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CRAgroup 2002/08/27 Prn:17/10/2002; 9:54 F:crass12571.tex by:Au p. 1

C. R. Acad. Sci. Paris, Ser. I 335 (2002) 1–6

Analyse numérique/*Numerical Analysis*

A particle model for fluid–structure interaction

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Abstract We propose a particle method to handle fluid–structure interactions on a 1D model problem. Interactions between fluid and solid particles implicitly enforce the continuity of stresses on the interface. Comparisons with results obtained by ALE methods allow one to evaluate the robustness and accuracy of the method. ***To cite this article: G.-H. Cottet, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 1–6.***

Une méthode particulaire pour le couplage fluide–structure

Résumé On propose dans cette Note une méthode particulaire pour le calcul d'interactions fluide–structure. La continuité des efforts entre les 2 milieux est prise en compte implicitement par les interactions entre particules fluides et solides. Des tests numériques permettent d'évaluer la robustesse et la précision de la méthode. ***Pour citer cet article : G.-H. Cottet, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 1–6.***

Version française abrégée

Le couplage fluide–structure vu sous l'angle des écoulements multi-phasiques. – Les problèmes d'intéractions fluide–structure sont difficiles tant du point de vue théorique que numérique. Sur le plan numérique, leur non-linéarité combinée à la nécessité de suivre 2 milieux différents avec une interface mobile rend leur simulation coûteuse. L'approche habituelle est de suivre le fluide de manière lagrangienne en déformant le maillage avec une vitesse compatible avec le suivi de la frontière du solide (méthodes dites ALE). Ceci ne permet cependant pas de prendre en compte des déformations importantes pouvant conduire à des changements de topologie.

La méthode que nous proposons est, dans l'esprit, une méthode de frontière immergée. C'est-à-dire que les 2 milieux sont traités par une loi de conservation unique qui prend en compte *implicitement* l'interface. Ce type d'approche offre l'avantage d'enlever a priori toute contrainte géométrique sur la forme de l'interface.

Pour définir la méthode on se concentre sur un problème modèle 1D, qui permettra en outre des comparaisons simples avec une méthode ALE. Le fluide est décrit par l'équation de Burgers et le solide

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S1631-073X(02)02571-2/FLA AID:2571 Vol.335(0)
CRAgroup 2002/08/27 Prn:17/10/2002; 9:54 F:crass12571.tex

par l'équation des ondes (1). On commence par remarquer que l'équation des ondes en la variable lagrangienne ξ, conduit à une équation de Burgers en variable Eulérienne x (formule (3) et équation (4)), et que la continuité des efforts entre le fluide et le solide (2) permet d'écrire les temes du second ordre sous forme de la divergence d'un unique opérateur D (formule (6)). On peut aussi voir les équations (5) et (6) comme une modélisation de type écoulement multi-phasique, où la fonction caractéristique du fluide décrit la phase liquide.

Le modèle particulaire : définition et inégalité d'énergie. – Partant de là, la méthode particulaire s'écrit naturellement : les particules, discrétisant aussi bien la partie fluide que la partie solide, bougent avec la vitesse locale et les termes d'ordre 2 sont pris en compte par une loi d'intéraction classique en méthodes particulaires pour traiter les opérateurs de diffusion [2] (formule (8)). Il est à remarquer que les intéractions entre particules fluides et solides prennent en compte implicitement l'equilibre des forces à l'interface. La consistance de ce modèle avec le problème de départ est attestée par le fait que l'on peut prouver une estimation d'énergie (10) qui met en évidence des analogues discrets naturels des énergies cinétique et élastique du système et de la dissipation visqueuse.

Validation numérique. – Des tests numériques permettent d'évaluer la précision de la méthode et de comparer ses performances avec celles d'une méthode ALE. Pour la mise en œuvre pratique de la méthode, on a choisi une approche de type *Particle In Cell*, c'est à dire que les vitesses, nécessaires pour advecter les particules, et leur dérivées, requises pour le calcul des volumes des particules, sont calculées sur une grille fixe avant d'être interpolées sur les particules. Le passage grille-particule est réalisé à l'aide d'une fonction d'interpolation spline cubique [1].

Comme cas test, on a choisi un cas où le rapport viscosité/coefficient de raideur élastique est très petit, ce qui conduit à des gradients très raides au niveau de l'interface. La méthode ALE choisie est basée sur un traitement centré des termes de convection. Ceci garantit la précision mais peut conduire à des problèmes d'instabilités non-linéaires. C'est ce qu'on observe près des bords et au voisinage de l'interface sur la Fig. 1. A contrario, les méthodes particulaires sont connues pour offrir un bon compromis entre précision et robustesse. Ceci se vérifie sur notre modèle. L'interface est capturée correctement, y compris lorsque les gradients y deviennent grands. Des raffinements successifs mettent en évidence la convergence de la méthode, dans le cas étudié, à un ordre compris entre 1 et 2.

Conclusion. – Les méthodes particulaires offrent une approche élégante et originale des problèmes d'intéraction fluide–structure. L'intérêt potentiel de ces méthodes réside dans la prise en compte de grandes déformations et de changements de topologie. Cette étude très préliminaire doit se poursuivre dans cette direction.

1. Introduction

Fluid–structure interaction problems are challenging both from the mathematical and numerical point of view. They are by essence nonlinear and questions of existence, uniqueness and regularity are still essentially open [4]. On the numerical side, the need to couple models of a different nature (fluid and solid) that are in general treated by different methods is evidently a source of difficulties. The moving interface between the models is one additional difficulty. The traditional way to tackle these problems is to use a Lagrangian formulation of the solid, with a Lagrangian tracking of the fluid. The fact that the fluid can be described along trajectories corresponding to velocities that only match solid deformations in the direction normal to the interface (the so-called *Arbitrary Lagrange Euler* methods) adds some flexibility for small deformation. This however does not allow one to track large deformations and change of topologies of the interface. Another class of potentially more flexible methods is based on the concept of immersed boundaries. The general idea is to get rid of any complex gridding requirements by treating boundaries as

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force terms in equations satisfied in a computational domain overlapping with boundaries. Clearly, when boundaries are moving, this approach is particularly appealing.

We consider in this Note a simple 1D model problem, where the fluid is described by the Burger's equation, and the solid by linear elasticity. More precisely the fluid velocity u and the solid displacement d satisfy the system

$$\frac{\partial u}{\partial t} + \frac{3}{2}u\frac{\partial u}{\partial x} - \nu\frac{\partial^2 u}{\partial x^2} = 0 \quad \text{for } x \in \big[0, \gamma(t)\big], \qquad \frac{\partial^2 d}{\partial t^2} - \mu\frac{\partial^2 d}{\partial \xi^2} = 0 \quad \text{for } \xi \in [1,2]. \tag{1}$$

In the above system ξ is the Lagrangian variable for the solid, ν and μ are respectively the fluid viscosity and solid elasticity coefficient. We assume that at the inital time the fluid and solid respectively occupy the intervals $[0,1]$ and $[1,2]$. The interface at time t is located at $\gamma(t) = 1 + d(1,t)$. For simplicity we assume fluid and solid at rest at their outer limits (left side of the fluid and right side of the solid):

$$u(0,t) = d(2,t) = \frac{\partial d}{\partial t}(2,t) = 0 \quad \text{for all } t > 0.$$

The interface condition translating forces equilibrium between fluid and solid can be expressed as

$$\nu\frac{\partial u}{\partial x}\big(\gamma(t),t\big) = \mu\frac{\partial d}{\partial \xi}(1,t). \tag{2}$$

These equations have of course to be supplemented by an initial state (velocity in the fluid and displacement and velocity in the solid). We recall that the coefficient $3/2$ in Eq. (1) is required to obtain energy equations and make this problem well-posed. It is merely an artefact of the 1D model and disappears for 2D or 3D models where the Burgers equation has to be replaced by the standard Navier–Stokes equation. We refer to [3,4] for a discussion and mathematical analysis of the model.

2. The particle model

The starting idea is an immersed boundary approach of the system (1), that is a description of both flows and fluids by a unique conservation equation without explicitely tracking the interface.

We first remark that linear elasticity can be written as a Burger's equation. If we relate Eulerian and Lagrangian coordinates in the solid by

$$x(\xi,t) = \xi + d(\xi,t) \tag{3}$$

then we have $u(x(\xi,t),t) = \partial d/\partial t(\xi,t)$ and linear elasticity gives

$$\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right)\big(x(\xi,t),t\big) = \mu\frac{\partial^2 d}{\partial \xi^2}(\xi,t). \tag{4}$$

Next we observe that if u and d are smooth functions respectively in the fluid and solid domains and statisfy the interface condition (2), then the function

$$\nu\frac{\partial u}{\partial x}\chi^F + \mu\frac{\partial d}{\partial \xi}\chi^S$$

is a continuous function of $x \in [0,2]$. In the above expression χ^F and χ^S respectively denote the characteristic function of the (moving with time) fluid and solid domains.

S1631-073X(02)02571-2/FLA AID:2571 Vol.335(0)
CRAgroup 2002/08/27 Prn:17/10/2002; 9:54 F:crass12571.tex

As a result, we can rewrite the system (1), (2) as a single equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\frac{1}{2}\chi^F u\frac{\partial u}{\partial x} + D \quad \text{for } x \in [0,2], \tag{5}$$

where

$$D = \left(\chi^F + \frac{\partial x}{\partial \xi}\chi^S\right)\frac{\partial}{\partial x}\left(\nu\frac{\partial u}{\partial x}\chi^F + \mu\frac{\partial d}{\partial \xi}\chi^S\right). \tag{6}$$

In the above equation $\partial x/\partial\xi$ denotes the Jacobian of the flow map (3) linking Lagrangian and Eulerian variables. Let us now describe how to build a particle approximation on this equation. We discretize velocities as $u(x,t) \simeq \sum_p u_p v_p \delta(x - x_p)$ where x_p are particle locations, u_p are local values of u and v_p are local volumes of particles in the physical space. We will write $p \in F, S$ to distinguish between particles in the fluid and in the solid (something which is entirely determined by the initial locations of particles). Then particle volumes are determined by

$$v_p = h \quad \text{if } p \in S; \qquad \frac{\mathrm{d}v_p}{\mathrm{d}t} = U_p'(x_p,t)v_p \quad \text{if } p \in F, \tag{7}$$

where h is the mesh size and U_p' denotes an approximation of the velocity derivative at x_p (see below how in practice we compute this quantity). The convection part of (5) is dealt with by pushing particles and modifying the weights of the fluid particles to account for the term $1/2\chi^F u\partial u/\partial x$. The delicate part, which is critically linked to the interface condition (2), is in the treatment of D. We use a Particle Strength Exchange algorithm [2] based on integral approximations for operators of the form $\mathrm{div}(a\nabla)$ in the variables x and ξ. We obtain directly from (6) $D(x_p) \simeq D_p$ where

$$D_p = \begin{cases} \sum_q \nu(u_q - u_p)\Lambda_h(x_p - x_q)v_q\frac{1+\chi_q^F}{2} + \frac{\mu}{2}\sum_{q\in S}(d_q - d_p)\Lambda_h(\xi_p - \xi_q)\frac{h^2}{v_p} & \text{if } p \in F, \\ \sum_q \mu(d_q - d_p)\Lambda_h(\xi_p - \xi_q)h\frac{1+\chi_q^S}{2} + \frac{\nu}{2}\sum_{q\in F}(u_q - u_p)\Lambda_h(x_p - x_q)\frac{v_p v_q}{h} & \text{if } p \in S, \end{cases} \tag{8}$$

where $\chi_p^{F,S}$ stands for $\chi^{F,S}(x_p)$ and $\Lambda_h(x) = h^{-3}\Lambda(x/h)$ where Λ is a positive symmetric kernel normalized to satisfy the moment condition $\int x^2\Lambda(x)\,\mathrm{d}x = 2$. Note that the terms h/v_p and v_p/h in the right-hand sides above come from the evaluation of $\partial x/\partial\xi$ and $\partial\xi/\partial x$ at the particle x_p. Displacements are related to particle locations by $d_p(t) = x_p(t) - x_p(0)$.

The particle method is finally defined by the following system of ODE's

$$\dot{x}_p = U_p, \quad \dot{v}_p = U_p' v_p, \quad \dot{u}_p = -\frac{U_p' u_p \chi_p^F}{2} + D_p. \tag{9}$$

A remarkable feature of this method is that it satisfies an energy inequality (to our knowledge ALE methods satisfy this property only up to discretization errors). At the continuous level the kinetic energy is defined as $E = \int_F u^2(x)\,\mathrm{d}x + \int_S u^2(x)\,\mathrm{d}\xi$. At the discrete particle level we thus consider

$$E^h = \sum_p u_p^2\left(v_p\chi_p^F + h\chi_p^S\right).$$

Multiplying the last equation in (9) by u_p, summing over p and using (7) yields

$$\dot{E}^h = 2\sum_{p\in F} v_p u_p \dot{u}_p + \sum_{p\in F}\dot{v}_p u_p^2 + 2\sum_{p\in S} h u_p\dot{u}_p = 2\sum_{p\in F} v_p D_p u_p + 2\sum_{p\in S} h D_p u_p.$$

We next multiply (8) by u_p. Rewriting $2u_p = (u_p + u_q) + (u_p - u_q)$, using the symmetry of Λ and recalling that $u_p = \dot{d}_p$, we obtain after some calculations

$$\dot{E}^h = -\nu \sum_{p\in F,\ q} |u_q - u_p|^2 \Lambda(x_p - x_q) v_p v_q \frac{\mu}{2} \sum_{p\in S,\ q} \frac{\mathrm{d}}{\mathrm{d}t} |d_p - d_q|^2 \Lambda(\xi_p - \xi_q) h^2. \tag{10}$$

The last term in the right-hand side above can be interpreted as a the time derivative of the solid elastic energy and Eq. (10) expresses the dissipation of the total energy of the fluid–structure system.

3. Implementation and numerical results

The results we now show concern a Particle-In-Cell implementation of the method. This means that velocities U_p and their derivatives U'_p needed to push particles and update their volumes are computed after interpolating particle velocities u_p on a regular fixed grid. This interpolation is done by a third order cubic spline (see [1]). The test case we have considered consists of a cosine initial velocity, initally pulling the solid to the left, with maximum amplitude 0.2. The interface is initially located at $x = 0.5$. Fig. 1 shows the velocity profiles for a sequence of times obtained by our method compared to an ALE method, which in the present case is very easy to implement. The viscosity was 10^{-3} and the elastic coefficient 1. Both method used 100 discretization points, a time-step of 0.005 and a fourth order Runge–Kutta time-stepping. For early times both methods give very similar results. The ALE method we used was based on centered finite-

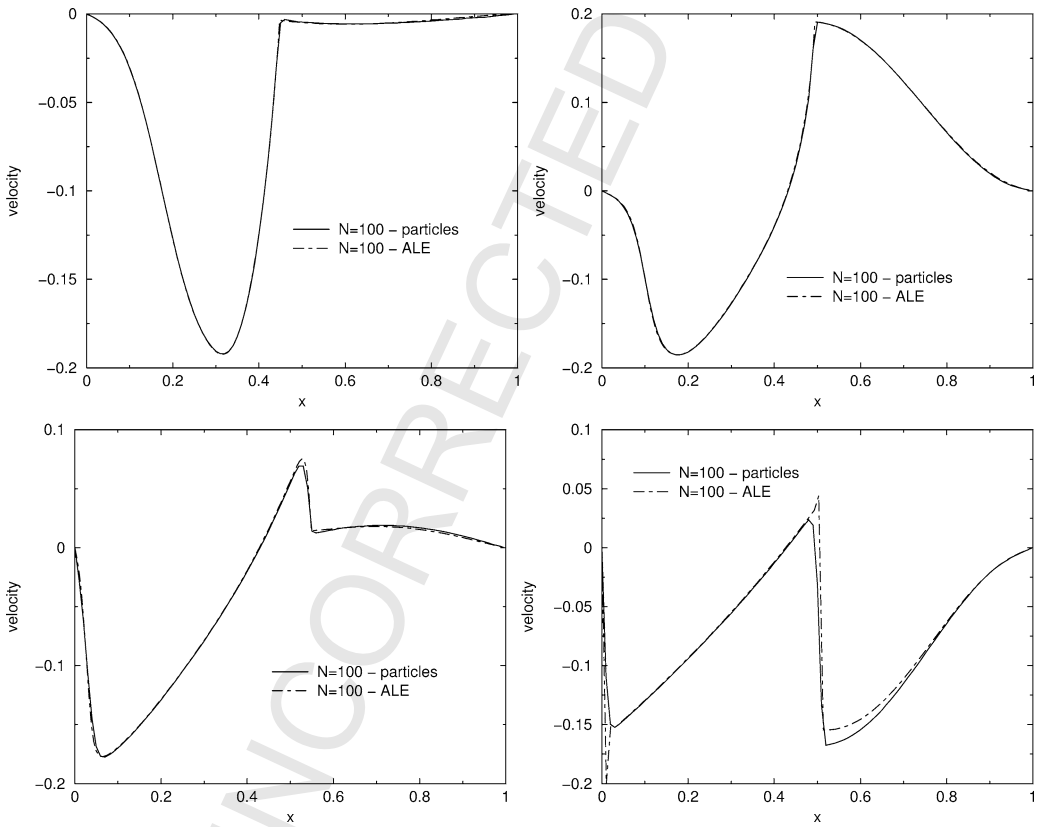

Figure 1. – Results of Particle and ALE methods for times 5, 10, 15 and 20 (left to right, top to bottom).

S1631-073X(02)02571-2/FLA AID:2571 Vol.335(0)
CRAgroup 2002/08/27 Prn:17/10/2002; 9:54 F:crass12571.tex

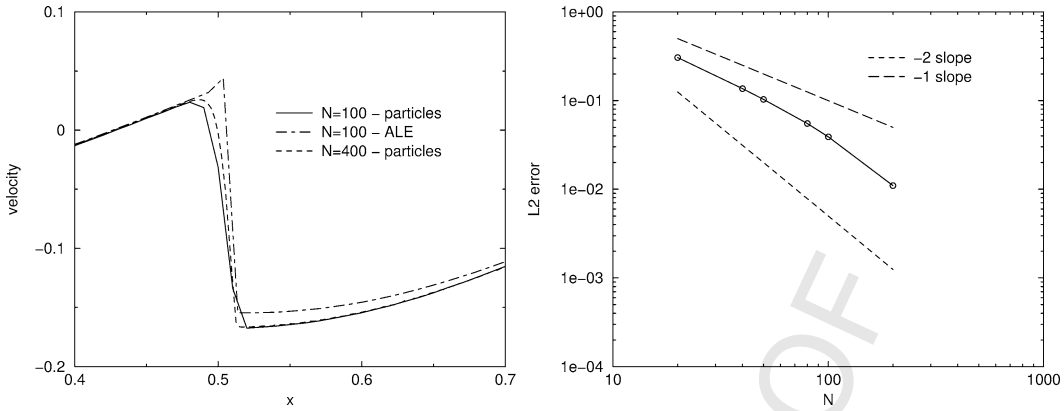

Figure 2. – Left picture: zoom around the interface at time $t = 20$. Right picture: Refinement study for the particle method at time $t = 20$. Velocity error in the L^2-norm as a function of the number of particles.

differences, which is the simplest choice for accuracy but at the expense of typical unphysical overshoots when the gradients get steep (for higher resolutions these instabilities disappear and the ALE solution gets closer to the particle solution). By contrast, the particle method is more robust. Finally, Fig. 2 is a refinement study on the same case at time 20. The left picture is a close up of the particle and ALE results around the interface. This demonstrates that the particle method, unlike the ALE method is almost converged with 100 points. On the right picture the convergence of the particle method is measured by comparison with a well converged calculation (with 400 points). This plot indicates a superlinear convergence rate.

4. Conclusion

We have shown how particle methods can handle in an elegant way fluid–structure interaction problems. Although the goal is to allow large displacements and change of topologies, a simple 1D model with linear elasticity illustrates the robustness and accuracy of the method. Future works will be to extend the idea to multi-dimensional problems and non-linear elasticity.

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