SUBSPACES OF MATRICES WITH SPECIAL RANK PROPERTIES

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Abstract. Let K be a field and let V be a vector space of finite dimension n over K. We investigate properties of a subspace $\mathcal{M}$ of $\mathrm{End}_K(V)$ of dimension $n(n-r+1)$ in which each non-zero element of $\mathcal{M}$ has rank at least r and show that such subspaces exist if K has a cyclic Galois extension of degree n. We also investigate the maximum dimension of a constant rank r subspace of $\mathrm{End}_K(V)$ when K is finite.

1. Introduction

Let K be a field and let m and n be positive integers with $m \leq n$. Let $M_{m\times n}(K)$ denote the vector space of $m \times n$ matrices with entries in K. When $m = n$, we write $M_n(K)$ in place of $M_{n\times n}(K)$. For any non-zero subspace U of $M_{m\times n}(K)$, we let $U^\times$ denote the subset of non-zero elements in U. Given a positive integer s, we let K^s denote the s-dimensional vector space of row vectors of size s over K. We consider the elements of $M_{m\times n}(K)$ as linear transformations from K^m into K^n, the action being defined by right multiplication on elements of K^m.

Let r be an integer satisfying $1 \leq r \leq m$. Much research has been devoted to the study of subspaces $\mathcal{M}$, say, of $M_{m\times n}(K)$ which satisfy one of the three following conditions:

(1) Each element of $\mathcal{M}$ has rank at most r.
(2) Each element of $\mathcal{M}^\times$ has rank r. We say that $\mathcal{M}$ is a *constant rank* r subspace in this case.
(3) Each element of $\mathcal{M}^\times$ has rank at least r.

Particular attention has been focused on finding the maximum value of $\dim \mathcal{M}$ in case $\mathcal{M}$ satisfies one of conditions (1), (2) or (3).

Concerning condition (1), note that if $\mathcal{N}$ denotes the subspace of $M_{m\times n}(K)$ consisting of all matrices whose bottom $m-r$ rows are zero rows, each element of $\mathcal{N}$ has rank at most r and $\dim \mathcal{N} = nr$. This simple observation enables us to obtain an upper bound for the dimension of a subspace satisfying condition (3).

Lemma 1. *Suppose that $\mathcal{M}$ is a subspace of $M_{m\times n}(K)$ with $\operatorname{rank} T \geq r$ for all $T \in \mathcal{M}^\times$. Then $\dim \mathcal{M} \leq n(m-r+1)$.*

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Proof. Replacing r by $r-1$, we know that there exists a subspace $\mathcal{R}$, say, of $M_{m\times n}(K)$ of dimension $n(r-1)$ whose elements all have rank at most $r-1$. Since $\mathcal{M}\cap\mathcal{R}=0$ and $\dim M_{m\times n}(K)=mn$, the desired inequality is immediate. □

While Lemma 1 is obtained by elementary means, it is interesting to note that for certain fields K, the bound we have obtained is optimal. For suppose that K admits a cyclic Galois extension of degree n. Then provided that $1\le r\le m\le n$, Guralnick has shown in [8], Lemma 3.2, that $M_{m\times n}(K)$ contains a subspace $\mathcal{M}$ with $\operatorname{rank} T\ge r$ for all $T\in\mathcal{M}^\times$ and $\dim\mathcal{M}=n(m-r+1)$. When K is finite, the existence of such a subspace had been shown by Delsarte in [4], Section 6 (in particular, Theorem 6.3).

In general, however, there do not exist subspaces of the type described in Lemma 1 which meet the upper bound for the dimension over arbitary fields K. Indeed, in unpublished work, referenced in [9], p.333, Meshulam has shown that if K is algebraically closed and if $m=n$, the bound in Lemma 1 can be improved to $\dim\mathcal{M}\le(n-r+1)^2$. Furthermore, Roth has shown in [9], p.333, that if K is infinite, there exists a subspace $\mathcal{M}$ of $M_n(K)$ with $\operatorname{rank} T\ge r$ for all $T\in\mathcal{M}^\times$ and $\dim\mathcal{M}=(n-r+1)^2$. Thus, Meshulam's bound is optimal for square matrices in the algebraically closed case.

We will investigate the structure of a subspace of $m\times n$ matrices of dimension $n(m-r+1)$ whose non-zero elements all have rank at least r, and show some interesting uniformity in properties of these subspaces. When K is finite, we will determine combinatorially the number of elements of any given rank in such a subspace, in terms of Gaussian coefficients. This enumeration has been performed previously by Delsarte, [4], Theorem 5.6, who used complex characters defined on a finite abelian group to obtain his conclusion.

Using Delsarte's character formula, we show in Section 4 that if K is finite, a constant rank r subspace of $M_{m\times n}(K)$ has dimension at most $m+n-r$. This bound is generally poor for r less than $m/2$, and a bound much closer to n should be expected. See, for example, Theorem 2 (and erratum) of [2]. Examples of Beasley, [1], and Boston, [3], show that there are $(n+1)$-dimensional constant rank $n-1$ subspaces of $M_n(\mathbb{F}_2)$ for $3\le n\le 5$. The existence of these subspaces implies that the bound above is optimal when $m=n$, $r=n-1$, $|K|=2$ and $3\le n\le 5$. We show in this paper that there is a 6-dimensional constant rank 3 subspace of $M_{4\times 5}(K)$ when $|K|=2$, so that the upper bound is again attained for $r=3$, $m=4$, $n=5$.

In Section 6, we briefly describe some aspects of rank questions as they relate to subspaces of skew-symmetric matrices. Our main result, Theorem 10, employs a character formula of Delsarte-Goethals, [5], to obtain a dimension bound when n is odd for a constant rank subspace of $n\times n$ skew-symmetric matrices over a finite field. Again, this bound is unlikely to be optimal in most cases.

2. Subspaces of matrices of rank bounded below

Let $\mathcal{M}$ be a subspace of $M_{m\times n}(K)$ each of whose non-zero elements has rank at least r. Suppose furthermore that $\dim\mathcal{M}$ equals $n(m-r+1)$, the upper bound provided by Lemma 1. We will derive some properties of $\mathcal{M}$ in this section.

Lemma 2. *Let $\mathcal{M}$ be a subspace of $M_{m\times n}(K)$ with $\operatorname{rank} T\ge r$ for all $T\in\mathcal{M}^\times$ and suppose that $\dim\mathcal{M}=n(m-r+1)$. Let $v_1,\ \dots,\ v_{m-r+1}$ be linearly independent* *elements of* K^m*. Then given any* $m-r+1$ *elements* $w_1, \ldots, w_{m-r+1}$ *of* K^m*, there exists a unique element* T*, say, of* $\mathcal{M}$ *with*

$$v_i T = w_i$$

for $1 \leq i \leq m-r+1$*.*

Proof. We may identify $K^{n(m-r+1)}$ with the direct sum of $m-r+1$ copies of K^n. Given the v_i as above, we define a linear transformation $\theta : \mathcal{M} \to K^{n(m-r+1)}$ by

$$\theta(T) = (v_1 T, \ldots, v_{m-r+1} T)$$

for all $T \in \mathcal{M}$. We claim that $\ker\theta = 0$. For let T be an element of $\ker\theta$. Then $\ker T$ has dimension at least $m-r+1$, since it contains the linearly independent vectors $v_1, \ldots, v_{m-r+1}$. This contradicts the supposition that $\operatorname{rank} T \geq r$ and we deduce that θ is injective. Since

$$\dim M = \dim K^{n(m-r+1)} = n(m-r+1),$$

it follows that θ is an isomorphism, and this establishes the lemma. □

Definition 1. Let $\mathcal{M}$ be a subspace of $M_{m\times n}(K)$ and let U be a subspace of K^m. We set

$$\mathcal{M}_U = \{T \in \mathcal{M} : uT = 0 \text{ for all } u \in U\}.$$

$\mathcal{M}_U$ is clearly a subspace of $\mathcal{M}$. The following result shows that the subspaces $\mathcal{M}_U$ exhibit uniform properties when $\mathcal{M}$ satisfies the hypotheses of Lemma 2.

Theorem 1. *Let* $\mathcal{M}$ *be a subspace of* $M_{m\times n}(K)$ *that satisfies the hypotheses of Lemma 2. Then we have* $\mathcal{M}_U = 0$ *if* $\dim U > m-r$ *and*

$$\dim \mathcal{M}_U = n(m+1-r-\dim U)$$

if $\dim U \leq m-r$*.*

Proof. Suppose that $T \in \mathcal{M}_U$. Then $U \leq \ker T$ and hence $r \leq \operatorname{rank} T \leq m - \dim U$. It follows that $\mathcal{M}_U = 0$ if $\dim U > m-r$.

Suppose now that $s = \dim U \leq m-r$. Let $u_1, \ldots, u_s$ be basis vectors of U. We define a linear transformation $\phi : \mathcal{M} \to K^{ns}$ by

$$\phi(T) = (u_1 T, \ldots, u_s T)$$

for all $T \in \mathcal{M}$. Lemma 2 implies that ϕ is surjective. Furthermore, it is clear that $\ker\phi = \mathcal{M}_U$. The formula for $\dim \mathcal{M}_U$ now follows from the rank-nullity theorem. □

We note without proof the following property of the subspaces $\mathcal{M}_U$.

Lemma 3. *Given subspaces* U *and* W *of* K^m*, we have*

$$\mathcal{M}_U \cap \mathcal{M}_W = \mathcal{M}_{U+W}\,.$$

Corollary 1. *Let* $\mathcal{M}$ *be a subspace of* $M_{m\times n}(K)$ *that satisfies the hypotheses of Lemma 2. Let* U *and* W *be subspaces of* K^m *such that* $\dim(U+W) > m-r$*. Then*

$$\mathcal{M}_U \cap \mathcal{M}_W = 0.$$

Corollary 2. *Let* $\mathcal{M}$ *be a subspace of* $M_{m\times n}(K)$ *that satisfies the hypotheses of Lemma 2. Let* U *and* W *be different subspaces of* K^m *such that* $\dim U = \dim W = m-r$*. Then each element of* $\mathcal{M}_U^\times$ *and of* $\mathcal{M}_W^\times$ *has rank* r *and*

$$\mathcal{M}_U \cap \mathcal{M}_W = 0.$$

Proof. Each element T of $\mathcal{M}_U^\times$ satisfies $\operatorname{rank} T \leq r$, since $U \leq \ker T$. However, $\operatorname{rank} T \geq r$, since $T \in \mathcal{M}_U^\times$, and we deduce that $\operatorname{rank} T = r$. Clearly, the same conclusion holds for each element of $\mathcal{M}_W^\times$. Now since $U \neq W$, it follows that $\dim(U+W) > m - r$ and therefore we obtain

$$\mathcal{M}_U \cap \mathcal{M}_W = 0.$$

by Corollary 1. □

We can now obtain our main conclusion about the structure of the special subspaces of endomorphisms studied in this section.

Theorem 2. *Let $\mathcal{M}$ be a subspace of $M_{m\times n}(K)$ that satisfies the hypotheses of Lemma 2. Then the subset of all elements of rank r in $\mathcal{M}$ is the disjoint union of all the subsets $\mathcal{M}_U^\times$, as U ranges over all subspaces U of dimension $m-r$ in K^m.*

Proof. We know that the union of the subsets $\mathcal{M}_U^\times$ is disjoint by Corollary 1. Now any element T of $\mathcal{M}$ with $\operatorname{rank} T = r$ lies in the subset $\mathcal{M}_W^\times$, where $W = \ker T$. Since $\dim W = m - r$, we see that the union described above comprises all elements of rank r. □

We note that Theorem 2 applies to $\mathcal{M} = M_{m\times n}(K)$, with $r = 1$.

3. Enumeration in the finite field case

Let q be a power of a prime and let $K = \mathbb{F}_q$. Let $\mathcal{M}$ be a subspace of $M_{m\times n}(\mathbb{F}_q)$, with $\operatorname{rank} T \geq r$ for all $T \in \mathcal{M}^\times$ and $\dim \mathcal{M} = n(m-r+1)$. We will enumerate the number of elements in $\mathcal{M}$ of nullity t (equivalently, of rank $m-t$), where $r \leq m-t \leq m$. The answer depends only on q, m, n, r and t, and not on how $\mathcal{M}$ is constructed. As we mentioned earlier, the enumeration was first performed by Delsarte in [4], Theorem 5.6, but our method is different, as we use only elementary linear algebra and counting techniques, rather than complex characters. We note that the formulae obtained generalize known results for the number of elements of $M_{m\times n}(\mathbb{F}_q)$ having a given nullity, since $M_{m\times n}(\mathbb{F}_q)$ is the unique subspace $\mathcal{M}$ when $r = 1$.

We start by introducing some notation. As we will be using concepts of nullity, rather than rank, we set $s = m - r$. We now define subsets $\mathcal{M}_t$ and $\mathcal{M}_{\geq t}$ of $\mathcal{M}$ by

$$\mathcal{M}_t = \{T \in \mathcal{M} : \operatorname{nullity} T = t\}, \quad \mathcal{M}_{\geq t} = \{T \in \mathcal{M} : \operatorname{nullity} T \geq t\}$$

for $0 \leq t \leq s$.

Using the notation of Definition 1, we have the following result.

Lemma 4.

$$\mathcal{M}_{\geq t} = \bigcup_{\dim U = t} (\mathcal{M}_U)^\times,$$

the union being taken over all subspaces U of dimension t.

Proof. Suppose that $T \in \mathcal{M}_U$. Then $U \leq \ker T$ and hence $\operatorname{nullity} T \geq t$. It follows that $T \in \mathcal{M}_{\geq t}$. Conversely, suppose that $T \in \mathcal{M}_{\geq t}$. Then $\dim \ker T \geq t$ and hence for any t-dimensional subspace U of $\ker T$, $T \in \mathcal{M}_U$. This implies the desired equality. □

The next result is the key to the evaluation of $|\,\mathcal{M}_t\,|$. We use the the familiar notation $\left[{n \atop k}\right]$ to denote the q-Gaussian coefficient which measures the number of subspaces of dimension k in an n-dimensional vector space over $\mathbb{F}_q$.

Lemma 5. *We have*

$$|\,\mathcal{M}_t\,| = \begin{bmatrix} m \\ t \end{bmatrix}(q^{n(s-t+1)} - 1) - \sum_{j=t+1}^{s} \begin{bmatrix} j \\ t \end{bmatrix} |\,\mathcal{M}_j\,|.$$

Proof. We have seen that

$$\mathcal{M}_{\geq t} = \bigcup_{\dim U = t} (\mathcal{M}_U)^{\times}.$$

Let $T \in \mathcal{M}$ be an element satisfying nullity $T = i \geq t$. Such a T occurs in exactly $\left[\begin{smallmatrix} i \\ t \end{smallmatrix}\right]$ subspaces $\mathcal{M}_U$, namely those that correspond to U being a t-dimensional subspace of $\ker T$. Thus, the elements of rank t in $\mathcal{M}$ are those that occur in exactly one subspace in the union. Counting the number of elements in the union above, and taking into account the multiplicities due to any given element belonging to more that one subspace, we obtain

$$\begin{bmatrix} m \\ t \end{bmatrix} |(\mathcal{M}_U)^{\times}\,| = \begin{bmatrix} m \\ t \end{bmatrix}(q^{n(s-t+1)} - 1).$$

If we subtract from this sum the multiple contributions due to elements of nullity greater than t, we obtain

$$\begin{bmatrix} m \\ t \end{bmatrix}(q^{n(s-t+1)} - 1) - \sum_{j=t+1}^{s} \begin{bmatrix} j \\ t \end{bmatrix} |\,\mathcal{M}_j\,|$$

and this expression measures the number of elements of nullity t, as required. □

We now use induction to find $|\,\mathcal{M}_t\,|$.

Theorem 3. *Let q be a power of a prime. Let $\mathcal{M}$ be a subspace of $M_{m\times n}(\mathbb{F}_q)$, each of whose non-zero elements has rank at least r, and suppose that $\dim \mathcal{M} = n(m-r+1)$. Let $s = m - r$. For $0 \leq t \leq s$, let $\mathcal{M}_t$ denote the subset of elements of nullity t in $\mathcal{M}$. Then we have*

$$|\,\mathcal{M}_t\,| = \sum_{i=1}^{s-t+1} \begin{bmatrix} m \\ s+1-i \end{bmatrix} \begin{bmatrix} s+1-i \\ t \end{bmatrix} (-1)^{s-t-i+1} q^{\binom{s-t-i+1}{2}} (q^{in} - 1).$$

Proof. We proceed by induction on $s - t$. When $s - t = 0$, which occurs when $t = m - r$, the formula given in Lemma 5 implies that

$$|\,\mathcal{M}_s\,| = \begin{bmatrix} m \\ s \end{bmatrix}(q^n - 1),$$

which is the value of $|\,\mathcal{M}_s\,|$ predicted by the formula we wish to prove.

Suppose now that we have obtained the formula for $|\,\mathcal{M}_{t+1}\,|, \ldots, |\,\mathcal{M}_s\,|$. We proceed to establish the corresponding formula for $|\,\mathcal{M}_t\,|$ by induction. We are thus assuming that

$$|\,\mathcal{M}_j\,| = \sum_{i=1}^{s-j+1} \begin{bmatrix} m \\ s+1-i \end{bmatrix} \begin{bmatrix} s+1-i \\ j \end{bmatrix} (-1)^{s-j-i+1} q^{\binom{s-j-i+1}{2}} (q^{in} - 1)$$

for $t + 1 \leq j \leq s$. The coefficient of $q^{in} - 1$ in this expression is

$$(-1)^{s-j-i+1} \begin{bmatrix} m \\ s+1-i \end{bmatrix} \begin{bmatrix} s+1-i \\ j \end{bmatrix} q^{\binom{s-j-i+1}{2}}.$$

Thus, applying the formula in Lemma 5, the coefficient of $q^{in}-1$ in $|\mathcal{M}_t|$ is

$$\sum_{j=t+1}^{s+1-i}(-1)^{s-j-i}\begin{bmatrix}j\\t\end{bmatrix}\begin{bmatrix}m\\s+1-i\end{bmatrix}\begin{bmatrix}s+1-i\\j\end{bmatrix}q^{\binom{s-j-i+1}{2}}$$

and we want to show that this equals

$$(-1)^{s-t-i+1}\begin{bmatrix}m\\s+1-i\end{bmatrix}\begin{bmatrix}s+1-i\\t\end{bmatrix}q^{\binom{s-t-i+1}{2}}.$$

Cancelling the common $(-1)^{s-i}$ terms, we therefore have to prove that

$$\sum_{j=t}^{s+1-i}(-1)^{j}\begin{bmatrix}j\\t\end{bmatrix}\begin{bmatrix}m\\s+1-i\end{bmatrix}\begin{bmatrix}s+1-i\\j\end{bmatrix}q^{\binom{s-j-i+1}{2}}=0.$$

This is equivalent to showing that the sum

$$\sum_{j=t}^{s+1-i}(-1)^{j}\begin{bmatrix}j\\t\end{bmatrix}\begin{bmatrix}s+1-i\\j\end{bmatrix}q^{\binom{s-j-i+1}{2}}$$

equals 0.

We now set $T=s-i+1$. The sum above is then

$$\sum_{j=t}^{T}(-1)^{j}\begin{bmatrix}j\\t\end{bmatrix}\begin{bmatrix}T\\j\end{bmatrix}q^{\binom{T-j}{2}}.$$

By a well known property of Gaussian coefficients, [6], Exercise 2.6.2, we have

$$\begin{bmatrix}j\\t\end{bmatrix}\begin{bmatrix}T\\j\end{bmatrix}=\begin{bmatrix}T\\t\end{bmatrix}\begin{bmatrix}T-t\\T-j\end{bmatrix}$$

and thus it suffices to prove that

$$\sum_{j=t}^{T}(-1)^{j}\begin{bmatrix}T-t\\T-j\end{bmatrix}q^{\binom{T-j}{2}}=0.$$

We now set $\ell=T-j$ and see that the sum above is transformed to

$$(-1)^{T}\sum_{\ell=0}^{T-t}(-1)^{\ell}\begin{bmatrix}T-t\\\ell\end{bmatrix}q^{\binom{\ell}{2}}.$$

This expression equals 0, by another well known property of Gaussian coefficients. See, for example, Formula 2, 2.6.12, in [6]. □

4. Constant rank subspaces of matrices

We begin by sketching a known construction of n-dimensional constant rank r subspaces. Suppose that L is a field extension of K of degree n. The regular representation of L over K provides us with a subspace $\mathcal{M}_n$, say, of $M_n(K)$ in which each non-zero matrix is invertible. Let T be an element of rank r in $M_{m\times n}(K)$. Then the subset $T\mathcal{M}_n$, consisting of all left multiples of elements of $\mathcal{M}_n$ by T, is a constant rank r subspace.

We summarize this argument as follows.

Theorem 4. *Suppose that the field K has an extension field of degree n. Then there exists a constant rank r subspace of $M_{m\times n}(K)$ of dimension n for each integer r with $1 \le r \le m$.*

We are interested here in obtaining a reasonable upper bound for the dimension of a constant rank r subspace of $M_{m\times n}(K)$. As far as we know, at the time of this writing, the best general result of this nature is the following, proved by Beasley and Laffey, [2], Theorem 2 (see also the erratum to this paper).

Theorem 5. *(Beasley and Laffey) Suppose that $1 \le r \le m$, $|K| \ge r+1$ and $n \ge 2r-1$. Then the dimension of a constant rank r subspace of $M_{m\times n}(K)$ is at most n.*

It is thus small finite fields which might cause problems in attempting to obtain a uniform result about the maximum dimension of a constant rank subspace, in line with what Theorems 4 and 5 suggest.

We present in Theorem 6 an upper bound for the dimension of a constant rank subspace over a finite field. In view of the result of Beasley and Laffey, our result is interesting in that involves no hypothesis about the size of the field (other than being finite), but it is clearly generally weak compared with Theorem 5 when $m > 2r-1$.

Let p be a prime and let q be a power of p. Suppose for the remainder of this section that $K = \mathbb{F}_q$, the finite field of order q. Under addition, $M_{m\times n}(\mathbb{F}_q)$ is an elementary abelian p-group of order q^{mn}, and thus has q^{mn} irreducible complex characters, which are homomorphisms from the additive group into the multiplicative group of the complex numbers. We may describe these characters in the following way.

Let $\mathrm{tr} : M_m(\mathbb{F}_q) \to \mathbb{F}_q$ denote the usual (matrix-theoretic) trace function. Let $\tau : \mathbb{F}_q \to \mathbb{F}_p$ denote the field-theoretic trace function. Let ω be a primitive p-th root of unity in the complex numbers. Then for each $S \in M_{m\times n}(K)$, we define an irreducible character λ_S of by $M_{m\times n}(\mathbb{F}_q)$

$$\lambda_S(T) = \omega^{\tau(\mathrm{tr}(ST'))}$$

for all $T \in M_{m\times n}(\mathbb{F}_q)$. (Here, T' denotes the transpose of T.) It is not difficult to show, using properties of the two trace functions, that we obtain all irreducible characters of $M_{m\times n}(\mathbb{F}_q)$ in this way.

For $0 \le k \le m$, let Ω_k denote the subset of all elements of rank k in $M_{m\times n}(\mathbb{F}_q)$. The function $P_k : M_{m\times n}(\mathbb{F}_q) \to \mathbb{C}$ defined by

$$P_k = \sum_{S\in\Omega_k} \lambda_S$$

is the character of a complex representation of $M_{m\times n}(\mathbb{F}_q)$ (usually a reducible representation). Although expressed in slightly different language, Delsarte proves the following result, [4], Theorem 3.1.

Lemma 6. *The characters P_k take equal (rational integral) values on elements of the same rank.*

Delsarte uses $P_k(r)$ to denote the value of P_k on an element of rank r. We will only need the values $P_1(r)$ in our work here. See [4], Theorem A2 (note that the summation index m in the printed formula must be replaced by k).

Lemma 7. *With the notation introduced above, we have*

$$P_1(r) = -\frac{(q^m - 1)}{q - 1} + \frac{q^n(q^{m-r} - 1)}{q - 1}.$$

We can now proceed to the proof of our main result of this section.

Theorem 6. *Let q be a power of the prime p and let $\mathcal{M}$ be a constant rank r subspace of $M_{m\times n}(\mathbb{F}_q)$. Then $\dim \mathcal{M} \leq m + n - r$.*

Proof. Let $t = \dim \mathcal{M}$. The restriction of the character P_1 of $M_{m\times n}(\mathbb{F}_q)$ to the subgroup $\mathcal{M}$ is also a character of $\mathcal{M}$ and therefore elementary character theory implies that the inner product of this character with the trivial character of $\mathcal{M}$ is an integer. Now this inner product is

$$\frac{P_1(0) + (q^t - 1)P_1(r)}{q^t},$$

since all non-zero elements of $\mathcal{M}$ have rank r by hypothesis. It follows that

$$P_1(0) \equiv P_1(r) \pmod{q^t}.$$

Using Delsarte's character formula, Lemma 7, we obtain

$$q^{m+n-r}(q^r - 1) \equiv 0 \pmod{q^t}.$$

It follows that $t = \dim \mathcal{M} \leq m + n - r$. □

The reader may compare our bound above with the upper bound $m + n - 2r + 1$ for the dimension of a constant rank r subspace of $M_{m\times n}(\mathbb{C})$, obtained by Westwick in [10]. This upper bound may be improved to $n - r + 1$ when $n - r + 1$ does not divide $(m - 1)!/((r - 1)!$.

The bound obtained in Theorem 6 is particularly poor when $r = 1$, since the correct upper bound for the dimension of a constant rank 1 subspace is n, a result which holds for all fields. Nonetheless, as we mentioned in the introduction, there are examples of $(n + 1)$-dimensional constant rank $n - 1$ subspaces of $M_n(\mathbb{F}_2)$ for $3 \leq n \leq 5$. The existence of these subspaces implies that the bound can be optimal in non-trivial ways.

Example 1. Consider the 6-dimensional subspace U of $M_{4\times 5}(\mathbb{F}_2)$ consisting of the linear span of the matrices

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

A consideration of all cases shows that U is a constant rank 3 space. By Theorem 5, a constant 3 rank subspace of $M_{4\times 5}(K)$ has dimension at most 5 when $|K| \geq 4$. Thus we see that the restriction in the field size in Theorem 5 is appropriate when $m = 4$, $n = 5$ and $r = 3$.

Example 2. If we adjoin an additional zero row at the top of each matrix of Example 3, the resulting matrices span a 6-dimensional constant rank 3 subspace of $M_5(\mathbb{F}_2)$.

5. A maximality result for certain constant rank subspaces

Suppose that $M_n(K)$ contains an n-dimensional constant rank n subspace, $\mathcal{N}$, say. This occurs, for example, if K admits a field extension of degree n. Let T be any element of rank $m-1$ in $M_{m\times n}(K)$. The subspace $\mathcal{M} = T\mathcal{N}$ is then an n-dimensional constant rank $m-1$ subspace of $M_{m\times n}(K)$. We intend to show in this section that such an n-dimensional constant rank $m-1$ subspace is maximal when K is finite, that is, the subspace is not contained in any $(n+1)$-dimensional constant rank $m-1$ subspace.

Definition 2. Let $\mathcal{R}$ be a subspace of $M_{m\times n}(K)$. Given any non-zero vector u in K^m, we set

$$\mathcal{R}_{\langle u\rangle} = \{T \in \mathcal{R} : uT = 0\}.$$

This is in accordance with the notation used in Definition 1.

Lemma 8. *Let $\mathcal{R}$ be a subspace of $M_{m\times n}(K)$ and let u be a non-zero element of K^m. Suuppose that* $\dim \mathcal{R} > n$. *Then*

$$\dim \mathcal{R}_{\langle u\rangle} > 0.$$

Proof. We define a K-linear transformation $\epsilon : \mathcal{R} \to K^n$ by

$$\epsilon(S) = uS$$

for all $S \in \mathcal{R}$. Since $\dim \mathcal{R} > \dim K^n$, ϵ is not injective, and hence $\dim \ker \epsilon > 0$. The desired result now follows, since ker $\epsilon = \mathcal{R}_{\langle u\rangle}$. □

We omit the proof of the following simple fact.

Lemma 9. *Let $\mathcal{M} = T\mathcal{N}$ be the n-dimensional constant rank $m-1$ subspace of $M_{m\times n}(K)$ described above. Let u be a basis vector for the kernel of T. Then each nonzero element of $\mathcal{M}$ has the same kernel $\langle u\rangle$.*

Lemma 10. *Let $\mathcal{R}$ be a constant rank $n-1$ subspace of $M_{m\times n}(K)$ and let $\langle u\rangle$, $\langle v\rangle$ be different one-dimensional subspaces of V. Then*

$$\mathcal{R}_{\langle u\rangle} \cap \mathcal{R}_{\langle v\rangle} = 0$$

Proof. This follows since a non-zero element in the intersection would annihilate the linearly independent vectors u and v and hence have rank at most $m-2$. □

Lemma 11. *Let $\mathcal{M} = T\mathcal{N}$ be the n-dimensional constant rank $m-1$ subspace of $M_{m\times n}(K)$ described in Lemma 9 and let u be a basis vector for* $\ker T$. *Suppose, if possible, that there exists an $(n+1)$-dimensional constant rank $m-1$ subspace $\mathcal{R}$ of $M_{m\times n}(K)$ containing $\mathcal{M}$. Then if $v \in K^m$ is not a scalar multiple of u, we have*

$$\dim \mathcal{R}_{\langle v\rangle} = 1,$$

while

$$\mathcal{R}_{\langle u\rangle} = \mathcal{M}_{\langle u\rangle} = \mathcal{M}.$$

Proof. Suppose that $v \notin \langle u\rangle$. Then we know that $\mathcal{M}_{\langle v\rangle} = 0$. We thus have

$$\mathcal{R}_{\langle v\rangle} \cap \mathcal{M} = \mathcal{M}_{\langle v\rangle} = 0.$$

It follows that $\dim \mathcal{R}_{\langle v\rangle} \leq 1$, since $\mathcal{M}$ has codimension 1 in $\mathcal{R}$. However, Lemma 8 implies that $\dim \mathcal{R}_{\langle v\rangle} \geq 1$ and hence $\dim \mathcal{R}_{\langle v\rangle} = 1$.

Finally, we know that $\mathcal{R}_{\langle u\rangle}$ contains $\mathcal{M}$. If $\mathcal{R}_{\langle u\rangle} = \mathcal{R}$, then $\mathcal{R}_{\langle u\rangle}$ has non-trivial intersection with any $\mathcal{R}_{\langle v\rangle}$. This contradicts Lemma 10 when u and v are linearly independent. □

We can now show that there is no subspace $\mathcal{R}$ fulfilling the requirements of Lemma 11 when K is finite.

Theorem 7. *Let $\mathcal{M} = T\mathcal{N}$ be the n-dimensional constant rank $m-1$ subspace of $M_{m\times n}(K)$ described in Lemma 9. Then if K is finite, $\mathcal{M}$ is not contained in any larger constant rank $m-1$ subspace of $M_{m\times n}(K)$.*

Proof. Suppose by way of contradiction that $\mathcal{M}$ is contained in the constant rank $m-1$ subspace $\mathcal{R}$ of dimension $n+1$. Then we have

$$\mathcal{R} = \bigcup \mathcal{R}_{\langle v\rangle},$$

where $\langle v\rangle$ ranges over the one-dimensional subspaces of K^m, and the subspaces $\mathcal{R}_{\langle v\rangle}$ have trivial intersection with each other. Moreover, one subspace in the union above is $\mathcal{M}$, the rest are one-dimensional. Thus if $|K| = q$, counting non-zero vectors in $\mathcal{R}$, we obtain

$$q^{n+1} - 1 = q^n - 1 + \frac{(q^m - q)}{(q-1)}(q-1).$$

This equality is clearly impossible when $m \leq n$.

□

The examples of Beasley, [1], and Boston, [3], imply that the spaces $M_n(\mathbb{F}_2)$ for $3 \leq n \leq 5$ contain two types of maximal constant rank $n-1$ subspaces. One type has dimension n, the other dimension $n+1$. Similarly, $M_{4\times 5}(\mathbb{F}_2)$ contains two types of maximal constant rank 3 subspaces, one type of dimension 5, the other of dimension 6.

6. Subspaces of skew-symmetric matrices with special rank properties

Specializing the rank theme described in the introduction to this paper, it is an interesting research problem to investigate subspaces $\mathcal{S}$, say, of skew-symmetric matrices which possess one of the following three properties.

(1) Each element of $\mathcal{S}$ has rank at most $2r$.
(2) Each element of $\mathcal{S}^\times$ has rank $2r$.
(3) Each element of $\mathcal{S}^\times$ has rank at least $2r$.

We note, as is well known, that a skew-symmetric matrix has even rank. Concerning condition 3, we have the following result, [7], Theorem 8.

Theorem 8. *Suppose that the field K has a cyclic Galois extension field of odd degree n and let r be an integer satisfying $2 \leq 2r \leq n-1$. Then there exists a subspace $\mathcal{S}$ of skew-symmetric matrices in $M_n(K)$ of dimension $n(n-2r+1)/2$ in which each element T of $\mathcal{S}^\times$ satisfies* $\operatorname{rank} T \geq 2r$.

Concerning Theorem 8, Delsarte and Goethals show in [5], Theorem 4, that if n is odd and K is finite, $n(n-2r+1)/2$ is the maximum dimension of a subspace of skew-symmetric matrices in $M_n(K)$ in which each non-zero element has rank at least $2r$. Their proof use characters defined on association schemes and we have not found a proof of this result working in the context of linear algebra, more especially, one which applies to infinite fields. (We do not know if the Delsarte-Goethals upper bound even holds for infinite fields.)

As far as constant rank subspaces of skew-symmetric matrices are concerned, the following existence theorem was established in [7].

Theorem 9. *Suppose that the field K has a cyclic Galois extension field of odd degree n and let $s>1$ be a divisor of n. Then there exists an n-dimensional constant rank $n(s-1)/s$ subspace of skew-symmetric matrices in $M_n(K)$.*

We do not know if there are values of r different from $n(s-1)/s$ for which there exists an n-dimensional constant rank r subspace of skew-symmetric matrices in $M_n(K)$.

We conclude this section by obtaining a version of Theorem 6 for constant rank subspaces of skew-symmetric matrices over finite fields. Our proof again uses characters of finite groups.

Theorem 10. *Let $\mathcal{S}$ be a constant rank $2r$ subspace of skew-symmetric matrices in $M_n(\mathbb{F}_q)$. Then $\dim \mathcal{S} \le 2n-2r-1$.*

Proof. Let $\mathcal{A}$ denote the vector space of $n\times n$ skew-symmetric matrices with entries in $\mathbb{F}_q$. We may consider $\mathcal{A}$ to be an elementary abelian p-group of order q^{nk}, and $\mathcal{S}$ to be a subgroup of order q^m, where $m=\dim\mathcal{S}$. For each integer k with $1\le k\le [n/2]$, Delsarte and Goethals construct a (reducible) complex character P_k of $\mathcal{A}$ which takes the same integral value on elements of the same rank in $\mathcal{A}$, [5], p.29. Following the notation of [5], we let $P_1(r)$ denote the value of P_1 on an element of rank $2r$. By the formula (15) in [5], we have

$$P_1(r) = -\frac{(q^{2t}-1)}{q^2-1} + \frac{q^n(q^{2t-2r}-1)}{q^2-1},$$

where $n=2t+1$ is odd. In the case that n is even, we have

$$P_1(r) = -\frac{(q^{n}-1)}{q^2-1} + \frac{q^{n-1}(q^{n-2r}-1)}{q^2-1}.$$

The rest of the proof is identical with that of Theorem 6. □

It may be of interest to comment on the sharpness of the dimension bound obtained above. When n is odd, there is an n-dimensional constant rank $n-1$ subspace of skew-symmetric matrices in $M_n(\mathbb{F}_q)$ and thus the bound given by the theorem is sharp in this case. When n is even, one can show that the maximum dimension of a constant rank n subspace of skew-symmetric matrices in $M_n(\mathbb{F}_q)$ is $n/2$, whereas the theorem gives an upper bound of $n-1$ for the dimension. Thus the bound given by the theorem is weak in this case. On the other hand, when m is an odd positive integer and $n=2m$, we can construct an m-dimensional constant rank $m-1$ subspace of skew-symmetric matrices in $M_m(\mathbb{F}_{q^2})$ and then use the trace map from $\mathbb{F}_{q^2}$ to $\mathbb{F}_q$ to obtain an n-dimensional constant rank $n-2$ subspace of skew-symmetric matrices in $M_n(\mathbb{F}_q)$. The theorem above gives $n+1$ as the maximum dimension for such a constant rank subspace, so that we are quite close to a sharp bound here. At the time of this writing, we have not found any constant rank subspace of skew-symmetric matrices in $M_n(\mathbb{F}_q)$ of dimension greater than n.

7. Construction of examples by field extension

Let L be a field extension of K of finite degree m. Recall that we may consider any vector space over L to be a vector space over K. In particular, let V be a vector space of finite dimension n over L and let V^K denote V considered as a vector space over K. We have then $\dim_K V^K = mn$.

Clearly, an L-linear endomorphism, σ, say, of V defines a K-linear endomorphism of V^K, which we shall denote by σ^K. It is elementary to see that the mapping $\sigma \to \sigma^K$ is K-linear and injective. Consequently, we have a K-linear monomorphism $\mathrm{End}_L(V) \to \mathrm{End}_K(V^K)$ whose image has dimension mn^2 (and hence the monomorphism is an isomorphism only when $L = K$).

The following fact about this monomorphism is well known, but we include a brief proof.

Lemma 12. *With the notation introduced above,*

$$\text{rank } \sigma^K = m \,\text{rank } \sigma.$$

Proof. Let N denote the kernel of σ. Clearly, N^K (N considered as a vector space over K) is the kernel of σ^K. We have then

$$\text{rank } \sigma = \dim_L V - \dim_L N$$

and

$$\text{rank } \sigma^K = \dim_K V^K - \dim_K N^K = m \dim_L V - m \dim_L N.$$

The result is immediate from these equalities. □

Suppose now that $\mathcal{M}$ is a subspace of $\mathrm{End}_L(V)$ and let $\mathcal{M}^K$ denote its image under the mapping just considered. Suppose that the different ranks of the non-zero elements of $\mathcal{M}$ are

$$r_1, r_2, \ldots, r_k.$$

Then $\mathcal{M}^K$ has dimension $m \dim_L \mathcal{M}$ and its non-zero elements have rank

$$mr_1, mr_2, \ldots, mr_k.$$

This simple observation provides a method for generating subspaces of matrices with special rank properties.

Suppose in particular that we have found an $(n+1)$-dimensional constant rank $n-1$ subspace in $M_n(\mathbb{F}_{q^m})$, where $m > 1$. Then the construction described above creates an $m(n+1)$-dimensional constant rank $m(n-1)$ subspace in $M_{mn}(\mathbb{F}_q)$. Theorem 6 shows that $m(n+1)$ is the maximum dimension for such a constant rank subspace. However, at this time of writing, the known examples are restricted to $\mathbb{F}_2$, and we do not know whether this small field is exceptional in this theory.

Let us assume for the rest of this section that L is a separable extension of K. Let T denote the trace function from L to K. The separability assumption then implies that T maps L onto K. Let $f : V \times V \to L$ be an L-valued bilinear form. We define a bilinear form $f^K : V^K \times V^K \to K$ by setting

$$f^K(u, v) = T(f(u, v))$$

for all u and v in V. We omit the formal proof of the following result, which follows that of Lemma 12.

Lemma 13. *With the notation introduced above,*

$$\text{rank } f^K = m \,\text{rank } f.$$

Furthermore, the mapping $f \to f^K$ is a K-linear monomorphism from the vector space of all L-valued bilinear forms on $V \times V$ (considered as a space over K) into the vector space of all K-valued bilinear forms on $V^K \times V^K$, whose image has dimension mn^2 over K. This monomorphism maps alternating bilinear forms into alternating bilinear forms and symmetric bilinear forms into symmetric bilinear forms.

Corollary 3. *Suppose that K has a separable field extension L of degree m. Then there exists a $3m$-dimensional constant rank $2m$ subspace of skew-symmetric matrices in $M_{3m}(K)$*

Proof. Let V be a 3-dimensional vector space over L. The space of L-valued alternating bilinear forms on $V \times V$ is 3-dimensional and each non-zero element in this space has rank 2. The result follows from Lemma 13. □

Thus, for example, taking $K = \mathbb{R}$ and $L = \mathbb{C}$, we see that there is a 6-dimensional constant rank 4 subspace of skew-symmetric matrices in $M_6(\mathbb{R})$. Similarly, as we observed in the previous section, when m is an odd positive integer and $n = 2m$, we can construct an m-dimensional constant rank $m - 1$ subspace of skew-symmetric matrices in $M_m(\mathbb{F}_{q^2})$ and then obtain an n-dimensional constant rank $n-2$ subspace of skew-symmetric matrices in $M_n(\mathbb{F}_q)$ for each prime power q.

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