value of the Bermudan option is given by

$$U_k = \operatorname{esssup}_{\tau\in\mathcal{T}_k} \mathbb{E}[Z_\tau|\mathcal{F}_k]$$

where $\mathcal{T}_k$ is the set of all $\mathcal{F}-$ stopping times with values in $\{k, k+1, \ldots, N\}$.

- From the Snell enveloppe theory, we derive the standard dynamic programming algorithm

$$\begin{cases} U_N = Z_N \\ U_k = \max\left(Z_k, \mathbb{E}[U_{k+1}|\mathcal{F}_k]\right) \end{cases} \tag{1}$$

The dual formulation of the price (1)

- The option price represents the value of the hedge portfolio. This is pointless if we do not know how to build the portfolio
- Dual representation (Rogers [2010, 2002], Haugh and Kogan [2004])

$$U_n = \inf_{M \in \mathbb{H}^p} \mathbb{E}\left[\max_{n \le j \le N} \{Z_j - (M_j - M_n)\} \middle| \mathcal{F}_n \right] \tag{2}$$

  where $\mathbb{H}^p$ is the set of $\mathcal{F}$-martingales that are L^2 integrable.
- From the Doob-Meyer decomposition

$$U_n = U_0 + M_n^\star - A_n^\star, \tag{3}$$

  where $M^\star \in \mathbb{H}^p$ vanishes at 0 and $A^\star$ is a predictable, nondecreasing and L^p-integrable process.
- $M^\star$ solves (2) and $U_n = \max_{n \le j \le N}\{Z_j - (M_j^\star - M_n^\star)\}$ (almost surely optimal martingales).

The dual formulation as an hedging portfolio

- Let $M \in \mathbb{H}^p$ be a martingale such that $M_0 = 0$ and $V_0 = \mathbb{E}[\max_{0 \le n \le N}\{Z_n - M_n\}] \ge U_0$.
- $V_0 + M_n$ can be interpreted as the value at time n of a self-financing portfolio
- We can prove that $\mathbb{E}[|Z_{\tau^\star} - (V_0 + M_{\tau^\star})|^p]^{1/p} \le 3\mathbb{E}[|M_N^\star - M_N|^p]^{1/p}$.
- As noticed by Rogers, if $M^\star$ is tradable, it is a perfect hedge.
- The dual problem is convex and admits many solutions. See Schoenmakers et al. [2013] for the characterization of almost surely optimal martingales.
- How to approximate $M^\star$? $\Rightarrow$ Find a new dual representation.

The excess reward representation (1)

With $\Delta M_n = M_n - M_{n-1}$,

$$
\begin{aligned}
&\max_{0 \le j \le N} \{Z_j - (M_j - M_0)\} \\
&= Z_N - (M_N - M_0) + \sum_{n=0}^{N-1} \max_{n \le j \le N} \{Z_j - M_j\} - \max_{n+1 \le j \le N} \{Z_j - M_j\} \\
&= Z_N - (M_N - M_0) + \sum_{n=0}^{N-1} \max_{n \le j \le N} \{Z_j - (M_j - M_n)\} - \max_{n+1 \le j \le N} \{Z_j - (M_j - M_n)\} \\
&= Z_N - (M_N - M_0) + \sum_{n=0}^{N-1} \left(Z_n + \Delta M_{n+1} - \max_{n+1 \le j \le N} \left\{ Z_j - \sum_{i=n+2}^{j} \Delta M_i \right\} \right)_+ .
\end{aligned}
$$

The excess reward representation (2)

By taking expectation,

$$\mathbb{E}\left[\max_{0\le j\le N}\{Z_j-(M_j-M_0)\}\right]$$
$$=\mathbb{E}[Z_N]+\sum_{n=0}^{N-1}\mathbb{E}\left[\left(Z_n+\Delta M_{n+1}-\max_{n+1\le j\le N}\left\{Z_j-\sum_{i=n+2}^{j}\Delta M_i\right\}\right)_+\right].$$

For $M=M^\star$, the red terms represents the values of having the right to exercise the option at time $n\in\{0,\dots,N-1\}$.

A sequence of optimization problems (1)

Introduce

$\mathcal{H}_n^p = \{Y \in \mathbb{L}^p(\Omega) \ : \ Y \text{ is real valued, } \mathcal{F}_n - \text{mesurable and } \mathbb{E}[Y|\mathcal{F}_{n-1}] = 0\}.$

It is tempting to solve backward from $n = N - 1$ to $n = 0$

$$\inf_{\Delta M_{n+1} \in \mathcal{H}_{n+1}^p} \mathbb{E}\left[\left(Z_n + \Delta M_{n+1} - \max_{n+1 \le j \le N}\left\{Z_j - \sum_{i=n+2}^{j} \Delta M_i\right\}\right)_+\right].$$

However, the non strict convexity of the positive part raises some issues in the back propagation of the minimisation problems.

A sequence of optimization problems (2)

Theorem

Let $\varphi : \mathbb{R} \to \mathbb{R}$ *be a convex function such that* $|\varphi(x)| \leq C(1 + |x|^p)$. *Then,*

$$\mathbb{E}[Z_N] + \sum_{n=0}^{N-1} \mathbb{E}\left[\varphi\left(Z_n + \Delta M_{n+1} - \max_{n+1 \leq j \leq N}\left\{Z_j - \sum_{i=n+2}^{j} \Delta M_i\right\}\right)\right]$$
$$\geq \mathbb{E}[Z_N] + \sum_{n=0}^{N-1} \mathbb{E}\left[\varphi\left(Z_n + \Delta M^\star_{n+1} - \max_{n+1 \leq j \leq N}\left\{Z_j - \sum_{i=n+2}^{j} \Delta M^\star_i\right\}\right)\right],$$

and $M^\star$ *is a solution of the following problems for* $n = N-1, \dots, 0$

$$(4) \qquad \inf_{\Delta M_{n+1} \in \mathcal{H}^p_{n+1}} \mathbb{E}\left[\varphi\left(Z_n + \Delta M_{n+1} - \max_{n+1 \leq j \leq N}\left\{Z_j - \sum_{i=n+2}^{j} \Delta M_i\right\}\right)\right].$$

When φ *is strictly convex,* $M^\star$ *is the unique solution of* (4).

Our theoretical algorithm

1. Take $p = 2$, $\phi(x) = x^2$
2. For each $n \in \{1, \dots, N\}$, choose a finite dimensional linear subspace $\mathcal{H}_n^{pr}$ of $\mathcal{H}_n^2$.
3. For $n = N - 1$ to $n = 0$, use an optimisation algorithm to minimise

$$\inf_{\Delta M_{n+1} \in \mathcal{H}_{n+1}^{pr}} \mathbb{E}\left[\left(Z_n + \Delta M_{n+1} - \max_{n+1 \le j \le N}\left\{Z_j - \sum_{i=n+2}^{j} \Delta M_i\right\}\right)^2\right].$$

ΔM_{n+1} solves a classical least square problem.

Two approximations are needed:

1. Use a finite dimensional subspace of $\mathcal{H}_n^{pr}$
2. Approximate $\mathbb{E}$ by Monte-Carlo.

Finite dimensional subspace approximation

We assume that the subspaces $\mathcal{H}_n$, $1 \leq n \leq N$, are spanned by $L \in \mathbb{N}^*$ martingale increments $\Delta X_{n,\ell} \in \mathcal{H}_n^2$, $1 \leq \ell \leq L$:

$$\mathcal{H}_n^{pr} = \left\{ \alpha \cdot \Delta X_n \ : \ \alpha \in \mathbb{R}^L \right\}.$$

The minimisation problem becomes

$$\inf_{\alpha \in \mathbb{R}^L} \mathbb{E}\left[\left(Z_n + \alpha \cdot \Delta X_{n+1} - \max_{n+1 \leq j \leq N} \left\{ Z_j - \sum_{i=n+2}^{j} \Delta M_i \right\} \right)^2 \right].$$

Monte Carlo approximation

Let $Q > 0$. For $1 \leq q \leq Q$, $(Z_n^q)_{1 \leq n \leq N}$ and $(\Delta X_n^q)_{1 \leq n \leq N}$ be independent sample paths of the underlying process Z and martingale increments ΔX.
Solve backward in time, the sequence of optimisation problems

$$\inf_{\alpha \in \mathbb{R}^L} \frac{1}{Q} \sum_{q=1}^{Q} \left(Z_n^q + \alpha \cdot \Delta X_{n+1}^q - \max_{n+1 \leq j \leq N} \left\{ Z_j^q - \sum_{i=n+2}^{j} \alpha_i^Q \cdot \Delta X_i^q \right\} \right)^2 .$$

Since the problem is strictly convex, it has a unique solution α_{n+1}^Q given by

$$\left(\sum_{q=1}^{Q} \Delta X_{n+1}^q (\Delta X_{n+1}^q)^T \right) \alpha_{n+1}^Q = \sum_{q=1}^{Q} \max_{n+1 \leq j \leq N} \left\{ Z_j^q - \sum_{i=n+2}^{j} \alpha_i^Q \cdot \Delta X_i^q \right\} \Delta X_{n+1}^q .$$

Convergence results

Proposition

Assume that for $1 \leq n \leq N$, the matrix $\mathbb{E}[\Delta X_n \Delta X_n^T]$ is invertible. Then,

- *For all $n \in \{1, \dots, N\}$, $\alpha_n^Q \to \alpha_n$ when $Q \to \infty$ a.s.*
- $U_0^Q = \frac{1}{Q} \sum_{q=1}^{Q} \max_{0 \leq j \leq N} \left\{ Z_j^q - \sum_{i=1}^{j} \alpha_i^Q \cdot \Delta X_i^q \right\} \to \mathbb{E}\left[\max_{0 \leq j \leq N} \left\{ Z_j - \sum_{i=1}^{j} \alpha_i \cdot \Delta X_i \right\} \right]$ *a.s.*

If we assume moreover that ΔX_i and Z_i have finite moments of order 4, then $\left(\sqrt{Q}(\alpha_n^Q - \alpha_n)\right)_{Q \geq 1}$ and $\left(\sqrt{Q}\left(U_0^Q - \mathbb{E}\left[\max_{0 \leq j \leq N}\left\{Z_j - \sum_{i=1}^{j} \alpha_i \cdot \Delta X_i\right\}\right]\right)\right)_{Q \geq 1}$ are tight.

The financial framework (1)

- A market with d assets $(S_t^k, t \geq 0)$, $k \in \{1, \dots, d\}$ and $(\mathcal{G}_t, t \geq 0)$ their usual filtration.
- For simplicity the interest rate r is deterministic
- Assume that the discounted assets $(\tilde{S}_t^k, t \geq 0)$ with $\tilde{S}_t^k = e^{-rt} S_t^k$ are square integrable $\mathcal{G}_t$-martingales.
- Consider a time horizon $T > 0$ and a Bermudan option with regular exercising dates

$$T_i = \frac{iT}{N}, \; i = 0, \dots, N.$$

The financial framework (2)

As perfect hedging is hung up to a martingale representation theorem, we further split each interval $[T_i, T_{i+1}]$ for $0 \leq i \leq N-1$ into $\bar{N}$ regular sub-intervals, and we set

(5) $$t_{i,j} = T_i + \frac{j}{\bar{N}} \frac{T}{N}, \text{ for } 0 \leq j \leq \bar{N}.$$

Consider a family of functions $u_p : \mathbb{R}^d \to \mathbb{R}$ for $p \in \{1, \dots, \bar{P}\}$ and a family of discounted assets $(\mathcal{A}^k)_{1 \leq k \leq \bar{d}}$. Then, we define the following elementary martingale increments:

(6) $$X^{p,k}_{t_{i,j}} - X^{p,k}_{t_{i,j-1}} = u^p_{i,j-1}(S_{t_{i,j-1}})(\mathcal{A}^k_{t_{i,j}} - \mathcal{A}^k_{t_{i,j-1}}),$$

for $1 \leq p \leq \bar{P}$ and $1 \leq k \leq \bar{d}$. Thus, $L = \bar{N} \times \bar{P} \times \bar{d}$ is the number of martingale increments between two exercising dates that span $\mathcal{H}^{pr}_i$.

The financial framework (3)

Decompose the martingale increments ΔM_{i+1}, $0 \leq i \leq N-1$ as follows

(7) $$\Delta M_{i+1} = \sum_{j=1}^{\bar{N}} \sum_{p,k} \alpha_{i,j}^{p,k} (X_{t_{i,j}}^{p,k} - X_{t_{i,j-1}}^{p,k}).$$

There are $L = \bar{N} \times \bar{P} \times \bar{d}$ coefficients to estimate
Between two exercising dates, the option is European and using the martingale property we can easily show that the coefficients on every sub-intervals can be computed independently.

The use of subticks induces a linear computational cost: instead of solving a linear system of size $L = \bar{N} \times \bar{P} \times \bar{d}$, we solve $\bar{N}$ linear systems of size $\bar{P} \times \bar{d}$.

Numerical experiments

$$U_0^Q = \frac{1}{Q} \sum_{q=1}^{Q} \max_{0 \leq j \leq N} \left\{ Z_j^q - \sum_{i=1}^{j} \alpha_i^Q \cdot \Delta X_i^q \right\}.$$

Because of overfitting, U_0^Q can significantly underestimate $\mathbb{E}\left[\max_{0\leq j\leq N}\left\{Z_j - \sum_{i=1}^{j} \alpha_i \cdot \Delta X_i\right\}\right]$ when Q is not sufficiently large, compared to the number of parameters to estimate.

$$\hat{U}_0^Q = \frac{1}{Q} \sum_{q=1}^{Q} \max_{0 \leq j \leq N} \left\{ \hat{Z}_j^q - \sum_{i=1}^{j} \alpha_i^Q \cdot \Delta \hat{X}_i^q \right\},$$

where $(\hat{Z}^q, \Delta\hat{X}^q)_{1\leq q\leq Q}$ is independent from the sample $(Z^q, \Delta X^q)_{1\leq q\leq Q}$ used to compute α^Q.

$\hat{U}_0^Q$ has a nonnegative biais. The difference $\hat{U}_0^Q - U_0^Q$ is a measure of the accuracy.

Comparison with Rogers' approach

Rogers directly solves

$$U_0 = \inf_{M \in \mathbb{H}^2} \mathbb{E}\left[\max_{0 \le j \le N} \{Z_j - (M_j - M_0)\}\right]$$

with $M_j = \lambda \frac{\partial}{\partial S_{t_j}} \tilde{P}(t_j, S_{t_j})$, for $\lambda \in \mathbb{R}$.

Rogers uses the continuous time European hedge.

The 1-dimensional put option

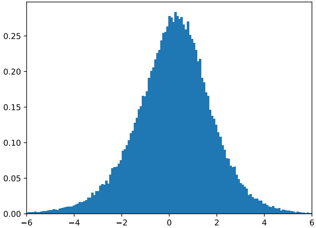
Consider a 1-dimensional put options in the Black Scholes models

Q	$\bar{N}$	P	Vanilla	U_0^Q	$\hat{U}_0^Q$
50000	1	1	True	9.91	9.91
100000	1	50	True	9.89	9.91
100000	1	50	False	10.32	10.33
100000	5	50	False	9.99	10.08
100000	10	100	False	9.82	10.19
500000	10	100	False	9.95	10.02
2000000	10	50	False	9.98	9.98
2000000	20	50	False	9.94	9.96

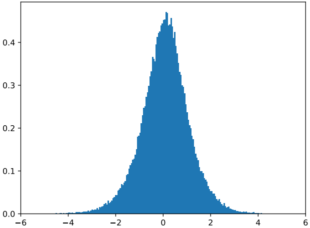
Table: Prices for a put option using a basis of P local functions with $K = S_0 = 100$, $T = 0.5$, $r = 0.06$, $\sigma = 0.4$ and $N = 10$ exercising dates. LS price with a polynomial approximation of order 6: 9.90.

The 1-dimensional put option

Empirical distribution of $\left(\hat{U}_0^Q + \sum_{i=1}^{\hat{\tau}^\star} \alpha_i^Q \cdot \Delta\hat{X}_i^q - \hat{Z}_{\hat{\tau}^\star}^q\right)_{1\leq q\leq Q}$



(a) $\bar{N} = 5, P = 50, Q = 10^5$.



(b) $\bar{N} = 10, P = 50, Q = 2 \times 10^6$.

Figure: P&L histograms of the hedging strategy for the Bermudan Put option for the stock only strategy.

The 1-dimensional put option

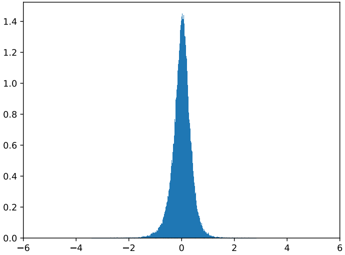


Figure: $\bar{N} = 1$, $P = 50$, $Q = 10^5$.

Figure: P&L histograms of the hedging strategy for the Bermudan Put option for the strategy using extra European options.

A butterfly option

$$\Psi(S) = 2\left(\frac{K_1 + K_2}{2} - S\right)_+ - (K_1 - S)_+ - (K_2 - S)_+ .$$

Using the European butterfly to hedge the Bermudan options gives a price way too high: 6.49 vs 5.65 (Longstaff Schwartz price)

Q	$\bar{N}$	P	Vanilla	P^Q	$\hat{P}^Q$
50000	1	50	False	6.54	6.54
50000	1	50	True	6.25	6.28
100000	10	50	False	5.97	6.00
100000	10	50	True	5.79	5.87
500000	20	50	False	5.86	5.87
500000	20	50	True	5.71	5.74

Table: Prices for a butterfly option with parameters using a basis of P local functions. The Longstaff-Schwartz algorithm with order 5 polynomials gives a price of 5.65.

The butterfly option

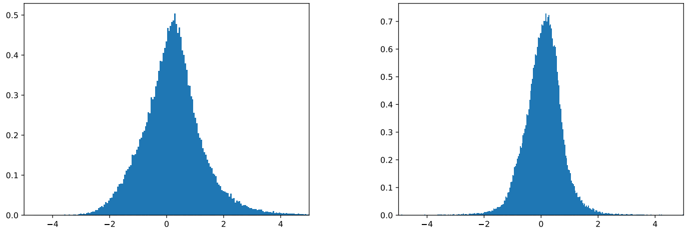


Figure: P&L histograms of the hedging strategy for the Bermudan Butterfly option obtained with $\bar{N} = 20$, $P = 50$, $Q = 5 \times 10^5$ for the stock only strategy (left) and the strategy using extra European options (right).

A basket option on 3 assets

Q	$\bar{N}$	P	Vanilla	U_0^Q	$\hat{U}_0^Q$
2000000	1	10	False	4.35	4.37
2000000	1	10	True	4.20	4.25
5000000	5	10	False	4.15	4.19
5000000	5	10	True	4.08	4.16
10000000	10	10	False	4.12	4.16
10000000	10	10	True	4.07	4.15

Table: Prices for a basket put option in dimension $d = 3$ using a basis of local functions with $K = S_0 = 100$, $T = 1$, $r = 0.05$, $\sigma^i = 0.2$, $\rho = 0.3$ and 10 exercising dates. The Longstaff Schwartz algorithm with a polynomial approximation of order 3 gives 4.03.

Conclusion

The key ingredients:

- The reward excess representation: a new dual formula.
- Strictly convexifying the optimisation problem: an algorithm to approximate $M^\star$.
- The use of sub-intervals and European options in the Bermudan portfolio.

We can compute a practical hedging strategy and its cost.

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