Moore-Penrose pseudo-inverse characterization*

1. Note that for all matrices M and N, even over a finite field, we have $rank(M) \geq rank(MN)$.
2. Over a finite field, Moore-Penrose pseudo-inverse satisfies
   (a) $AA^\dagger A = A$
   (b) $A^\dagger AA^\dagger = A^\dagger$
   (c) $(A^\dagger A)^T = A^\dagger A$
   (d) $(AA^\dagger)^T = AA^\dagger$

Theorem 1. *Let $A \in \mathbb{K}^{m\times n}$, $A^\dagger$ exists iff $rank(A) = rank(A^T A) = rank(AA^T)$.*

Proof. 1. On the one hand, suppose $A^\dagger$ exists.

- From the definition of transpose we have $(A^\dagger A)^T = A^T A^{\dagger T}$ so that from (2c) we also have $A^\dagger A = A^T A^{\dagger T}$.
- Then from (2a) we have $A = AA^\dagger A = A(A^\dagger A) = AA^T A^{\dagger T}$.
- Now, from (1) twice and the latter we get $rank(A) \geq rank(AA^T) \geq rank(AA^T A^{\dagger T}) = rank(A)$.
- Therefore $rank(A) = rank(AA^T)$.

Similarly

- From the definition of transpose we have $(AA^\dagger)^T = A^{\dagger T} A^T$ so that from (2d) we also have $AA^\dagger = A^{\dagger T} A^T$.
- Then from (2a) we have $A = AA^\dagger A = (AA^\dagger)A = A^{\dagger T} A^T A$.
- Now, from (1) twice and the latter we get $rank(A) \geq rank(A^T A) \geq rank(A^{\dagger T} A^T A) = rank(A)$.
- Therefore $rank(A) = rank(A^T A)$.

2. On the other hand, now suppose $rank(A) = rank(AA^T) = rank(A^T A) = r$.

- Using Gaussian elimination (see e.g. [1, 6.5.5] and references therein), there exists two full-rank matrices $L_r \in \mathbb{K}^{m\times r}$ and $U_r \in \mathbb{K}^{r\times n}$, and two permutation matrices P and Q such that $A = PL_r U_r Q$.
- Then from (1) twice and the hypothesis, we have $rank(U_r U_r^T) \geq rank(PL_r(U_r QQ^T U_r^T)L_r^T P^T) = rank(AA^T) = rank(A) = r$.
- Therefore, as $U_r U_r^T \in \mathbb{K}^{r\times r}$, it is full-rank and invertible.
- Similarly $rank(L_r^T L_r) \geq rank(Q^T U_r^T (L_r^T P^T PL_r)U_r Q) = rank(A^T A) = rank(A) = r$.
- Therefore, as $L_r^T L_r \in \mathbb{K}^{r\times r}$, it is full-rank and invertible.
- Finally $A^\dagger = Q^T U_r^T (U_r U_r^T)^{-1}(L_r^T L_r)^{-1} L_r^T P^T$ satisfies all four Equations (2).

□

References

[1] Jean-Guillaume Dumas, Pascal Giorgi, and Clément Pernet. Dense linear algebra over prime fields. *ACM Transactions on Mathematical Software*, 35(3):1–42, November 2008. URL: http://hal.archives-ouvertes.fr/hal-00018223, doi:10.1145/1391989.1391992.

*Jean-Guillaume Dumas

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