Algorithms for the splitting of formal series; applications to alien differential calculus

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Abstract

We present algorithms which involve both the splitting of formal series solutions to linear ordinary differential equations with polynomial coefficients into a finite sum of subseries which themselves will be solutions of linear ODEs, and the simplification of the recurrence relations satisfied by their coefficients.When coping with series that are solutions of a given differential equation at an irregular – singular point of rank $k \geq 2$, it enables us to reduce the computations to series solutions of an ODE with an irregularity of rank one. In particular, we are able to conduct effective calculations with Écalle's alien derivations for these series. We apply our techniques to some "accelerating functions" of Écalle.

Introduction

The question of decomposing ("splitting") a given formal series which is a solution of some linear differential equation into a finite sum of series which are also solutions of some differential equation is a classical one and is encountered e. g. in formal calculations for dynamical systems or in the quest for obtaining effective estimates for some generating functions. However, even if the theoretical framework for this simple question is well known (Ore extensions, holonomic functions, effective D-modules, etc) the explicit calculations that one might wish to conduct with computer algebra systems are in practice very explosive, even for simple examples. In this paper we describe a pragmatic approach to this problem: we introduce algorithms that imply new procedures, which rely and articulate with existing ones. The calculations have been performed in Maple. These procedures have been first tested on "academic examples", for which they improved significantly upon the existing techniques. Then, we have applied them to series that appear in the complete solution of some linear differential equations with polynomial coefficients which present a particular mathematical interest.

In the neighborhood of an irregular singular point, say at $z \sim \infty$, a linear differential equation with analytic coefficients has a basis of solutions of the form:

$$y(z) = e^{Q(u)} u^{\alpha} f(u) \tag{1}$$

In that expression:

- $u = z^{\frac{1}{\nu}}$ (ν is an integer, $\alpha \in \mathbb{C}$)
- Q is a polynomial in u with a vanishing constant coefficient
- f is a formal series in u^{-1}, with possibly logarithms, more precisely: $f(u) \in \mathbb{C}[[u^{-1}]][\log u]$

The (optimal) ramified variable u, the polynomial Q and the exponent α are formal invariants (in the sense that they depend on the class of the equation modulo a transformation with formal coefficients) of the equation, for which algebraic algorithms have been designed and implemented during the last two decades ([5], [14]). The formal series f, whose coefficients are computable through recurrence relations are generically divergent but the growth of these coefficients is no worse that some Gevrey order ([16]) and resummation techniques are at hand to get approximate solutions of the given equation, with errors that are exponentially small. When the Newton polygon ([16, 15]) of such an equation has just one slope, equal to one, the series can be resummed by applying the Borel–Laplace transforms; when it has one slope equal to k, we have to apply the Borel transform to a function of the new variable z^k. In the case of a multiplicity of slopes the series will be multisummable ([11, 15, 10]) and they can be treated by applying the same mechanism, but with a –finite – succession of stages (there are several *critical times*, in Écalle's language), using convolution operators that involve some special functions : Écalle's "accelerating functions".

In the present paper, we present calculations for situations of a single slope, equal to $k \geq 2$. We are able to compute effectively the action of alien derivations on these series, generalizing the ones made in [8] for equation of rank one (and single level). In rank ≥ 2, such formal–numerical calculations are completely new and in fact, let alone numerical computations such as ours, almost no examples of calculations on resurgent functions of "level k" (meaning divergent series, which are resurgent as functions of some z^k), or worse with a multiplicity of levels, can be found in the litterature (see however [10], [2]). Note, that although we cope with relatively simple examples, results of this sort are not anecdotic: even in the case of linear ODEs with polynomial coefficients, it is only in last 25 years, thanks to Ramis' works and Écalle's theory of acceleration of resurgent functions that the asymptotics of such solutions have completely been elucidated. The great majority of so called classical functions fall within this class. The paper is organized as follows : in section 1, we introduce very briefly the context that is relevant for the examples we work on : irregular–singular points of linear ODEs, Stokes phenomenon, Ramis' theorem in differential Galois theory, Écalle's alien derivations and accelerating functions. The algorithms for the splitting of series are introduced and described in section 3 and, in section 4, we explicit the calculations we have performed for the accelerating functions $C_{1/3}$ and $C_{1/4}$.

1 Stokes phenomenon, differential Galois groups, accelerating functions

1.1 Stokes phenomenon; alien derivations

The formal series contained in a basis of formal solutions such as in(1) are generically divergent, but they are asymptotic, in some sectors, to some analytic solutions. The comparision of these analytic resummations of the series on various sectors gives rise, for a finite number of critical directions, to the Stokes phenomenon ([16]).
An important object attached to such an equation is its (local) Galois differential group, for which we refer to [15]. J. -P. Ramis proved that it is characterized by the "exponential torus", the "formal monodromy" (which are formal invariants that we don't describe here, referring to [11], [16]), and the Stokes matrices. Now, the "important part" in the Galois differential group comes from the Stokes matrices, which are difficult to calculate and involve transcendental constants. These matrices are unipotent, their logarithms are nilpotent matrices that correspond to operators that are derivations acting on the space of series we are working on. These derivations, in turn, can be decomposed into elementary components : the so called alien derivations of Écalle, which can be defined independantly of Galois considerations. Alien derivations are operators, introduced by Jean Écalle ([6]), acting on some spaces of holomorphic functions; they are in fact derivations relatively to a convolution product. A series f solution to an ODE, at an irregular singular point for which the Newton polygon has a single slope, equal to one, will have a convergent Borel transform $\varphi = \hat{B}(f)$ and this germ φ of analytic function can in fact be continued along any broken line from the origin, and we define resurgent functions and alien operators in this context. This property of analytic continuation along any broken line γ starting from the origin of $\mathbb{C}$, going around a finite number of singularities possibly met on γ leads to the general definition of resurgent functions. Let us introduce them in the most simple setting, which is already enough to treat many natural but non trivial examples. The whole point is that our procedures for the splitting of series enable us to decompose a divergent series into a finite sum of series which, as functions of some new variable z^k, are resurgent in the sense explained below.

Definition 1.1. An analytic germ, continuable as above, is called a simple resurgent function if the behaviour of its analytic continuation at any singularity ω is of the form

$$\frac{1}{2i\pi}log(\zeta-\omega)s(\zeta-\omega)+r(\zeta-\omega)$$

where r, s are regular germs.
We denote by $\mathcal{R}$ the space of simple resurgent functions.

Let ω a non zero complex number and d_θ the half line through ω. We are going to define an operator:

$$\Delta_\omega : \mathcal{R} \longrightarrow \mathcal{R}$$

For series belonging to $\mathcal{R}$ and if ω is the only singularity on d_θ, Δ_ω amounts to extracting the singular part s at the singularity ω, up to a factor $2i\pi$ (and thus acts as 0, if there is no singularity at the point ω):

$$\Delta_\omega\, f(\zeta) = s(\zeta)$$

which is also equal to the *variation* of $2i\pi f$, namely the difference between the continuation by the right and by the left of $2i\pi f$ at the singular point ω. For the general case, the definition of Δ_ω involves an average of the singular parts obtained at ω relatively to the way we avoid the singularities between the origin and ω, when performing the analytic continuation.

Alien derivations Δ_ω thus constitute a family of linear operators, indexed by points ω of the complex plane but they are also derivations of the convolutive algebras of resurgent functions and moreover they satisfy a simple commutation relation with the ordinary derivation ∂, namely: $[\partial, \Delta_\omega] = \omega\, \Delta_\omega$. For all this we refer to [8] and of course to the original papers of Ecalle. We denote by the same symbol the operators that are the pullbacks in the space of formal series $\mathbb{C}[[z^{-1}]]$ or of more general algebras of ramified formal series, which are indispensable in applications, by (the extended)– inverse Borel transform.

For solutions of linear ODEs, the generic case involves only, for each critical time, one singularity as in Definition 1.1 on each singular direction and this will be the situation for the examples described in detail below.

Now, these derivations and also the can be seen as acting on a formal basis of solutions to an equation such as the one above. It acts in the following way, where we denote by $Y(z) = \sum_{i=1}^{n} u_i \exp(\lambda_i z) f_i(z)$ the general solution of the equation (supposed non resonant. see [8] or [6]) :

$$\Delta_{\lambda_i - \lambda_j} Y = A_{(\lambda_i - \lambda_j)} u_i \frac{\partial}{\partial u_j} Y.$$

Thus there is only a finite number of Δ_ω that can act non–trivially on Y and that action is expressed as the action of an *ordinary* (meaning non alien!) differential operator in the variables u_i, on the formal integral Y, constituting a simple example of resurgence equation called by Écalle a *bridge equation*, as it throws a bridge between alien and ordinary differential calculus. We thus get a Lie algebra of Galois derivations acting on the vector space of solutions. In fact, they "belong" to the Lie algebra of the differential Galois group of the equation (later on we shall call it the *Lie–Galois algebra*, to be short), and constitute the most important – and most difficult to determine – part of that algebra.

1.2 Accelerating functions

Écalle's accelerating functions can be defined by an integral formula:

$$C_\alpha(t) = \int_\gamma \exp\left(u - t u^{\frac{1}{\alpha}}\right) \quad \text{where } \gamma \text{ is a Hankel contour.}$$

The accelerating functions are used to define convolution operators to sum divergent series with several critical times; they come together with decelerating functions and it was observed by Anne Duval 15 years ago (see [12], in which the results mentionned in the present section can be found. see also [7]) that these (decelerating) functions fall within the class of so-called Faxen integrals and are particular cases of G-functions of Meijer. As such, they can be written in explicit, though complicated, expansions involving the Gamma function. Each C_α is an entire function, with an expansion at the origin :

$$C_\alpha(t) = 2i \sum_{n\geq 0} \sin \frac{n\pi}{\beta} \frac{\Gamma(1+n/\alpha)}{\Gamma(1+n)} t^n \quad \text{with } 1/\alpha + 1/\beta = 1.$$

An accelerating function with a rational α satisfies a simple linear differential equation with polynomial coefficients, namely $C_{\frac{p}{q}}$ is a solution of $A = 0$, where A is the following operator:

$$D^q - (-1)^{q-p} \prod_{j=1}^{p} (\frac{p}{q} tD + j) \quad \text{where } D = \tfrac{d}{dt}.$$

In fact the operator A admits a simple order one left factor, which entails that $C_{\frac{p}{q}}$ is in the kernel of the following operator, of order $q-1$:

$$q \prod_{j=1}^{q-1} (\delta - j) - (-1)^{q-p} p t^q \prod_{j=1}^{p-1} (\frac{p}{q}\delta + j) \quad \text{where } \delta \text{ is the Euler operator } t\tfrac{d}{dt}.$$

Such an equation has a single slope at ∞: the formal series solutions at $z \sim \infty$ will be "k –summable", and resurgent with respect to some variable z^k.

The family of accelerating functions $(C_{p/q})$ constitutes an interesting object of study *per se*, and it was already remarked in [12] that it deserves a thorough study. We show below how our algorithms for the splitting of series make possible calculations that pave the way for the determination of the differential Galois groups of the equations above, of low degree.

2 Simplification tools for recurrence equations

2.1 Ore polynomials and series equality

Let $\mathbf{R}$ be a ring and $\sigma : \mathbf{R} \to \mathbf{R}$ be an injective endomorphism of $\mathbf{R}$.
Let δ be a pseudo-derivation w.r.t. σ, that is a map from $\mathbf{R}$ to $\mathbf{R}$ satisfying:

$$\delta(a+b) = \delta a + \delta b, \delta(ab) = \sigma(a)\delta b + \delta a b \quad \text{for any } a, b \in \mathbf{R}.$$

Definition 2.1. The left skew polynomial ring given by σ and δ is the ring $(\mathbf{R}[x], +, .)$ of polynomials in x over $\mathbf{R}$ with the usual polynomial addition, and multiplication given by:

$$xa = \sigma(a)x + \delta a \quad \text{for any } a \in \mathbf{R}.$$

This ring is denoted $\mathbf{R}[x; \sigma, \delta]$ and its elements are called skew polynomials or Ore polynomials.

We refer to [4] for this definition and the first properties of this ring, and to [3] for the arithmetic and algorithmic point of view, when $\mathbf{R}$ is a field (in particular, greatest common right divisor, extended right Euclidean algorithm).

In the following, we will deal with $\mathbf{R} = \mathbb{C}[n]$, τ the automorphism of $\mathbf{R}$ over $\mathbb{C}$ that takes n to $n+1$, and $\mathbf{R}[x;\tau,0]$ the ring of linear ordinary recurrence operators (with polynomial coefficients). In this ring, the multiplication is given by:

$$xa = \tau(a)x, \quad \text{for any } a \in \mathbf{R}.$$

We are interested in formal series $\sum_{n\geq 0} a_n x^n$, such that the coefficients $(a_n)_{n\geq 0}$ are defined by a finite difference equation with polynomial coefficients. That means that the coefficients of the series are defined by:

- $a_0, a_1, \ldots, a_{m-1}$, the first m terms (also called initial conditions), $a_i \in \mathbb{C}$;
- and a recurrence equation

$$P_0(n)a_n + \cdots + P_r(n)a_{n+r} = 0, \forall n \geq m - r,$$

with $P_0, \ldots, P_r \in \mathbb{C}[n]$ and $P_r(\lambda) \neq 0, \forall \lambda \geq m - r \qquad (1).$

An equivalent manner of writing the condition (1) is to define

$$\begin{aligned} \overline{\lambda} &= \max\{\lambda \in \mathbb{N}, P_r(\lambda) = 0\} \\ &= -1 \quad \text{if } P_r(\lambda) \neq 0, \forall \lambda \in \mathbb{N}, \end{aligned}$$

and to suppose that $m > \overline{\lambda} + r$.
In the following, this series will be represented by the initial conditions $a_0, \ldots, a_{\overline{\lambda}+r}$ and the skew polynomial:

$$x^{\overline{\lambda}+1}(P_0 + P_1x + \cdots + P_rx^r) = (\tau^{\overline{\lambda}+1}(P_0) + \cdots + \tau^{\overline{\lambda}+1}(P_r)x^r)x^{\overline{\lambda}+1}.$$

Our objective is to *simplify* the skew polynomial defining the previous series: reduce its degree, simplify the coefficients P_i (in particular reduce their degree in n).

2.2 redundant initial conditions

Proposition 2.2. *Let a be the series defined by $a_0, \ldots, a_{m-1}$ and the skew polynomial $P = (P_0 + P_1x + \cdots + P_rx^r)x^{m-r}$, with $P_r(\lambda) \neq 0, \forall \lambda \in \mathbb{N}$.*
Suppose that $m > r$, $P_r(-1) \neq 0$ and $P_0(-1)a_{m-r-1} + P_1(-1)a_{m-r} + \cdots + P_r(-1)a_{m-1} = 0$. Let b be the series defined by $b_0 = a_0, \ldots, b_{m-2} = a_{m-2}, Q = \tau^{-1}(P_0) + \tau^{-1}(P_1)x + \cdots + \tau^{-1}(P_r)x^r)x^{m-r-1}$. Then $a = b$.

2.3 right factor

Proposition 2.3. *Let a be the series defined by $a_0, \ldots, a_{m-1}$ and a skew polynomial P, of degree m, with $P_m(\lambda) \neq 0, \forall \lambda \in \mathbb{N}$.*
Let b be the series defined by $b_0 = a_0, \ldots, b_{r-1} = a_{r-1}$ and a skew polynomial Q, of degree r.
Suppose that Q is a right factor of P, and that $Q(a) = 0, \forall j \leq m - r - 1$.
Then $a = b$.

Proof. $r < m$; $P = RQ$ with $Q_r(\lambda) \neq 0, \forall \lambda \in \mathbb{N}$ and $\deg(R) = m - r$.

$$Q(a)(j) = Q(b)(j), \forall j \leq m - r - 1,$$
$$R(Q(a)) = RQ(a) = P(a) = 0,$$
$$\text{and} \qquad R(Q(b)) = R(0) = 0,$$
$$\text{then} \qquad Q(a) = Q(b).$$

So $a_0 = b_0, \ldots, a_{r-1} = b_{r-1}$, and $Q(a) = Q(b) = 0$, then $a = b$. □

2.4 guessing a right factor

Let a be the series defined by $a_0, \ldots, a_{m-1}$ and the skew polynomial $P = P_0 + P_1 x + \cdots + P_m x^m$, with $P_m(\lambda) \neq 0, \forall \lambda \in \mathbb{N}$.
Knowing $a_0, a_1, \ldots, a_{m-1}, a_m, \ldots a_{N-1}$, we determine another polynomial $\overline{Q}$ of degree $\overline{r}$, such that $\overline{Q}(a)(n) = 0, 0 \leq n \leq N - \overline{r} - 1$. We will explain in the next section how to do that.
We consider the polynomials P and $\overline{Q}$ as polynomials with rational coefficients (in $\mathbb{C}(n)[x; \tau, 0]$) and compute a greatest common right divisor $\tilde{Q}$. Suppose that $\tilde{Q}$ is not trivial, and of degree r.
We denote by $\tilde{c}_1$ and $\tilde{c}_2$ two polynomials (in $\mathbb{C}(n)[x; \tau, 0]$) such that $\tilde{c}_1 P + \tilde{c}_2 \overline{Q} = \tilde{Q}$. We put v the lcm of the denominators of the coefficients of $\tilde{c}_1, \tilde{c}_2$ and $\tilde{Q}$. After multiplication by v, we obtain polynomials c_1, c_2 and $\hat{Q}$ in $\mathbb{C}[n][x; \tau, 0]$ such that $c_1 P + c_2 \overline{Q} = \hat{Q}$. So $\hat{Q}(a)(j) = 0, \forall j \leq N - \overline{r} - 1$.
We compute the $\hat{w}$ the gcd of the coefficients of $\hat{Q}$ and divide $\hat{Q}$ by $\hat{w}$ to obtain $Q = Q_0 + Q_1 x + \cdots + Q_r x^r$. Define

$$\begin{aligned} \overline{\lambda} &= \max\{\lambda \in \mathbb{N}, \hat{w} Q_r(\lambda) = 0\} \\ &= -1 \quad \text{if } \hat{w} Q_r(\lambda) \neq 0, \forall \lambda \in \mathbb{N}, \end{aligned}$$

Consider now the series b defined by $b_0 = a_0, \ldots, b_{r+\overline{\lambda}} = a_{r+\overline{\lambda}}$ and the skew polynomial $x^{\overline{\lambda}+1} Q$.
Suppose that $N \geq m - r + \overline{r} + \overline{\lambda} + 1$. Then $a = b$.

Proof. $Q(a)(j) = 0, \forall \overline{\lambda} < j \leq N - \overline{r} - 1$, so $x^{\overline{\lambda}+1}Q(a)(j) = 0, \forall j \leq N - \overline{r} - \overline{\lambda} - 2$.
Moreover $x^{\overline{\lambda}+1}Q$ divides (right, in $\mathbb{C}(n)[x;\tau,0]$) $x^{\overline{\lambda}+1}P$, and the coefficients of the quotient have for denominator some shifted of $\tau^{\overline{\lambda}+1}(Q_r)$, so $x^{\overline{\lambda}}Q$ is a right factor of $x^{\overline{\lambda}+1}wP$, for a polynomial $w \in \mathbb{C}[n]$ without root in $\mathbb{N}$. We can apply Proposition 2. □
Conclusion: we can apply the first subsection "redundant conditions" to the new polynomial $x^{\overline{\lambda}+1}Q$ in order to reduce its degree.

2.5 guessing a new polynomial Q

At the present time, it is done by two ways:

- solving *by hand* a linear system, for fixed values of r, the degree of $\overline{Q}$ (in x) and of max_d, the maximum of the degrees of the coefficients of $\overline{Q}$ (in n). This is done by the function *diminue_syst* for all values $r \leq \deg(P)$ and $max_d \leq \max_i \deg(P_i)$;

- using the *gfun* package [17] and the function *listtorec*; this is done by the function *diminue_gfun* ; the number of terms N used can be given by the user as parameter, by default it is assigned to a value depending on the degree of P (in x) and the max of the degrees of the coefficients of P in n.

2.6 examples

1. consider the series $\sum_{n\geq 0} n!(x^{2n} + x^{2n+1})$. As a solution of the differential linear homogeneous equation

   $$(2 + 6x + 3x^2 + x^3)y(x) + (3x^2 + 8x^3 + 3x^4 - 4x - 2)y'(x) + (x^3 + 2x^4 + x^5)y''(x) = 0,$$

   it is defined by DESIR by the 4 first terms $a_0 = a_1 = a_2 = a_3 = 1$ and the skew polynomial

   $$P = (4+j)^2 - 15 - 6j + (-21 - 6j + 2(4+j)^2)x + (-2 - 2j + (4+j)^2)x^2 + (-10 - 4j)x^3 + (-8 - 2j)x^4.$$

   With *diminue_syst*, we find $\overline{Q} = j + j^2 - 2x + (2 - 2j)x^2 = \tilde{Q} = \hat{Q} = Q$ and $\hat{w} = 1, \overline{\lambda} = 1$.
   That means that the series can be defined by the first terms $[1, 1, 1, 1]$ and x^2Q.
   With *diminue_gfun* and $N = 20$, we obtain an other polynomial $\overline{Q}$, but which leads to the same Q as before.
   On this example, we didn't reduce the degree of the skew polynomial, but we simplify the coefficients.

2. the Ramis-Sibuya equation is the following differential linear homogeneous equation of order 3:

   $$eq := (3x^3 - 10x^2 - 2x - 4)x^6 y'''(x) + (12x^5 - 47x^4 - 16x^3 - 50x^2 - 8x - 8)x^3 y''(x)$$

   $$+2(3x^6 - 14x^5 - 12x^4 - 5x^3 - 14x^2 - 6x - 4)xy'(x) + (12x^4 - 14x^3 + 60x^2 + 12x + 8)y(x) = 0.$$

This equation admits a series solution

$$\hat{f}(t) = t(1 + 2t^2 - 7t^3 + 24t^4 + \dots) = \hat{g}(t) + \hat{g}(t^2),$$

where $\hat{g}$ denotes the Euler series $\hat{g}(t) = \sum_{n\geq 0}(-1)^n n! t^n$.

We first consider the series $\frac{\hat{f}}{t}$ (the *regular part* of the series in the internal data furnished by DESIR) defined by the first terms $[1, 0, 2, -7, 24, -118]$ and a skew polynomial of degree 6

$$P = P_0 + P_1 x + P_2 x^2 + P_3 x^3 + P_4 x^4 + P_5 x^5 + P_6 x^6$$

with coefficients P_i of maximum degree 3.

Such a skew polynomial is represented in MAPLE by $OrePoly(P_0, P_1, \dots, P_6)$.
We find a polynomial

$$\overline{Q} = OrePoly\left(\frac{7}{2} + \frac{25}{4}j + \frac{13}{4}j^2 + \frac{1}{2}j^3, \frac{5}{2} + \frac{9}{4}j + \frac{1}{2}j^2, 7 + \frac{11}{2}j + j^2, \frac{5}{2} + j\right),$$

which divides P: $\overline{Q} = \tilde{Q}$. $v = 4$, so that $\hat{Q} = 4\tilde{Q}$, $\hat{w} = 1$ and $Q = \hat{Q}$, $\overline{\lambda} = -1$.
The same series is defined by the first terms $[1, 0, 2]$ and the skew polynomial $Q = 14 + 25j + 13j^2 + 2j^3 + (5 + 2j)(j + 2)x + (28 + 22j + 4j^2)x^2 + (10 + 4j)x^3$.
On this example, we reduce the degree of the polynomial from 6 to 3.

The series $\hat{f}$ is defined by the first terms $[0, 1, 0, 2, -7, 24]$ and the skew polynomial $P_1 = \sum_{i=0}^{6} \tau^{-1}(P_i)x^i$.
We find a poynomial $\overline{Q}_1$ which is not a factor of P_1, but the right gcd of P_1 and $\overline{Q}_1$ is not trivial. Finally, we find that the series $\hat{f}$ can be defined by the first terms $[0, 1, 0, 2]$ and the polynomial

$$x(j\left(2j^2 + 7j + 5\right) + (2j + 3)(1 + j)x + (4j^2 + 14j + 10)x^2 + (4j + 6)x^3).$$

So we obtain a polynomial of degree 4.

Remark 2.4. on both these examples, the results are given by *diminue_syst*. The function *diminue_gfun* doesn't give any new polynomial $\overline{Q}$.

3 Splitting of a series

We consider a series $\hat{f}(x) = \sum_{n\geq 0} a_n x^n$, defined by its first terms $a_0, \dots, a_{m-1}$ and the skew polynomial $x^{m-r}(P_0 + P_1 x + \dots + P_r x^r)$, with $P_0 \neq 0$ and $P_r(\lambda) \neq 0, \forall \lambda \in \mathbb{N}, \lambda \geq m - r$. Our goal is to split the series.
Let $\alpha \in \mathbb{N}^*$. Let $q \in \mathbb{N}, 0 \leq q \leq \alpha - 1$ and $\hat{f}_q(x) = \sum_{j\geq 0} a_{j\alpha+q} x^j$, so that

$$\hat{f}(x) = \hat{f}_0(x^\alpha) + \dots + x^{\alpha-1}\hat{f}_{\alpha-1}(x^\alpha).$$

We want to build a recurrence equation with polynomial coefficients satisfied by each of the α subseries $\hat{f}_q$. An algorithmic process to do this has been first introduced by F. Naegele [13] and implemented in $A^{\#}$.

3.1 description of the method

The principle is the following: if we suppose that $\hat{f}$ is defined by a recurrence equation of the form

$$Q_0(j)a_j + Q_1(j)a_{j+\alpha} + \cdots + Q_{\overline{r}}(j)a_{j+\overline{r}\alpha} = 0, \qquad (*)$$

then each equation

$$Q_0(j\alpha+q)a_{j\alpha+q}j + Q_1(j\alpha+q)a_{(j+1)\alpha+q} + \cdots + Q_{\overline{r}}(j\alpha+q)a_{(j+\overline{r})\alpha+q} = 0$$

will define the qth subseries.
So it is now sufficient to describe an algorithm that permits to transform the recurrence equation

$$P_0(j)a_j + P_1(j)a_{j+1} + \cdots + P_r(j)a_{j+r} = 0,$$

into a recurrence equation of type $(*)$.
For this, we consider the system of $r(\alpha-1)+1$ linear equations:

$$P_0(j+i)a_{j+i} + P_1(j+i)a_{j+i+1} + \cdots + P_r(j+i)a_{j+i+r} = 0, \text{for } i = 0, \ldots, r(\alpha-1),$$

and we solve it considering as unknown the $r(\alpha-1)$ coefficients a_{j+k} such that α doesn't divide k, that is the terms:

$$a_{j+1}, \ldots, a_{j+\alpha-1}, a_{j+\alpha+1}, \ldots, a_{j+2\alpha-1}, \ldots, a_{j+(r-1)\alpha+1}, \ldots, a_{j+r\alpha-1}.$$

After Gaussian elimination, the system becoms $Ax=b$, with a matrix A in which there is at least one null row for which the corresponding element in b gives the wanted equation.

3.2 examples

1. we consider the first example of section 2.6, with $\alpha = 2$.
   If the series is defined by the four first terms $[1,1,1,1]$ and the skew polynomial P, the function *scindage* returns two series defined by skew polynomials of degree 4 and maximum degree of the coefficients 8. But after applying *diminue_syst*, we obtain that both subseries are defined by $a_0 = 1$ and the skew polynomial $-1-j+x$.
   If the same series is defined by the four first terms $[1,1,1,1]$ and the skew polynomial x^2Q, the function *scindage* returns two series defined by skew polynomials of degree 2 and maximum degree of the coefficients 2. Of course, after applying *diminue_syst*, we obtain the same reduced result as before.

2. we consider the second example of section 2.6, with $\alpha = 2$. More precisely, we split in two subseries the series defined by $[1,0,2]$ and the skew polynomial Q. We find that $\hat{f}_0$ is defined by $a_0 = 1$ and the skew polynomial $-2-6j-4j^2+x$, and that $\hat{f}_1$ is defined by $[0,-7]$ and the skew polynomial

   $$x\left(-16j^4 - 116j^3 - 302j^2 - 334j - 132 - (16j^3 - 96j^2 - 187j - 118)x + (7+4j)x^2\right).$$

3. we can split the series defined by $[0,1,0,2]$ and the skew polynomial Q_1 for big values of α. With $\alpha = 9$ or 10, the result is immediate. For $\alpha = 10$, the recurrence equation for the corresponding first subseries $\hat{f}_0$ is of degree 2 and maximum degree of the coefficient 20. Unfortunately, the function *diminue_syst* takes more time to give no better result.

3.3 From recurrence equation to differential equation

In this section, our goal is to compare our results to the results obtained by M. Barkatou and al. in [1]. In this paper, the authors proposed different methods to find, given a series $\hat{f} = \sum_{n\geq 0} a_n x^n$ solution of an ordinary linear differential equation and given an integer α, a differential equation satisfied by the subseries $\hat{f}^j(x) = \sum_{n\geq 0} a_{j+mr} x^{j+mr}$. So, we treat the same problem, modulo the fact that $\hat{f}^q(x) = x^q \hat{f}_q(x^q)$.
As an example, they take the equation of Ramis-Sibuya, with $\alpha = 2$, and obtain a differential equation of order 5 (and maximum degree of the coefficient 13), satisfied by the two subseries $\hat{f}^0$ and $\hat{f}^1$. If they apply similar methods to non homogeneous linear differential equations, they obtain two (homogeneous) linear differential equations of order 4 (and maximum degree of the coefficients 12) satisfied by each subseries $\hat{f}^0$ and $\hat{f}^1$.
With our method, knowing the initial conditions and a recurrence equation, we use the package *gfun* and the function *diffeqtorec* to perform the Mellin transform and obtain a non homogeneous differential equation satisfied by each subseries. Then an homogeneous differential equation is simply found by differentiating the non homogeneous one divided by the right member.
By this way, we find that the series $\hat{f}_1$ is solution of the following linear equation:

$$-4x^3 y'''(x) - 22x^2 y''(x) + (1 - 22x) y'(x) - 2y(x) = 0.$$

For the series $\hat{f}_0$, we find the following operator of degree 5:

$$16\,x^7 \frac{d}{dx^5} y\,(x) + \left(16\,x^5 + 212\,x^6\right) \frac{d}{dx^4} y\,(x) + \left(80\,x^4 + 762\,x^5\right) \frac{d}{dx^3} y\,(x)$$
$$+ \left(768\,x^4 + 59\,x^3 - 4\,x^2\right) y''\,(x) + \left(9\,x + 132\,x^3\right) y'\,(x) \; - 10\,y\,(x)\,.$$

4 Examples

4.1 Description of the calculations

We are now ready to apply the techniques described above to cope with series S that appear in a basis of formal solutions to some linear differential equation. The calculations that we perform for the reduced equations of order 2 and 3 respectively, of which $C_{1/3}$ and $C_{1/4}$ are solutions are summed up by the following stages:
– We split each series S in subseries.
– For each subseries, we generate a linear ODE with an irregular–regular point with one

critical time, and of rank one at the origin, of which the given subseries is solution.
– Then the Borel transform of this series is also solution of a linear ODE with polynomial coefficients, in *the Borel plane*, by Fourier duality ([8]).
– Finally, we are in a position to apply numerical procedures to estimate the analytic continuation of the subseries, as in [8].

The calculations are detailed and exhaustive for $C_{1/3}$; $C_{1/4}$ we have only room to show the initial stages, but the rest is straighforward, as we have now satisfactory tools for the splitting.

Remark 4.1. We express the equations in the variable x, with $x = 1/z$, in order to use the procedures (Desir, etc) which are designed for a singularity at the origin.

4.2 The accelerating $C_{1/3}$

This function is solution of the following equation:

$$acc3 := 12\,xy'(x) + 3\,x^2 y''(x) + 6\,y(x) - \frac{y(x)}{x^3} = 0$$

A basis of formal solutions is given by Desir under this form:

```
> desir(acc3,y(x),t,10,res);
```

$$[[x(t) = 1/3\,t^2, y(t) = e^{-2\,t^{-3}} \left(1 - \tfrac{5}{144}\,t^3 + \tfrac{385}{41472}\,t^6 - \tfrac{85085}{17915904}\,t^9 + O\left(t^{10}\right)\right) t^{-3/2}]]$$

The variable *ser* contains the regular part of one solution. The function *prepa_scindage* extracts the summability properties and transforms the series into a suitable data structure: it is described by the initial conditions and the polynomial coefficients of the recurrence equation.

```
> ser_prep:=prepa_scindage(ser);
```

the series is 3 summable

$$ser\_prep := [r, [1,0,0], [-1/12 + 1/12\,(3/2+r)^2, 0, 0, 1/3\,(1+1/3\,r)\sqrt{3}]]$$

Here is a readable form of the series:

$$1 - \tfrac{5}{48}\sqrt{3}t^3 + \tfrac{385}{1536}\,t^6 - \tfrac{85085}{221184}\sqrt{3}t^9 + O\left(t^{10}\right)$$

Here is the result of the splitting:

```
> S_S := scindage(3,ser_prep);
```

$$S\_S := [[r, [1], [15 + 108\,r + 108\,r^2, 48\sqrt{3} + 48\sqrt{3}r]], [r, [], [1]], [r, [], [1]]]$$

We try to simplify the recurrence equation for the non null series:

```
> s1_s := diminue_syst(S_S[1]);
```

$$s1\_s := [r, [1], [5 + 36\,r + 36\,r^2, 16\sqrt{3} + 16\sqrt{3}r]]$$

Here is the readable form of the splitted series:

$$y_1(t) = 1 - \tfrac{5}{48}\sqrt{3}t + \tfrac{385}{1536}\,t^2 - \tfrac{85085}{221184}\sqrt{3}t^3 + \tfrac{37182145}{14155776}\,t^4 + O\left(t^5\right)$$

Differential equation satisfied by the previous series multiplied by t:

$$equa := \left(5\,x - 16\sqrt{3}\right) f(x) + 16\sqrt{3}x \tfrac{d}{dx} f(x) + 36\,x^3 \tfrac{d^2}{dx^2} f(x)$$

```
> desir(equa,f(x),t,3);
```

$[[x(t) = t, f_1(t) = t\left(1 - \frac{5}{48}\sqrt{3}t + \frac{385}{1536}t^2 + O\left(t^3\right)\right)],$
$[x(t) = t, f_2(t) = e^{4/9\frac{\sqrt{3}}{t}}t\left(1 + \frac{5}{48}\sqrt{3}t + \frac{385}{1536}t^2 + O\left(t^3\right)\right)]]$

For the following, we denote by y_2 the regular part of f_2.
Differential equation satisfied by its Borel transform:

$$equaborel := \left(16\sqrt{3} + 72\zeta\right)\frac{d}{d\zeta}\phi(\zeta) + \left(16\sqrt{3}\zeta + 36\zeta^2\right)\frac{d^2}{d\zeta^2}\phi(\zeta) + 5\phi(\zeta) = 0$$

This equation presents two singularities:

$$sing := [0, -4/9\sqrt{3}]$$

A basis of formal solutions at the origin:

```
> frobenius(equaborel,phi(zeta),zeta=0,3);
```

$[\phi_1(\zeta) = 1 - \frac{5}{48}\sqrt{3}\zeta + \frac{385}{3072}\zeta^2 + O\left(\zeta^3\right),$
$\phi_2(\zeta) = \ln(\zeta)\left(1 - \frac{5}{48}\sqrt{3}\zeta + \frac{385}{3072}\zeta^2\right) - \frac{13}{24}\sqrt{3}\zeta + \frac{719}{1024}\zeta^2 + O\left(\zeta^3\right)]$

The series ϕ_1 is the Borel transform of $ty_1 = f_1$.
A basis of solutions in the neighborhood of the non null singularity:

```
> frobenius(equaborel,phi(zeta),zeta=sing[2],3);
```

$[\psi_1(\zeta) = 1 + \frac{5}{48}\sqrt{3}\zeta + \frac{385}{3072}\zeta^2 + O\left(\zeta^3\right),$
$\psi_2(\zeta) = \ln(\zeta)\left(1 + \frac{5}{48}\sqrt{3}\zeta + \frac{385}{3072}\zeta^2\right) + \frac{13}{24}\sqrt{3}\zeta + \frac{719}{1024}\zeta^2 + O\left(\zeta^3\right)]$

We verify that the series coefficient of $log(\zeta)$ in ψ_2 is $\hat{B}y_2$, the Borel transform of y_2.
By analytic continuation, we perform the connection:

$$\phi_1 = \lambda_1\psi_1 + \lambda_2\psi_2, \text{ with } \{\lambda_2 = -0.1591549446, \lambda_1 = 0.9241811708\}$$

The *alien derivative* of $\phi_1 = \hat{B}f_1$ at the singularity $-4/9\sqrt{3}$ is $2i\pi\lambda_2\hat{B}y_2$.

```
> evalf(2*I*Pi*lambda2);
```

$$-1.000000010\,i$$

This result is coherent with our knowledge about the relationship between the studied differential equation and the Airy equation on one hand, and the exact computation of the Stokes constants for the Airy equation ([11, 9]) on the other hand. Anne Duval noticed 15 years ago the very simple relation between $C^{1/3}$ and the Airy function ([12]). The Galois group of Airy's equation is $SL_2(\mathbb{C})$, one of the rare groups that had been calculated prior to Ramis's theorem (and of N. Katz's results).

4.3 The accelerating $C_{1/4}$

We study now the following equation:

$$acc4 := -36x^2y''(x) - 60xy'(x) - 4x^3y^{(3)}(x) - 24y(x) + \frac{y(x)}{x^4} = 0$$

A basis of formal solutions is given by Desir under this form:

```
> desir(acc4,y(x),t,15,res);
```

$$[[x(t) = 1/4\,t^3, y(t) = e^{-3t^{-4}}\left(1 + \frac{5}{36}t^4 - \frac{313}{5184}t^8 + \frac{15181}{559872}t^{12} + O\left(t^{15}\right)\right)t^{-2}]]$$

> `ser_prep:=prepa_scindage(ser);`

the series is 4 summable

$ser\_prep := [r, [1, 0, 0, 0, 5/9\sqrt[3]{4}, 0, 0, 0],$
$[16\left(-\frac{41}{864} + \frac{1}{72}r + \frac{1}{288}(6+r)^2 - \frac{1}{1728}(6+r)^3\right)\sqrt[3]{4}, 0, 0, 0, \frac{8}{9} - 1/12\,(6+r)^2,$
$0, 0, 0, 1/4\,(-2 - 1/4\,r)\,4^{2/3}]]$

Here is a readable form of the series:

$$1 + 5/9\sqrt[3]{4}t^4 - \frac{313}{324}4^{2/3}t^8 + O\left(t^{10}\right)$$

Now we perform the splitting:

> `S_S := scindage(4,ser_prep);`

$S\_S := [[r, [1, 5/9\sqrt[3]{4}, -\frac{313}{324}4^{2/3}],$
$[0, -1544\,2^{2/3} - 2496\,2^{2/3}r - 1536\,2^{2/3}r^2 - 256\,2^{2/3}r^3, -3216$
$-2880\,r - 576\,r^2, -648\sqrt[3]{2} - 216\sqrt[3]{2}r]], [r, [], [1]], [r, [], [1]], [r, [], [1]]]$

In this case, the simplification of the recurrence relation defining the non null series permits to reduce by 1 the degree of the recurrence equation:

> `diminue_syst(S_S[1]);`

$[r, [1, 5/9\sqrt[3]{4}], [41\,2^{2/3} + 24\,2^{2/3}r + 96\,2^{2/3}r^2 + 32\,2^{2/3}r^3, 114 + 216\,r + 72\,r^2, 54\sqrt[3]{2} + 27\sqrt[3]{2}r]]$

Here is the readable form of the splitted series:

$$1 + 5/9\,2^{2/3}t - \frac{313}{162}\sqrt[3]{2}t^2 + \frac{15181}{2187}t^3 - \frac{2166725}{157464}2^{2/3}t^4 + O\left(t^5\right)$$

Differential equation satisfied by the previous series multiplied by t:

$equa := \left(728\,2^{2/3}x^2 + 288\,x + 27\sqrt[3]{2}\right)g''(x) + \left(345\,2^{2/3}x + 114\right)g'(x)$
$+ \left(320\,2^{2/3}x^3 + 72\,x^2\right)g^{(3)}(x) + 41\,2^{2/3}g(x) + 32\,2^{2/3}x^4g^{(4)}(x)$

> `desir(equa,g(x),t,3);`

$[[x(t) = t, g_1(t) = 1/9\,t^2\left(486\sqrt[3]{2} - 1026\,t + 1388\,2^{2/3}t^2 + 9\,O\left(t^3\right)\right)],$
$[x(t) = t, g_2(t) = 1/6\,t\left(162\sqrt[3]{2} + 180\,t - 313\,2^{2/3}t^2 + 6\,O\left(t^3\right)\right)],$
$[x(t) = t, g_3(t) = \frac{1}{324}e^{-3/16\frac{\sqrt[3]{2}(-3+i\sqrt{3})}{t}}t$
$\left(324 - 90\,t2^{2/3} - 90\,it\sqrt{3}2^{2/3} + 313\,t^2\sqrt[3]{2} - 313\,it^2\sqrt{3}\sqrt[3]{2} + 324\,O\left(t^3\right)\right)]]$

For the following, we denote by y_3 the regular part of g_3.

Differential equation satisfied by its Borel transform:

$equaborel := \left(54\sqrt[3]{2}\zeta + 288\,\zeta^2 + 192\,\zeta^3 2^{2/3}\right)\phi''(\zeta) + \left(27\sqrt[3]{2}\zeta^2 + 72\,\zeta^3 + 32\,2^{2/3}\zeta^4\right)\phi^{(3)}(\zeta)$
$+ \left(114\,\zeta + 152\,2^{2/3}\zeta^2\right)\phi'(\zeta) + 41\,2^{2/3}\zeta\,\phi(\zeta) = 0$

This equation presents three singularities:

$$sing := [0, 3/16\sqrt[3]{2}\left(-3 + i\sqrt{3}\right), -3/16\sqrt[3]{2}\left(3 + i\sqrt{3}\right)]$$

A basis of formal solutions at the origin:

$[\phi_1(\zeta) = \zeta\left(108\sqrt[3]{2} - 114\,\zeta + \frac{1388}{27}2^{2/3}\zeta^2 - \frac{68273}{1944}\sqrt[3]{2}\zeta^3 + \frac{712307}{218700}\zeta^4 + O\left(\zeta^5\right)\right),$
$\phi_2(\zeta) = 27\sqrt[3]{2} + 30\,\zeta - \frac{313}{12}2^{2/3}\zeta^2 + \frac{15181}{486}\sqrt[3]{2}\zeta^3 - \frac{2166725}{69984}\zeta^4 + O\left(\zeta^5\right),$

$\phi_3(\zeta) = 2\ \ln(\zeta)\left(27\sqrt[3]{2} + 30\,\zeta - \frac{313}{12}\,2^{2/3}\zeta^2 + \frac{15181}{486}\sqrt[3]{2}\zeta^3 - \frac{2166725}{69984}\,\zeta^4\right)$
$+\,108\sqrt[3]{2} - 144\,\zeta + \frac{107}{3}\,2^{2/3}\zeta^2 + \frac{5257}{729}\sqrt[3]{2}\zeta^3 - \frac{10870291}{209952}\,\zeta^4 + O\left(\zeta^5\right)]$
The series ϕ_2 is the Borel transform of g_2.
A basis of solutions in the neighborhood of the first non null singularity:
$[\psi_1(\zeta) = \zeta\ \left(-6912\sqrt[3]{2} - 6912\,i\sqrt[3]{2}\sqrt{3} + \left(7296 - 7296\,i\sqrt{3}\right)\zeta + O\left(\zeta^2\right)\right),$
$\psi_2(\zeta) = -1728\,i\sqrt[3]{2}\sqrt{3} - 1728\sqrt[3]{2} + \left(-1920 + 1920\,i\sqrt{3}\right)\zeta + O\left(\zeta^2\right),$
$\psi_3(\zeta) = 2\ \ln(\zeta)\left(-1728\,i\sqrt[3]{2}\sqrt{3} - 1728\sqrt[3]{2} + \left(-1920 + 1920\,i\sqrt{3}\right)\zeta\right) - 6912\sqrt[3]{2}$
$-\,6912\,i\sqrt[3]{2}\sqrt{3} + \left(-9216\,i\sqrt{3} + 9216\right)\zeta + O\left(\zeta^2\right)]$
We verify that the series coefficient of $log(\zeta)$ in ψ_3 is proportional to $\hat{B}y_3$, the Borel transform of y_3 : an illustration of the bridge equation (1.1).

5 Conclusion

The algorithms for the manipulation and in particular the splitting of formal series that we have described in the present paper have enabled us to perform calculations with alien derivations for series solution of some linear ODEs of rank $k \geq 1$, with one critical time. Numerical results of this sort are completely new, even in simple situations. The numerical results we give for the accelerating functions $C_{1/3}$ and $C_{1/4}$ yield data that can be used in a straightforward way:
• to perform numerical approximations to the constants of structure of the Lie–Galois algebra of the corresponding equations
or, in another direction,
• to obtain, as in [9], resummation of formal solutions in a "large" sector, past the Stokes lines, as we have a control of the Stokes phenomenon through the coefficient A_ω of the bridge equation.

We stress the fact that the examples that we treat, in the present work and in a continuation of it for other equations of single rank or for cases with several critical times, are neither innocent nor anecdotic : a great number of the so called classical functions are indeed solutions of linear ODEs, and more specifically, the most interesting among them are solutions to equations with one irregular-singular point and at most another singular point which is at worst regular–singular (the class of Hamburger equations; in particular the confluent hypergeometric equations).
In a way, the present computations, although they rely on mathematics that are now considered as well known (Gevrey asymptotics and the like), use in a certain sense the DESIR code up to its limits, and thus yield new results that are consequences of the seminal works made by Jean Della Dora 25 years ago.

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