Hybrid Systems: Verification and Controller Synthesis

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VERIMAG

Plan

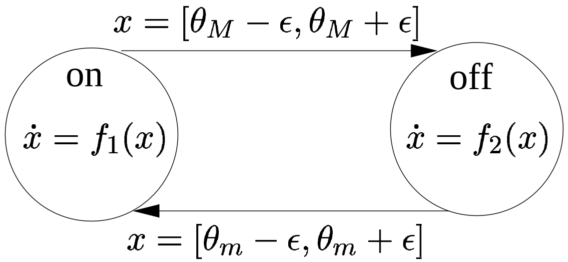
1. Algorithmic verification of hybrid systems
2. Reachability analysis of continuous systems
3. Safety verification of hybrid systems
4. Controller synthesis
5. Abstraction

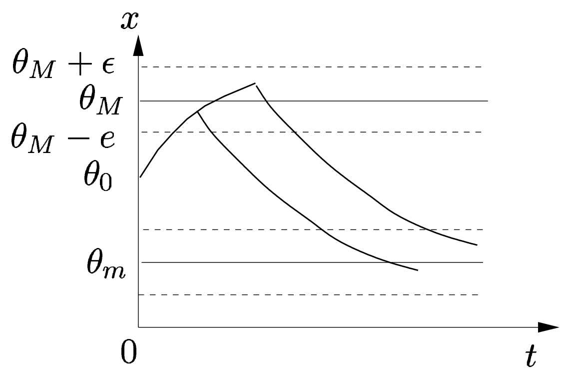
Algorithmic Analysis of Hybrid Systems

- Formal verification: prove that the system satisfies a given property
- Controller synthesis: design controllers so that the controlled system satisfies a desired property
- We concentrate on invariance properties: all trajectories of the system stay in a subset of the state space

Algorithmic Analysis of Hybrid Systems

Thermostat example





Difficulties in analysis of hybrid systems

- Two-phase evolution
- Non-deterministic behavior
- Set of initial states

$\Rightarrow$ How to characterize and represent set of trajectories (or tubes of trajectories) generated by continuous dynamics and discrete transitions

Algorithmic Analysis of Hybrid Systems

- Exact symbolic methods: applicable for restricted classes of hybrid systems (linear dynamics with special eigenstructures) [PappasLafferriereYovine 99]
- Approximate methods: using a variety of set representation
  - Level set method (using Hamilton-Jacobi partial differential equation formulation) [TomlinLygerodSastry00]
  - Polyhedral approximations [GreenstreetMitchell98, DangMaler98, ChutinanKrogh99, AsarinDangMaler01]
  - Ellipsoidal calculus [KurzhanskiVaraiya00, BotchkarevTripakis00]

In our work, we use convex and orthogonal polyhedra to represent and compute reachable sets of hybrid systems (see later).

Abstract Verification Algorithm

$R^0 := Y;$

repeat $k = 0, 1, 2, \dots$

if $(R^k \cap \mathcal{B} \neq \emptyset)$ **return unsafe** /* $\mathcal{B}$: *bad set* */

$R^{k+1} := R^k \cup \delta(R^k);$

until $R^{k+1} = R^k$

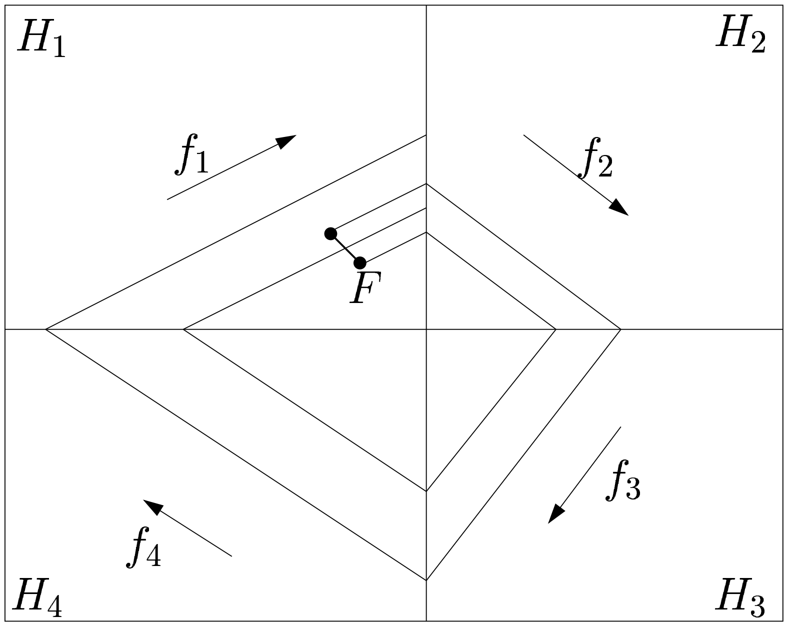
return safe

$\Rightarrow$ Computation of the following *functions over subsets* of the state space of hybrid systems: successor, union and intersection, emptiness checking.

Termination is not guaranteed.

Example of a non-terminating computation

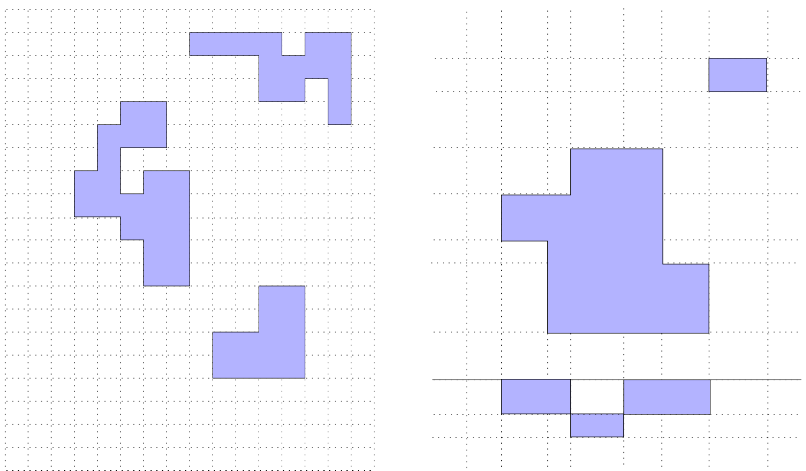
A 4-state PCD (piecewise-constant derivative) system



Our approach: polyhedral approximation

To represent reachable sets, we use orthogonal polyhedra (unions of closed full-dimensional hyper-rectangles)

- Canonical representation $\Rightarrow$ effective computations of Boolean operations, equivalence and emptiness checking, membership testing, and other geometric operations (face detection, etc.).
- Appropriate for over- and under-approximations of non-convex sets



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2. Reachability analysis of continuous systems
   - **Linear systems**
   - Non-linear systems
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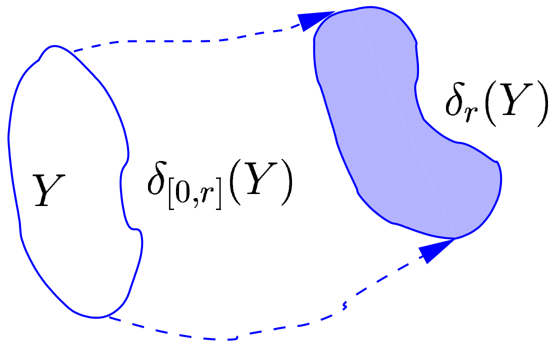
Reachability operators

Continuous system $\dot{\mathbf{x}} = f(\mathbf{x})$ where $\mathbf{x} \in \mathcal{X}$; $f : \mathcal{X} \to \mathbb{R}^n$ continuous vector field. Let $\phi_{\mathbf{x}}(t)$ be the solution of the diff eq with $\mathbf{x}$ as initial condition.

Given a time interval I and a set of states Y, successor operator $\delta_I(Y) = \{\mathbf{y} \mid \exists \mathbf{x} \in Y\ \exists t \in I\ \mathbf{y} = \phi_{\mathbf{x}}(t)\}$.

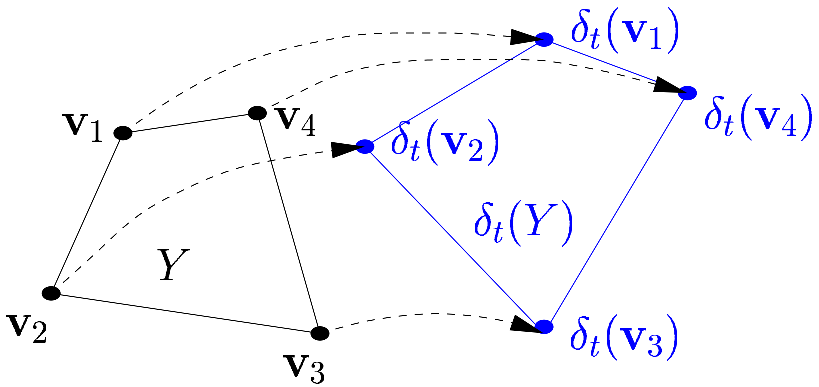
The reachable set from Y is $\delta(Y) = \delta_{[0,\infty)}(Y)$ (all states reachable after any non-negative amount of time).



Reachability analysis of linear systems

A continuous linear system $\dot{\mathbf{x}} = A\mathbf{x}$. Initial set Y is a convex bounded polyhedron $Y = conv(V)$ where $V = \{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ is a finite set of vertices

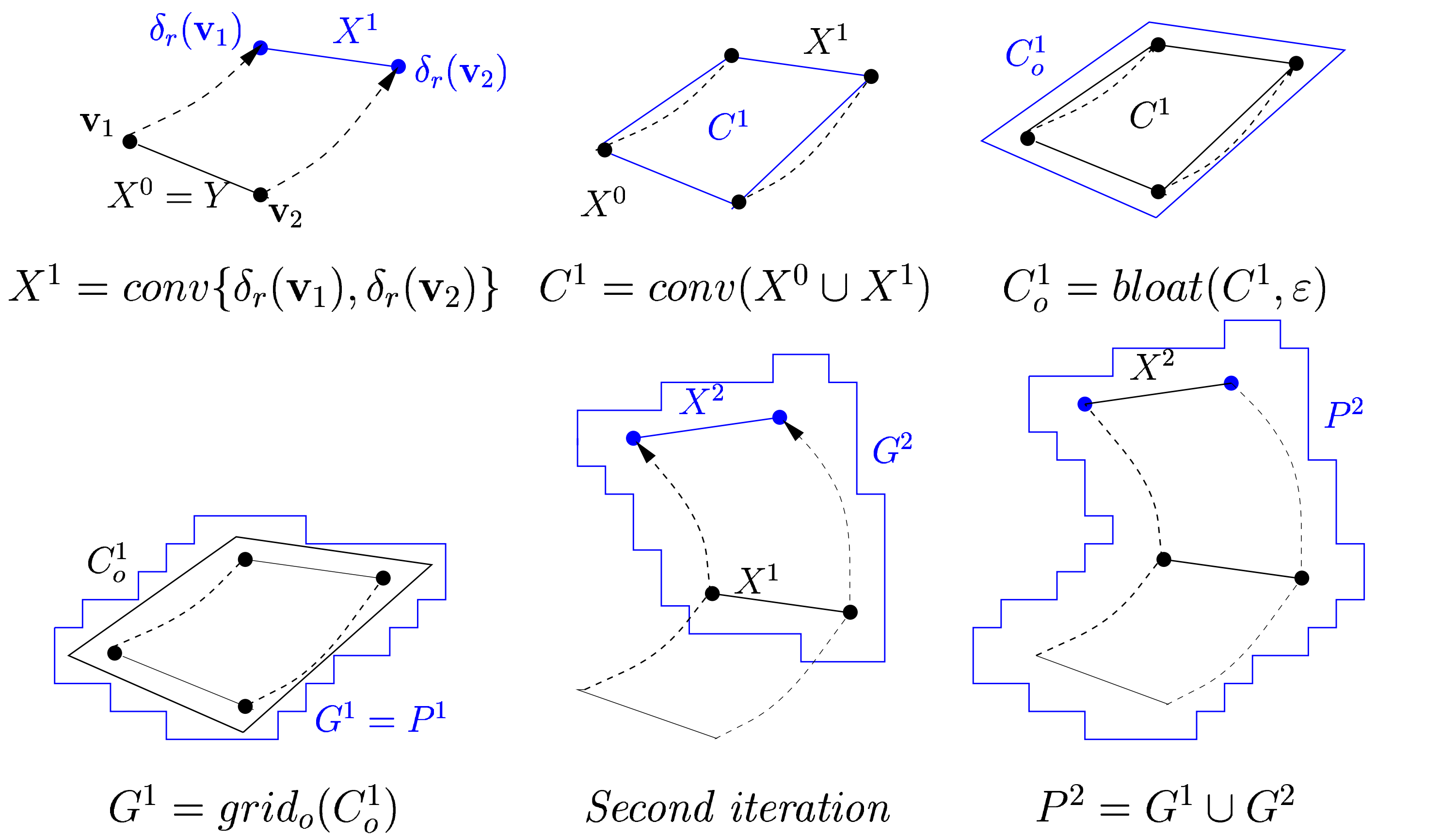
• Reachable set at time r $\delta_t(Y) = conv\{\delta_t(\mathbf{v}_1), \ldots, \delta_t(\mathbf{v}_m)\}$, and the successor of a point $\mathbf{v}$ is $\delta_t(\mathbf{v}) = e^{At}\mathbf{v}$



• Reachable set during time interval $[0, r]$,

Lemma: Given a time step $r \geq 0$, there exists $\varepsilon = \mathcal{O}(r^2)$ such that $\delta_{[0,r]}(Y) \subseteq conv(Y \cup \delta_r(Y)) \oplus \varepsilon B$ (ε-neighborhood of the convex hull of Y and $\delta_r(Y)$).

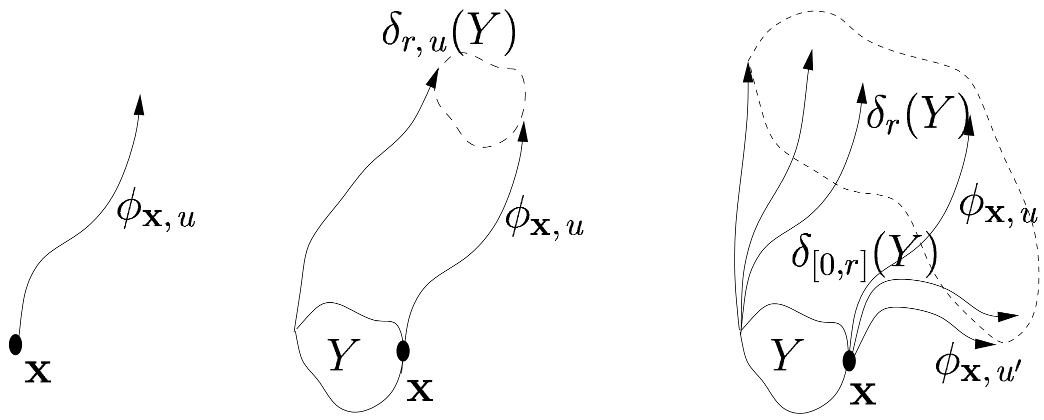
Reachability analysis of linear systems (cont'd)



No accumulation of error, approximation error is of order $\mathcal{O}(r^2)$.

Linear systems with uncertain input

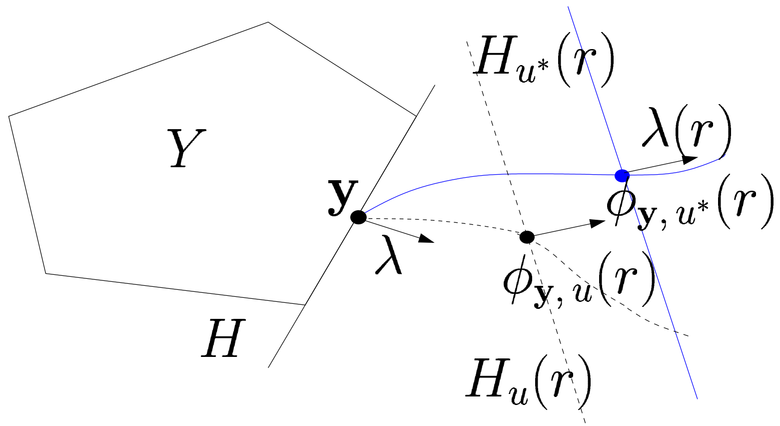
- System $\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + \mathbf{u}(t)$ where $\mathbf{x} \in \mathcal{X}$ and $\mathbf{u}(\cdot) \in \mathcal{U}$.
- Admissible input function $\mathbf{u}(\cdot) : \mathbb{R}^{+} \to U$ and U is a *convex bounded polyhedron*.
- Input can represent under-specified control or external disturbance



Linear systems with uncertain input (cont'd)

Computing reachable set $\delta_t(Y)$ at time r using the Maximal Principle

- The initial polyhedron can be written as intersection of half-spaces. Each half-space $H = \{\mathbf{x} \mid \langle \lambda, \mathbf{x} \rangle \leq \langle \lambda, \mathbf{y} \rangle\}$; λ: normal vector, $\mathbf{y}$: supporting point
- For every half-space H, there exists an input u^* s.t. calculating its successors under u^* is sufficient to derive a *tight polyhedral approximation* of $\delta_t(Y)$.
- Evolution of normal vector $\dot{\lambda}(t) = -A^T \lambda(t)$ (adjoint system) independent of input, $u^*(r) \in \arg\max\{\langle \lambda(r), \mathbf{u} \rangle \mid \mathbf{u} \in U\}$.



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Reachability analysis of non-linear systems

Consider a system $\dot{\mathbf{x}} = f(\mathbf{x})$, f is Lipschitz.

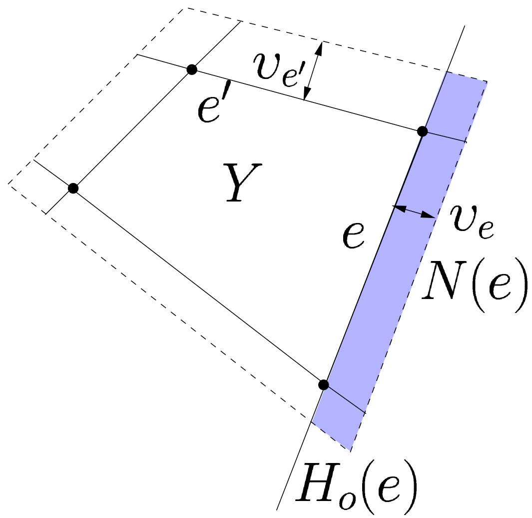
We propose two techniques

- 'Face lifting': Propagate the boundary of the reachable set, extension of Euler scheme for sets
- 'Hybridization': Extension of the simulation method based on simplicial decomposion of the state space (the previous talk) [Girard et al 02].

Face lifting technique

Continuity of trajectories: trajectory from a point $\mathbf{x} \in Y$ either *remains in Y forever* or *traverses the boundary ∂Y* after some time $\Rightarrow$ it suffices to compute from ∂Y.

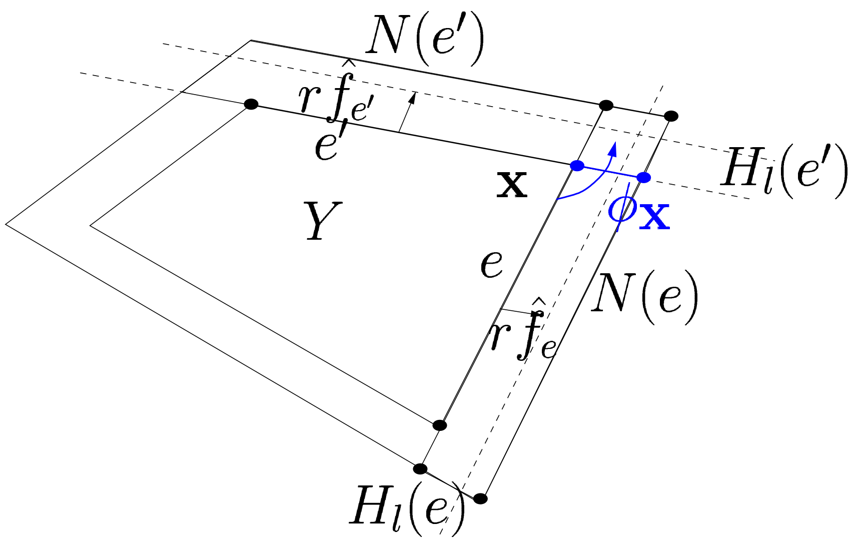
Lemma: Give a time step r, for each face e there exist v_e s.t. all trajectories starting from e stay in the neighborhood $N(e)$ for at least r time.



Face lifting technique: Principe

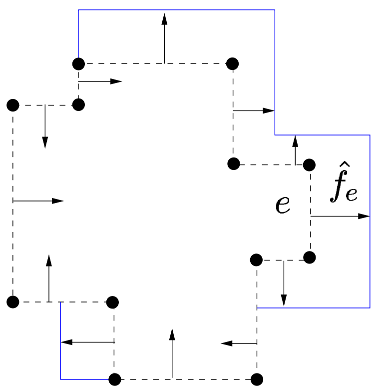
Over-approximating $\delta_{[0,r]}(Y)$ (reachable set during $[0, r]$)

1. For every face e of Y, construct the neighborhood $N(e)$
2. *Lifting operation*: For every face e of Y, $\hat{f}_e = \max\{f_e(\mathbf{x}) \mid \mathbf{x} \in N(e)\}$ where $f_e(\mathbf{x})$ projection of $f(\mathbf{x})$ on the *outward normal* of face e
   - If $\hat{f}_e$ is positive, lift $H(e)$ outward by the amount $r\hat{f}_e$ to obtain $H_l(e)$.
3. Intersect all the new half-spaces $H_l(e)$ $\Rightarrow$ **over-approximation** of $\delta_{[0,r]}(Y)$.



Face lifting technique on orthogonal polyhedra

- To avoid excessively conservative approximations, some faces must be split a priori $\Rightarrow$ the result of the lifting operation is non-convex.
- Use orthogonal polyhedra which offer the advantages:
  - Orthogonal polyhedra are closed under lifting operation
  - Faces of an orthogonal polyhedron can be systematically enumerated.
  - Efficient algorithms for the union operation and other required geometric operations.



Example: Airplane Safety [LygerosTomlinSastry97]

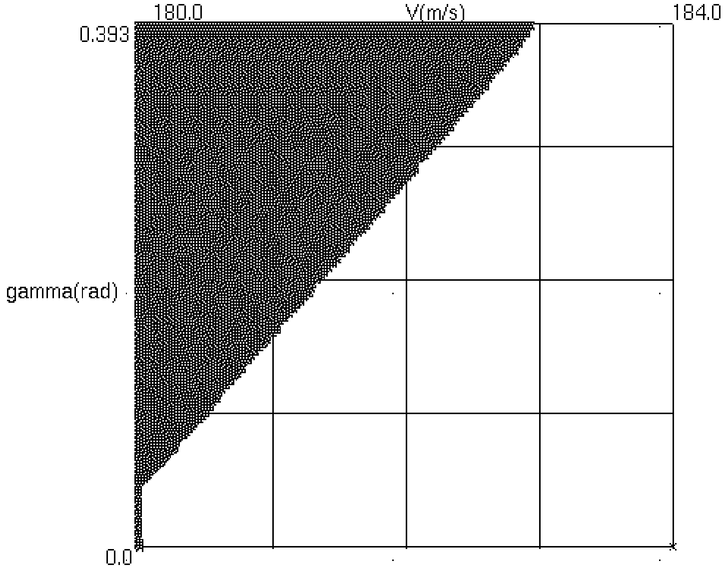
State variables x_1, x_2 represent velocity and flight path angle of an aircraft

u_1: thrust, u_2: pitch angle.

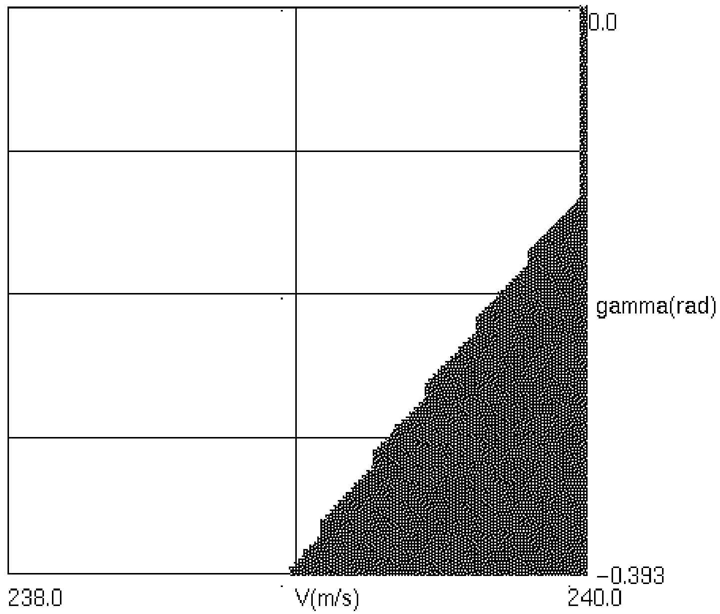
Safe set $P = [V_{min}, V_{max}] \times [\gamma_{min}, \gamma_{max}]$

$$\begin{aligned}
\dot{x}_1 &= -\frac{a_D x_1^2}{m} - g \sin x_2 + \frac{u_1}{m} \\
\dot{x}_2 &= \frac{a_L x_1 (1 - c x_2)}{m} - \frac{g \cos x_2}{x_1} + \frac{a_L c x_1}{m} u_2
\end{aligned}$$

EXAMPLE: AIRPLANE SAFETY [LygerosTomlinSastry97]



$u_1 = T_{max}, u_2 = \Theta_{min}$



$u_1 = T_{min}, u_2 = \Theta_{max}$

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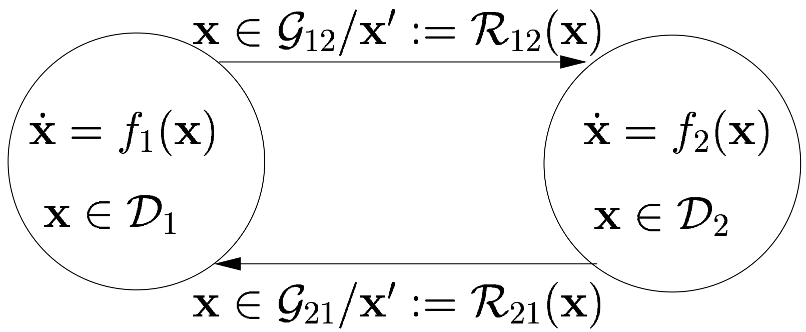
Hybrid systems

Hybrid automata

- Staying conditions of each mode $\mathcal{D}_q$, transition guard $\mathcal{G}_{qq'}$: convex polyhedra
- Reset maps: affine $\mathcal{R}_{qq'}(\mathbf{x}) = K_{qq'}\mathbf{x} + P_{qq'}$

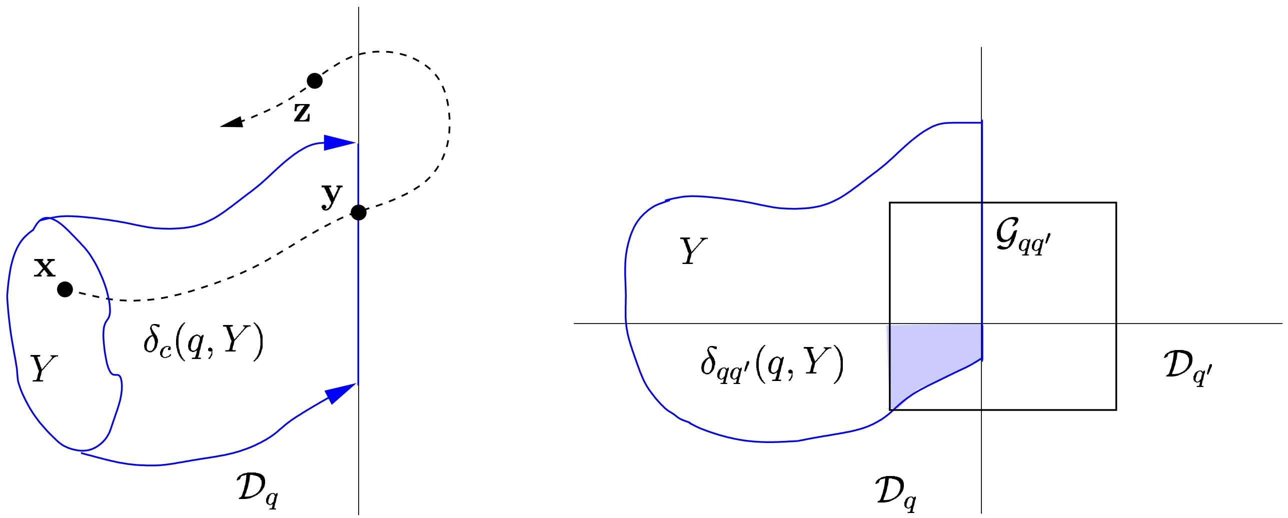


Reachability of Hybrid Automata

The state $(q, \mathbf{x})$ of the system can change in two ways:

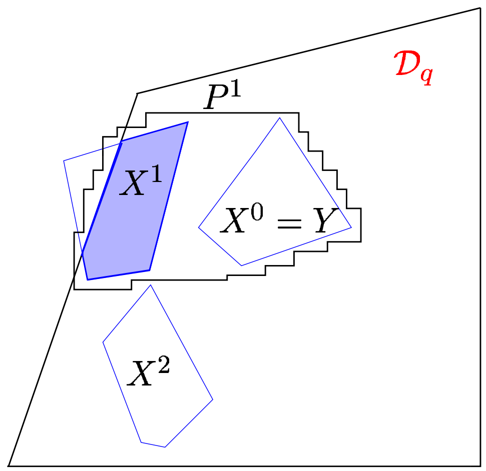
- continuous evolution: q remains constant, and $\mathbf{x}$ changes continuously according to the diff. eq. at q
- discrete evolution (by making a transition): q changes, and $\mathbf{x}$ changes according to the reset function.

$\Rightarrow$ continuous-successor δ_c and discrete-successor $\delta_{qq'}$



Reachability computation

- Computation of continuous-successors



- Computation of discrete-successors

$$\delta_{qq'}(q, Y) = \{(q', \mathcal{R}_{qq'}(Y \cap \mathcal{G}_{qq'}) \cap \mathcal{D}_{q'})\}$$

$\Rightarrow$ Boolean and geometric operations over orthogonal polyhedra

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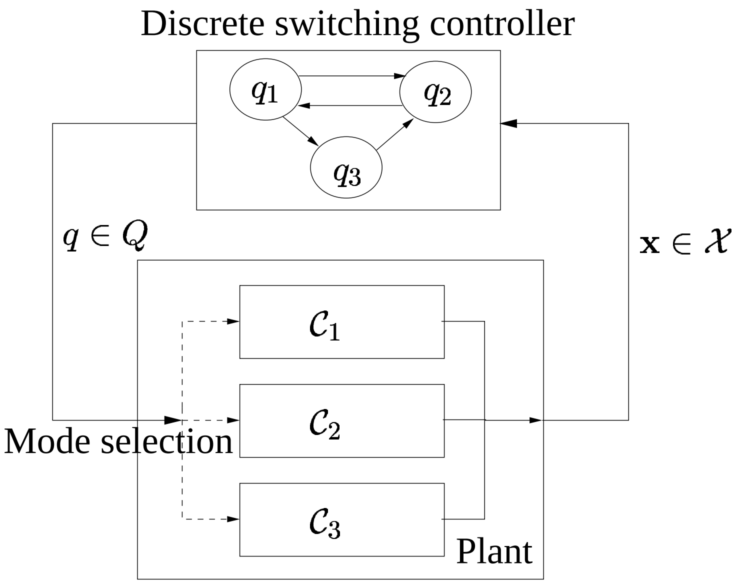
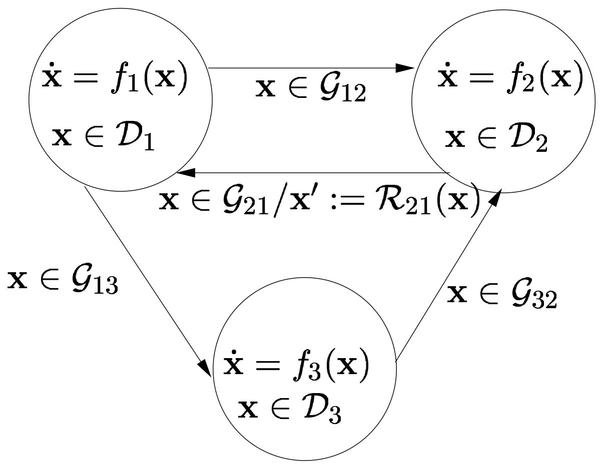
Switching Controller Synthesis: Setting

Plant: several 'continuous modes'

- Discrete switching controller continuously observes the state of the plant and decides which mode to select. We assume complete observability.
- Controller is non-deterministic, and feedback map $s : Q \times \mathcal{X} \to 2^Q$
- The overall system can be modeled as a hybrid automaton





Safety Controller Synthesis: Problem

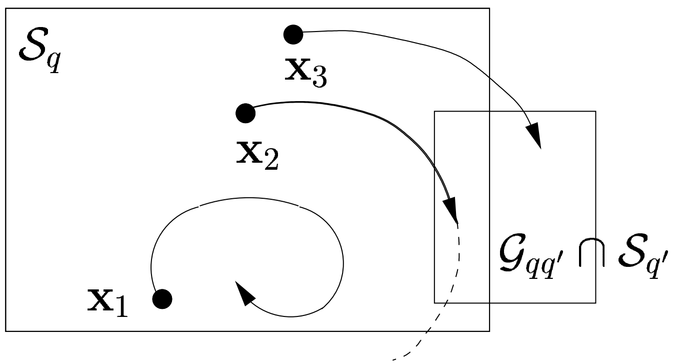
- Problem:
  - Given a hybrid automaton $\mathcal{A}$ and a safe set $\mathcal{S}$
  - How to restrict the guards and the staying conditions of $\mathcal{A}$ so that all trajectories of the resulting automaton $\mathcal{A}^*$ stay in $\mathcal{S}$.
- Solution [TomlinLygerosSastry00, AsarinDangMaler00]:
  - Compute the maximal invariant set, that is the set of winning state.
  - Winning states are the states from which the controller, by switching properly, ensures that all the trajectories of the controlled system lie within $\mathcal{S}$.

One Step Predecessor operator

The one step predecessor operator $\pi : 2^{Q\times\mathcal{X}} \to 2^{Q\times\mathcal{X}}$

Given a set $\mathcal{S} = \{(q, \mathcal{S}_q) \mid q \in Q\}$, $\pi(\mathcal{S})$ is the set of all states from which all trajectories

- stay indefinitely in $\mathcal{S}$ without switching OR
- stay in $\mathcal{S}$ for some time and then make a transition to another location and still in $\mathcal{S}$



Computation of the Maximal Invariant Set

$$\mathcal{P}^0 := \mathcal{S};$$

$$\textbf{repeat } k = 0, 1, 2, \ldots$$

$$\mathcal{P}^{k+1} := \mathcal{P}^k \cap \pi(\mathcal{P}^k);$$

$$\textbf{until } \mathcal{P}^{k+1} = \mathcal{P}^k$$

$$\mathcal{P}^* := \mathcal{P}^k;$$

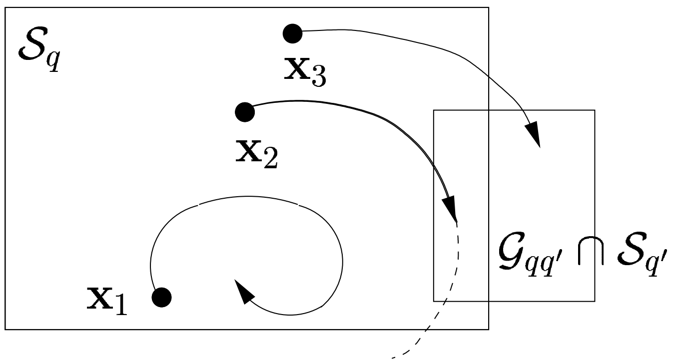
$\mathcal{P}^*$: maximal invariant set

$\mathcal{A}^*$: $\mathcal{D}^* = \mathcal{D} \cap \mathcal{P}^*$, $\mathcal{G}^* = \mathcal{G} \cap \mathcal{P}^*$.

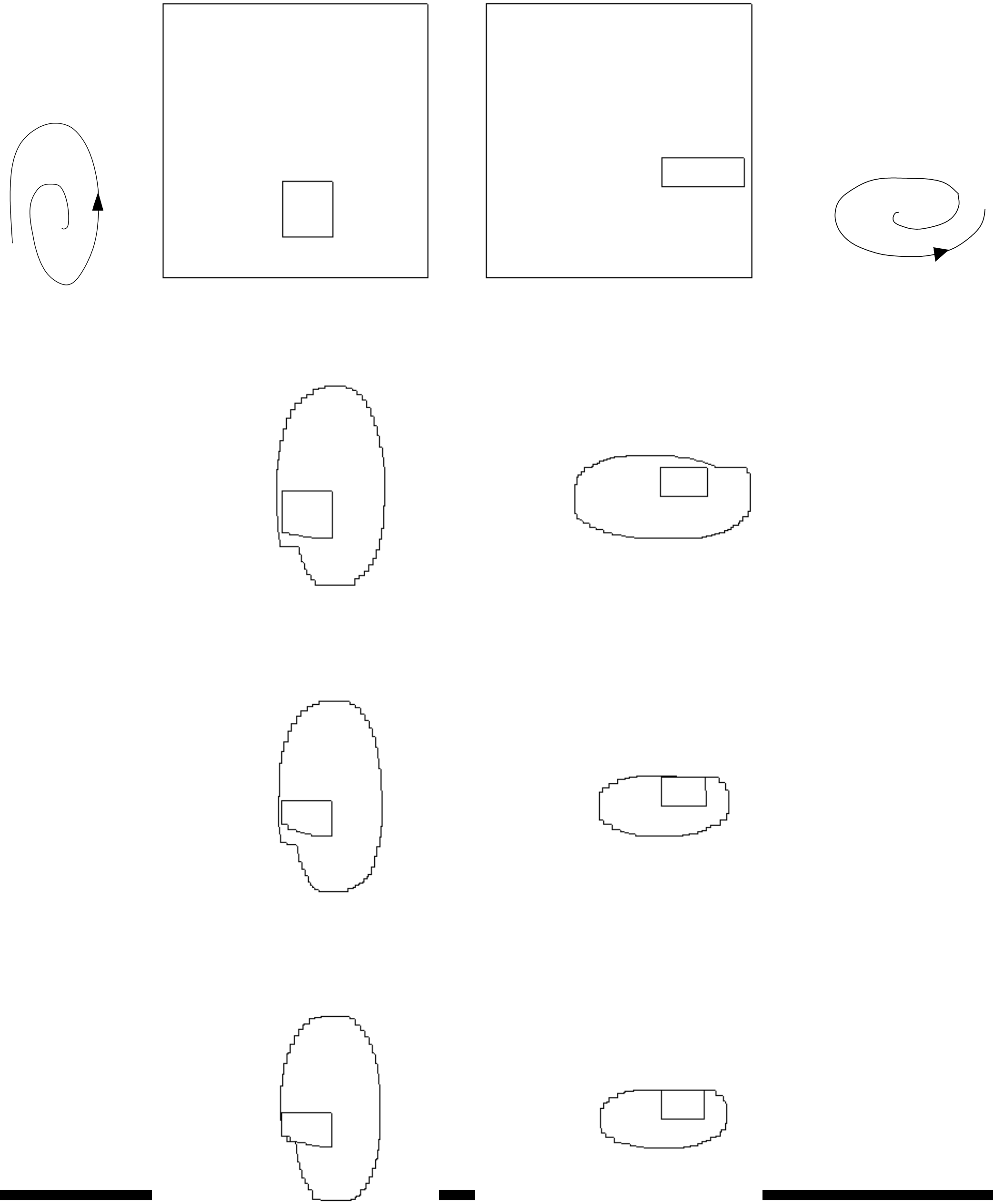
$\mathcal{A}^*$ is the least restrictive automaton satisfying the desired safety property.

Computation of the operator π

- States from which the system stay indefinitely in $\mathcal{S}_q$ without switching $\Rightarrow$ backward reachable set from the complement of $\mathcal{S}_q$
- States from which the system stay in $\mathcal{S}_q$ for some time and then make a transition to q' and still in $\mathcal{S}_{q'}$ $\Rightarrow$ continuous-predecessors from $\mathcal{G}_{qq'} \cap \mathcal{S}_{q'}$ with staying condition $\mathcal{S}_{q'}$
- Under-approximations



Example: Two-spiral system



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Abstraction by Projection: Introduction

- Dimension reduction method for continuous systems
- Basic idea: project away some variables the evolution of which is modeled as input in the dynamics of remaining variables
- A 'hybridization' method using ideas of qualitative simulation
- Goal:
  - more precise than qualitative simulation
  - less expensive than analysis of the original systems

Projection

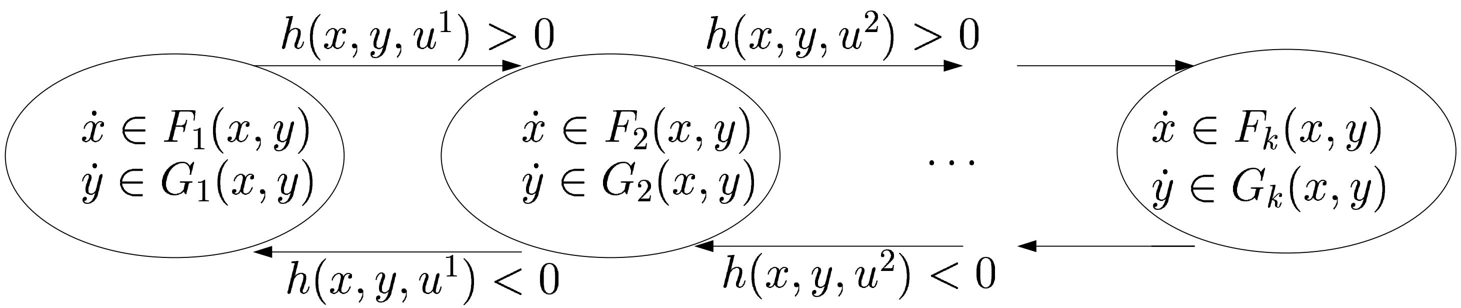
$$\begin{cases} \dot{x} &= f(x, y, z) \\ \dot{y} &= g(x, y, z) \\ \dot{z} &= h(x, y, z) \end{cases}$$

- f, g, h are Lipschitz continuous. We want to abstract away variable z
- Partition the domain of z into k disjoint intervals $\{[l^1, u^1), [l^2, u^2), \ldots [l^k, u^k]\}$, $l^{i+1} = u^i$ for all i
- If $z \in I_z^i = [l^i, u^i]$, the dynamics of x and y can be approximated by *differential inclusion*:

$$\begin{cases} \dot{x} &\in F_i(x, y) = \{f(x, y, z) \mid z \in I_z^i\} \\ \dot{y} &\in G_i(x, y) = \{g(x, y, z) \mid z \in I_z^i\} \end{cases}$$

Hybridization

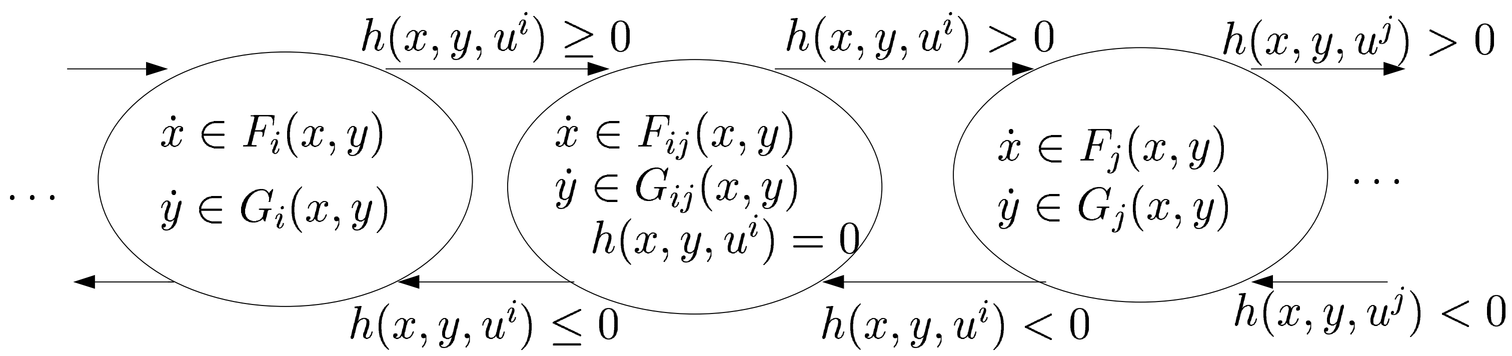
- The original system is thus approximated by 2-dimensional hybrid system with k different continuous dynamics
- Switchings between continuous dynamics correspond to the reachability relation between adjacent intervals I_z^i:
  - Transition from $I_z^i = [l^i, u^i)$ to $I_z^{i+1} = [l^{i+1}, u^{i+1})$ $(u^i = l^{i+1})$ is possible if at the boundary the derivative of z is positive, i.e. $h(x, y, u_i) > 0$
  - Similarly, transition from I_z^{i+1} to I_z^i if $h(x, y, u_i) < 0$
  - These switching conditions are not sufficient $\Rightarrow$ conservative approximation



Remedy Discontinuities

- Our hybridization method introduces discontinuities
- We will "convexify" the dynamics at switching surfaces (to guarantee existence of solution, error bound)
- Between adjacent intervals I_z^i and I_z^j $(j = i+1)$, add a location with dynamics:

$$\begin{cases} \dot{x} & \in F_{ij}(x,y) = co\{F_i(x,y), F_j(x,y)\} \\ \dot{y} & \in G_{ij}(x,y) = co\{G_i(x,y), G_j(x,y)\} \end{cases}$$



Convergence Result

- Resulting abstract system $(\dot{x}', \dot{y}') \in \mathcal{F}(x', y')$ is *upper semi-continuous* and *one-sided Lipschitz* $\Rightarrow$ We can prove error bound:
  - Distance between trajectories of the original system and the abstract system is bounded:

$$|(x(t), y(t)) - (x'(t), y'(t))| \leq |(x(0), y(0)) - (x'(0), y'(0))|e^{Lt} + \frac{\Delta}{L}(e^{Lt} - 1)$$

  - Δ: bound on the distance between the derivatives (which depends on the size of z mesh)
- First order method

Abstraction with Timing Information

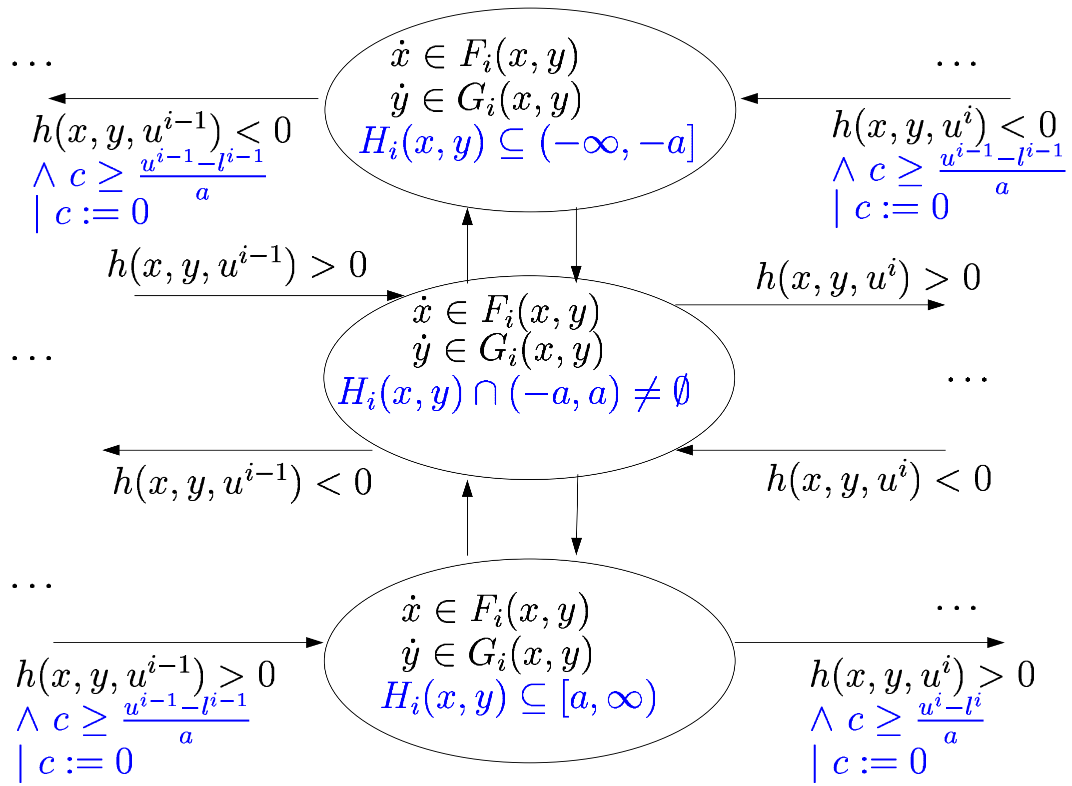
- So far, we use only the sign of the derivative of z to determine switching conditions
- The time the system can stay with a dynamics (staying time) is omitted
- To inlude more timing information to obtain more precise abstraction
  - Linear dynamics: staying time can be approximated numerically [Girard 03]
  - For nonlinear dynamics: discretize $\dot{z}$ into intervals and then estimate bounds on staying time.

Abstraction with Timing Information: nonlinear dynamics

- Additionally discretize the derivative of z into disjoint intervals
- Each location of the approximating automaton corresponds to an interval I_z^i of z and an interval $I_{\dot{z}}^j$ of $\dot{z}$
- Then, based on the intervals of derivatives of z we can estimate the bounds on the staying time and then embed this information in the switching conditions.

Abstraction with Timing Information: nonlinear dynamics

The domain of $\dot{z}$ is partitioned into 3 intervals: $I^1_{\dot{z}} = (-\infty, a], I^2_{\dot{z}} = (-a, a), I^3_{\dot{z}} = [a, \infty)$ where $a > 0$.



Computation Issues

- Linear Systems: abstract system is a linear system with uncertain input.
- Nonlinear systems: abstract system is a more general differential inclusions
- We focus on the case of multi-affine systems (which have numerous applications in biology, economy)

Abstraction of multi-affine systems

$$\begin{cases} \dot{x}_1 & = a_1 x_1 + b_1 x_2 + c_1 x_1 x_2 \\ \dot{x}_2 & = a_2 x_1 + b_2 x_2 + c_2 x_1 x_2 \end{cases}$$

Abstract away $x_2 \Rightarrow$

$$\begin{cases} \dot{x}_1 & = a_1 x_1 + b_1 u + c_1 u x_2 \\ ||u(\cdot)|| \leq \delta \end{cases}$$

We obtain a bilinear control system

Reachability analysis of Bilinear Control Systems

Consider a bilinear control system with additive and multipicative inputs

$$\dot{x}(t) = f(x(t), u(t)) = Ax(t) + \sum_{j=1}^{l} u_j(t) B_j x(t) + Cu(t)$$

$x(t) \in \mathbb{R}^n$: state variables, input $u : \mathbb{R}^+ \to U$ and $U \subset \mathbb{R}^l$ is a bounded convex polyhedron.

Basic idea: Applying the Maximum principle to find the 'optimal' input u^* which can be used to over-approximate the reachable set $\Rightarrow$ require solving an optimal control problem for a bilinear system. For tractability purposes,

1. Restrict to piecesiwe constant inputs $u(t) = \bar{u}(t_k), t \in [t_k, t_{k+1}) \Rightarrow$ error in solution of order $O(r^2)$, $r = \max\{t_{k+1} - t_k\}$ time step
2. To solve bilinear diff equations, treat the bilinear term as independent input (see next)

Applying the Maximum Principe

Represent the initial set Y as intersection of half-spaces.
For each half-space H with normal v and supporting point p.

$$
\begin{aligned}
\dot{\tilde{x}} &= A\tilde{x} + \sum_{j=1}^{l} \tilde{u}_j B_j \tilde{x} + C\tilde{u} \\
\dot{\tilde{q}} &= -\frac{\partial H}{\partial x}(\tilde{x}, \tilde{q}, \tilde{u}) \text{ where } H(q, x, u) = \langle q, Ax + \sum_{j=1}^{l} u_j B_j x + Cu \rangle \\
\tilde{u}(t) &\in argmax\{\langle \tilde{q}(t), \sum_{j=1}^{l} u_j B_j \tilde{x}(t) + Cu \rangle \mid u \in U\}
\end{aligned}
$$

with initial conditions: $\tilde{q}(0) = v$, $\tilde{x}(0) = p$.
Then,

- for all $t > 0$, the half-space $H(t)$ defined by normal $\tilde{q}(t)$ and supporting point $\tilde{x}(t)$ contains the reachable set $\delta_t(Y)$,
- and the corresponding hyperplane is a supporting hyperplane of $\delta_t(Y)$.

Reachability analysis of Bilinear Control Systems

• Solving the optimal control problem for arbitrary inputs is hard $\Rightarrow$ restrict to piecewise constant inputs $u(t) = \bar{u}(t_k), t \in [t_k, t_{k+1})$.

• Solving bilinear systems with piecewise constant input:

$$x(t_{k+1}) = e^{Ar}x(t_k) + \int_0^r a^{A(r-\tau)} b\bar{u}_k d\tau + \int_0^r a^{A(r-\tau)} Bx(t_k+\tau)\bar{u}_k d\tau$$

We approximate $x(t_k + \tau)$ for $\tau \in [0, r)$ by: $\pi(\tau) = \alpha\tau^3 + \beta\tau^2 + \gamma\tau + \sigma$ satisfying Hermite interpolation conditions:

$$\pi(0) = x(t_k), \quad \dot{\pi}(0) = \dot{x}(t_k), \quad \pi(r) = x(t_{k+1}), \quad \dot{\pi}(r) = \dot{x}(t_{k+1})$$

- We obtain the coefficients of $\pi(\tau)$ as linear functions of $x(t_k)$ and $x(t_{k+1})$
- Replace $x(t_k + \tau)$ by $\pi(\tau)$ in the integral we obtain an algebraic equation: $M_k x(t_{k+1}) = m_k$ allowing to determine the map between $x(t_{k+1})$ and $x(t_k)$
- We can prove that the error is quadratic $O(r^2)$

Example: A biological system

A multi-affine system, used to model the gene transcription control in the *Vibrio fischeri* bacteria [Belta et al 03].

$$\begin{cases} \dot{x_1} &= k_2 x_2 - k_1 x_1 x_3 + u_1 \\ \dot{x_2} &= k_1 x_1 x_3 - k_2 x_2 \\ \dot{x_3} &= k_2 x_2 - k_1 x_1 x_3 - n x_3 + n u_2 \end{cases} \qquad (1)$$

State variables x_1, x_2, x_3 represent cellular concentration of different species

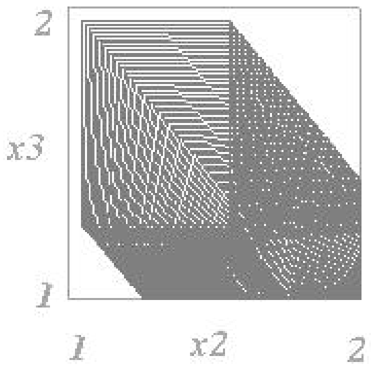
Parameters k_1, k_2, n are binding, dissociation and diffusion constants.

Control variables u_1 and u_2 are plasmid and external source of autoinducer.

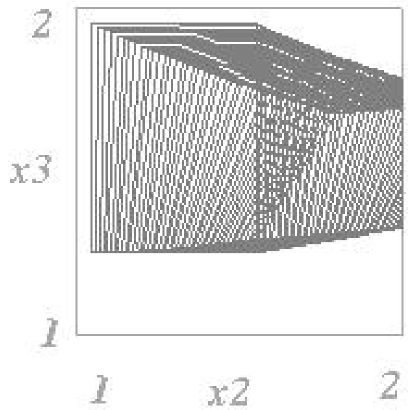
Goal: drive the system through to the face $x_2 = 2$

Example: A biological system (cont'd)

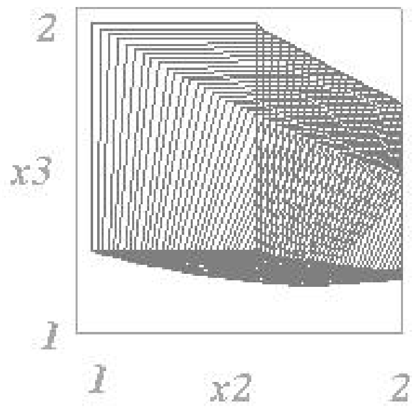
Reachability results obtained by abstracting away the variable x_1.



uncontrolled system ($u = 0$)



location $x_1 \in [1.0, 1.5]$



location $x_1 \in [1.5, 2.0]$