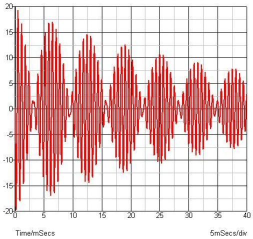
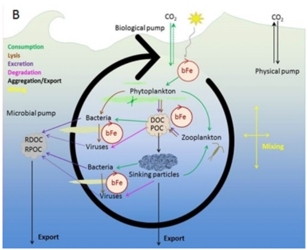
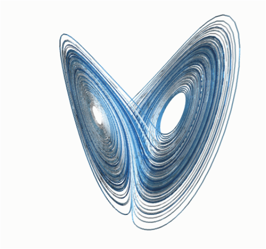
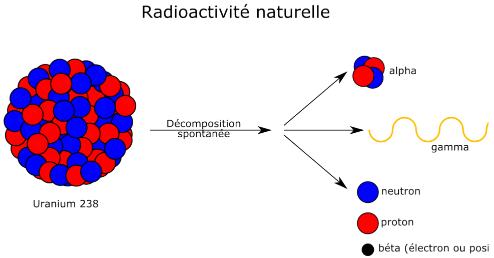
Ordinary Differential Equations







Example #1 : radioactivity



Observation : the variation in the quantity of radioactive nuclei is proportional to their quantity and to the elapsed time.

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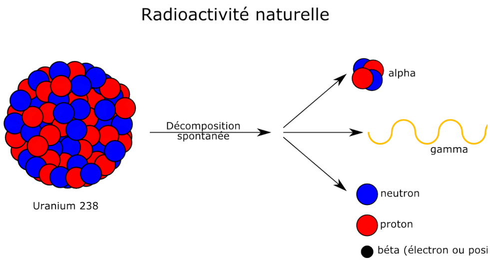

Observation : the variation in the quantity of radioactive nuclei is proportional to their quantity and to the elapsed time.

$$N(t+\Delta t) - N(t) \propto \Delta t \, N(t) \quad \underset{\Delta t \to 0}{\Longrightarrow} \quad N'(t) = -\lambda \, N(t)$$

$N(t)$: quantity of radioactive nuclei at time t, and $\lambda > 0$

Example #1 : radioactivity

Multiplication by $e^{\lambda t}$: $N'(t)\,e^{\lambda t} + \lambda N(t)\,e^{\lambda t} = 0,$ i.e. $\left(N(t)\,e^{\lambda t}\right)' = 0$

Hence $N(t) = C\,e^{-\lambda t} = N(0)\,e^{-\lambda t}$

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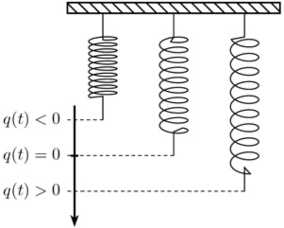
Half-life time : $N(t) = \dfrac{N(0)}{2}$ for $e^{-\lambda t} = \dfrac{1}{2}$, i.e. $t_{\text{half}} = \dfrac{\ln 2}{\lambda}$

- Iodine 131: $t_{\text{half}} \simeq 8$ days
- Cesium 137: $t_{\text{half}} \simeq 30$ years
- Plutonium 239: $t_{\text{half}} \simeq 24\,110$ years

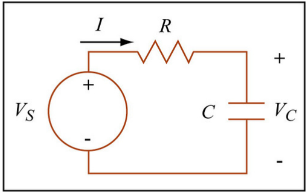
Example #2 : mass - spring system



Second law of dynamics: $m\, q''(t) + k\, q(t) = 0$ ($k > 0$: spring stiffness)

$$q(t) = A \cos \omega t + B \sin \omega t \quad , \quad \text{with } \omega = \sqrt{\frac{k}{m}}$$

Example #3: RC circuit



$I(t)$: intensity

R: resistance, C: capacitance

$I(t) = \dfrac{dq(t)}{dt}$ with $q(t)$ the electrical charge

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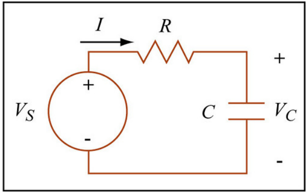

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$$\begin{aligned} V_S(t) &= V_R(t) + V_C(t) \\ &= R\,I(t) + \frac{q(t)}{C} = R\,\frac{dq}{dt}(t) + \frac{q(t)}{C} \end{aligned}$$

$$R\,\frac{dq}{dt}(t) + \frac{1}{C}\,q(t) = V_S(t)$$

Example #4 : viral epidemic (system of ODEs)

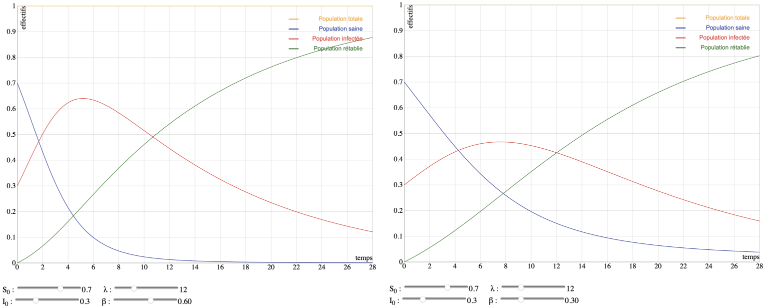
The SIR model

Susceptible individuals

Infectious individuals

Removed (and immune) or deceased individuals

$$\begin{cases} S'(t) &= -\beta\, I(t) S(t) \\ I'(t) &= \beta\, I(t) S(t) - \lambda\, I(t) \\ R'(t) &= \lambda\, I(t) \end{cases}$$



Further information:

https://interstices.info/modeliser-la-propagation-dune-epidemie/

Some vocabulary about differential equations

- A **differential equation** is a relationship linking a function and its successive derivatives. **The unknown is therefore a function**. In very general terms, it can be written as $F(x, y, y', y'', \dots, y^{(n)}) = 0$, where $y(x)$ is the unknown function.

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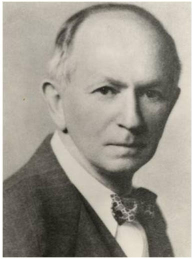
  Otherwise (E) is said to be **nonlinear**.

Example #5: predator-prey models



Alfred Lotka
(1880 - 1949)

Vito Volterra
(1860 - 1940)

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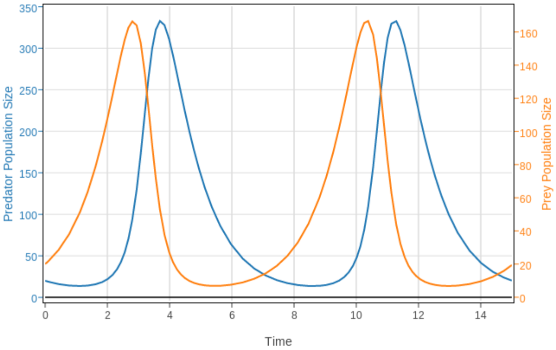


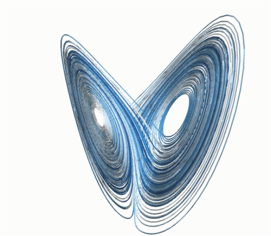
Example #6: chaos and butterfly effect

Chaotic systems: small initial perturbations may lead to huge final differences (atmosphere, ocean, climate are chaotic systems).

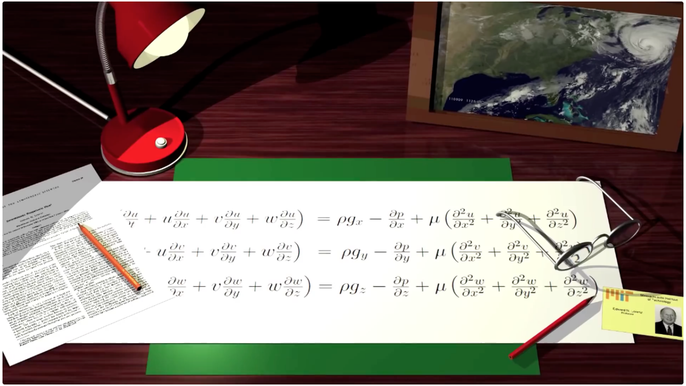
Edward Lorenz
(1917-2008)

Ellen Fetter
(1940 –)

$$\begin{cases} \dfrac{dx}{dt} = \sigma(y-x) \\ \dfrac{dy}{dt} = \rho x - y - xz \\ \dfrac{dz}{dt} = xy - \beta z \end{cases}$$



Does the flap of a butterfly's wings in Brazil set off a tornado in Texas?
(139th meeting of the American Association for the Advancement of Science, 1972)



http://www.chaos-math.org

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Vocabulary of differential equations

		linear (yes/no)	homogeneous (yes/no)	order
1.	$z'(x) + x^3\, z(x) = \sqrt{x}$			
2.	$y'(t)\, y(t) - t\; y(t) = \cos t$			
3.	$y'(t)^2 + 3t^2\, y(t) - t = 0$			
4.	$z^3\, z' = 5\, z$			
5.	$e^x\, u''(x) - x\, u(x) = 0$			
6.	$y\, y' + y - t = 0$			
7.	$z^2\, z' = \sqrt{z}$			
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First-order linear ODEs

Solutions of a first-order linear ODE

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where $a(t), b(t), c(t)$ are given functions.

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Principle of superposition

Let $u_p(x)$ a particular solution of (E).
The solutions of (E) are the functions $u(x) = u_p(x) + u_0(x)$, where u_0 represents the solutions of (E_0).

In other words, the set of solutions of (E) is $\mathcal{S} = u_p + \mathcal{S}_0$, where $\mathcal{S}_0$ denotes the set of solutions of (E_0).

Solving a first-order linear ODE

- **Step 0**: on which domain?

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- **Step 0**: on which domain? $\longrightarrow$ each interval where $a(t)$ does not cancel. Dividing by $a(t)$, the equation becomes

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- **Step 1**: solution of the associated homogeneous equation $(E_0) \quad y_0'(t) + \alpha(t)\, y_0(t) = 0$
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- **Step 2**: determination of a particular solution y_p of (E)
  - either by analogy (simple and intuitive method, but does not work systematically)
  - or by variation of constants (always works, but is a little bit more demanding in terms of calculations)

The set of solutions is then $\mathcal{S} = y_p + \mathcal{S}_0 = \{\, y_p + y_0, y_0 \in \mathcal{S}_0 \}$

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- **Step 3** (possibly): connection of solutions between different intervals where $a(t)$ does not cancel out : given solutions on $]t_1, t_2[$ and $]t_2, t_3[$, does it exist $\mathcal{C}^1$ solutions on $]t_1, t_3[$?

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- Let $A(x)$ <u>a</u> primitive of $\alpha(x)$.

  Multiplying (E_0) by $e^{A(x)}$: $\quad e^{A(x)}\, y_0'(x) + \alpha(x)\, e^{A(x)}\, y_0(x) = 0$

  i.e. $\quad \left(e^{A(x)}\, y_0(x)\right)' = 0$

  hence $\quad e^{A(x)}\, y_0(x) = \text{cste}$

  Solutions are thus of the form: $\quad y_0(x) = K\, e^{-A(x)} \quad$ with $K \in \mathbb{R}$.

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$$(E_0) \quad y_0'(x) + \alpha(x)\, y_0(x) = 0 \qquad x \in I$$

- Let $A(x)$ a primitive of $\alpha(x)$.

  Multiplying (E_0) by $e^{A(x)}$: $e^{A(x)}\, y_0'(x) + \alpha(x)\, e^{A(x)}\, y_0(x) = 0$

  $$\text{i.e.} \qquad \left(e^{A(x)}\, y_0(x)\right)' = 0$$

  $$\text{hence} \qquad e^{A(x)}\, y_0(x) = \text{cste}$$

  Solutions are thus of the form: $y_0(x) = K\, e^{-A(x)}$ with $K \in \mathbb{R}$.

- Conversely, any function of this form is a solution:

if $y_0(x) = K\, e^{-A(x)}$, then $y_0'(x) = -K\,\alpha(x)\, e^{-A(x)}$, hence $y_0'(x) + \alpha(x)\, y_0(x) = 0$

Step 1: solution of the associated homogeneous equation

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- In summary: the set of solutions on I is $\quad \mathcal{S}_0 = \left\{ y_0 / y_0(x) = K\, e^{-A(x)} \quad \text{with } K \in \mathbb{R} \right\}$

Solution of the homogeneous ED: example #1

$$(E_0) \qquad (1+x)\, y_0'(x) + y_0(x) = 0$$

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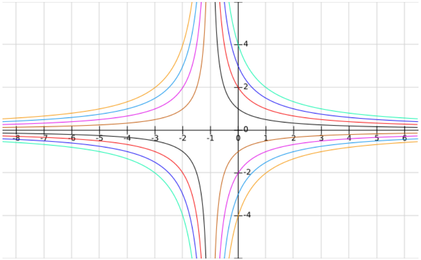
$$(E_0) \qquad (1+x)\,y_0'(x) + y_0(x) = 0$$

On $I =]-\infty;-1[$ or $I =]-1;+\infty[$: $\quad (E_0)$ becomes $\quad y_0'(x) + \dfrac{1}{1+x} y_0(x) = 0.$

A primitive of $\alpha(x) = \dfrac{1}{1+x}$ is $A(x) = \ln|1+x|$. Hence $e^{-A(x)} = \dfrac{1}{|1+x|}$

Hence the solutions on I:

$$y_0(x) = \frac{K}{1+x} \qquad K \in \mathbb{R}$$



Solution of the homogeneous ED: example #2

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Solution of the homogeneous ED: example #2

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A primitive of $\alpha(t) = \tan(t)$ is $A(t) = -\ln|\cos(t)|$. Hence $e^{-A(t)} = |\cos(t)|$

Hence the solutions on I :

$$u_0(t) = K \cos t \qquad K \in \mathbb{R}$$

Remark: reminder on integration

Three main tools for the computation of primitives and integrals:

- **handbook of usual primitives**

	Fonction		Primitives
	x^α	$(\alpha \in \mathbb{R}, \alpha \neq -1)$	$\frac{x^{\alpha+1}}{\alpha+1} + C$
et aussi	$(x-a)^\alpha$	$(\alpha \in \mathbb{R}, \alpha \neq -1)$	$\frac{(x-a)^{\alpha+1}}{\alpha+1} + C$
	$\frac{1}{x}$		$\ln\|x\| + C$
et aussi	$\frac{1}{x-a}$		$\ln\|x-a\| + C$
	e^x		$e^x + C$
et aussi	$e^{\alpha x}$	$\alpha \neq 0$	$\frac{1}{\alpha} e^{\alpha x} + C$
	$\cos x$		$\sin x + C$
et aussi	$\cos(\alpha x)$	$\alpha \neq 0$	$\frac{1}{\alpha} \sin(\alpha x) + C$
	$\sin x$		$-\cos x + C$
et aussi	$\sin(\alpha x)$	$\alpha \neq 0$	$\frac{-1}{\alpha} \cos(\alpha x) + C$
	1 $= 1 + \tan^2 x$		$\tan x + C$

Remark: reminder on integration

Three main tools for the computation of primitives and integrals:

- **handbook of usual primitives**
- **integration by parts**

$(uv)' = u'v + uv'$. Thus $\int (uv)' = uv = \int u'v + \int uv'$.

i.e. $\int u'v = uv - \int uv'$

- $\int \ln x \, dx$ — *En posant* $u = x$ *et* $v = \ln x$ *:*

$$\int \ln x \, dx = x \ln x - \int x \frac{1}{x} dx = x \ln x - \int dx = x \ln x - x + C$$

Remark: reminder on integration

Three main tools for the computation of primitives and integrals:

- **handbook of usual primitives**
- **integration by parts**
- **change of variable**

- $I = \int_a^b x e^{x^2}\, dx$ — *On pose* $u = x^2$. *Alors* $du = 2x\ dx$, *d'où :*

$$I = \int_{x=a}^{x=b} x e^{x^2}\, dx = \frac{1}{2} \int_{u=a^2}^{u=b^2} e^u\, du = \frac{1}{2} [e^u]_{a^2}^{b^2} = \frac{e^{b^2} - e^{a^2}}{2}$$

Solving a first-order linear ODE

- **Step 0**: on which domain? $\longrightarrow$ each interval where $a(t)$ does not cancel. Dividing by $a(t)$, the equation becomes
$$(E) \quad y'(t) + \alpha(t)\, y(t) = \beta(t) \qquad \text{where } \alpha(t) = b(t)/a(t) \text{ and } \beta(t) = c(t)/a(t)$$
- **Step 1**: solution of the associated homogeneous equation $(E_0) \quad y_0'(t) + \alpha(t)\, y_0(t) = 0$
$\longrightarrow$ computation of a primitive.
One gets a set of solutions $\mathcal{S}_0$
- **Step 2**: determination of a particular solution y_p of (E)
  - either by analogy (simple and intuitive method, but does not work systematically)
  - or by variation of constants (always works, but is a little bit more demanding in terms of calculations)

  The set of solutions is then $\mathcal{S} = y_p + \mathcal{S}_0 = \{\, y_p + y_0, y_0 \in \mathcal{S}_0\}$
- **Step 3** (possibly): connection of solutions between different intervals where $a(t)$ does not cancel out : given solutions on $]t_1, t_2[$ and $]t_2, t_3[$, does it exist $\mathcal{C}^1$ solutions on $]t_1, t_3[$?

Particular solution: method by analogy

$$(E) \qquad y'(x) + \alpha(x)\, y(x) = \beta(x)$$

If the right-hand side $\beta(x)$ is a polynomial, or a linear combination of exponentials, or a linear combination of sine and cosine functions, and if $\alpha(x)$ is constant or of similar nature as $\beta(x)$, then it may exist a particular solution $u_p(x)$ in a form similar to that of $\beta(x)$.

$\longrightarrow$ simple method, but which is not always successful

Particular solution: method by analogy

Example: $y'(t) + (2t+1)\,y(t) = 6t^2 - t + 1$

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The r.h.s. being of degree 2, one can look for $y_p(t)$ as a degree 1 pol.

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Let inject $y_p(t) = at + b$ in (E) :

$$\begin{aligned} a + (2t+1)(at+b) &= 6t^2 - t + 1 \\ \text{thus} \quad 2a\,t^2 + (a+2b)\,t + a + b &= 6t^2 - t + 1 \end{aligned}$$

Hence $2a = 6, a + 2b = -1, a + b = 1$. Thus $a = 3, b = -2$.

$y_p(t) = 3t - 2$ is a particular solution of (E).

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It's miraculous : 3 equations for 2 unknowns (a, b).

If one changes a coefficient in the r.h.s., it does not work anymore.

For instance: $y'(t) + (2t+1)\,y(t) = 6t^2 - t$

$$2a = 6, a + 2b = -1, a + b = 0 \quad \longrightarrow \text{no solution.}$$

Method by analogy: examples

In what form can particular solutions of the following equations be found?

1. $(t+1)y' + (2t-1)\,y = 2t^3 + t^2 + 1$

2. $f'(t) - f(t) = \sin t + 2\,\cos t$

3. $z'(x) - 3\,z(x) = \sin 3x$

4. $u' + 3u = 5e^{2x} + 6e^{-x}$

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   $u_p(x) = e^{2x} + 3\,e^{-x}$

Variation of constants method: general principle

$$(E) \quad y'(x) + \alpha(x)\, y(x) = \beta(x)$$

Solutions of (E_0): $y_0(x) = K\, e^{-A(x)}$ where $K \in \mathbb{R}$ and $A(x)$ is a primitive of $\alpha(x)$

Idea: look for a particular solution under the form $y_p(x) = K(x)\, e^{-A(x)}$.

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Then $y_p'(x) = K'(x)\, e^{-A(x)} - K(x)\, \alpha(x)\, e^{-A(x)}$. Thus:

$$\begin{aligned} y_p'(x) + \alpha(x)\, y_p(x) &= \underbrace{K'(x)\, e^{-A(x)} - K(x)\, \alpha(x)\, e^{-A(x)}}_{y_p'} + \alpha(x)\, \underbrace{K(x)\, e^{-A(x)}}_{y_p} \\ &= K'(x)\, e^{-A(x)} \end{aligned}$$

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Hence $K'(x) = \beta(x)\, e^{A(x)}$. Hence $K(x)$ by integration. Hence $y_p(x)$.

Variation of constants method: example

$$(E) \quad y'(x) + 2x\, y(x) = 2x\, e^{-x^2}$$

Variation of constants method: example

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Solutions of (E_0) $\quad y_0'(x) + 2x\, y_0(x) = 0$: $\quad y_0(x) = K\, e^{-\int 2x} = K\, e^{-x^2} \quad K \in \mathbb{R}$

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Particular solution: let $y_p(x) = K(x)\,e^{-x^2}$. Thus, injecting in (E):

$$\underbrace{K'(x)\,e^{-x^2} + K(x)\,(-2x)\,e^{-x^2}}_{y_p'} + 2x\,\underbrace{K(x)\,e^{-x^2}}_{y_p} = 2x\,e^{-x^2}$$

thus $K'(x)\,e^{-x^2} = 2x\,e^{-x^2}$, then $K'(x) = 2x$.

Variation of constants method: example

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Finally: $\quad y_p(x) = x^2\, e^{-x^2}$

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The solutions of (E): $\quad y(x) = x^2\,e^{-x^2} + K\,e^{-x^2} = (x^2 + K)\,e^{-x^2} \quad K \in \mathbb{R}$

Variation of constants method: example

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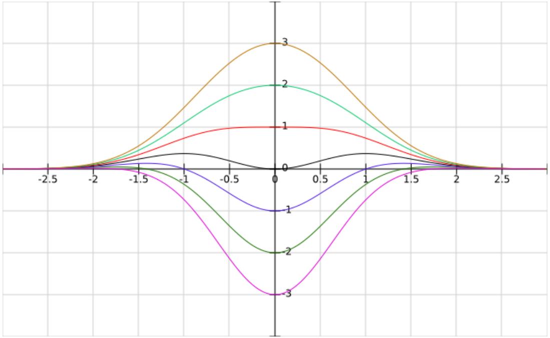


Solutions : $\quad y(x) = x^2\, e^{-x^2} + K\, e^{-x^2} = (x^2 + K)\, e^{-x^2} \quad K \in \mathbb{R}$

Solving a first-order linear ODE

- **Step 0**: on which domain? $\longrightarrow$ each interval where $a(t)$ does not cancel. Dividing by $a(t)$, the equation becomes

$$(E) \quad y'(t) + \alpha(t)\, y(t) = \beta(t) \qquad \text{where } \alpha(t) = b(t)/a(t) \text{ and } \beta(t) = c(t)/a(t)$$

- **Step 1**: solution of the associated homogeneous equation $(E_0) \quad y_0'(t) + \alpha(t)\, y_0(t) = 0$
  $\longrightarrow$ computation of a primitive.
  One gets a set of solutions $\mathcal{S}_0$
- **Step 2**: determination of a particular solution y_p of (E)
  - either by analogy (simple and intuitive method, but does not work systematically)
  - or by variation of constants (always works, but is a little bit more demanding in terms of calculations)

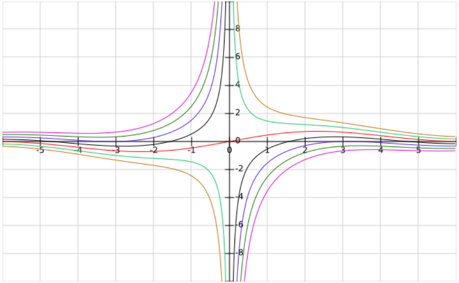
  The set of solutions is then $\mathcal{S} = y_p + \mathcal{S}_0 = \{\, y_p + y_0, y_0 \in \mathcal{S}_0\}$
- **Step 3** (possibly): connection of solutions between different intervals where $a(t)$ does not cancel out : given solutions on $]t_1, t_2[$ and $]t_2, t_3[$, does it exist $\mathcal{C}^1$ solutions on $]t_1, t_3[$?

Connection of solutions: example #1

$$(E) \quad tz'(t) + z(t) - \sin t = 0$$

Solutions de (E) are:

- on $]-\infty, 0[$: $\quad z_-(t) = -\dfrac{\cos t}{t} + \dfrac{K_-}{t} \qquad K_- \in \mathbb{R}$
- on $]0, +\infty[$: $\quad z_+(t) = -\dfrac{\cos t}{t} + \dfrac{K_+}{t} \qquad K_+ \in \mathbb{R}$



What are the solutions of (E) on $\mathbb{R}$?

i.e. what are the possible continuous and differentiable connections at $t = 0$ between a function z_- and a function z_+?

Connection of solutions: example #1

- on $]-\infty, 0[$: $\quad z_-(t) = -\dfrac{\cos t}{t} + \dfrac{K_-}{t} \qquad K_- \in \mathbb{R}$
- on $]0, +\infty[$: $\quad z_+(t) = -\dfrac{\cos t}{t} + \dfrac{K_+}{t} \qquad K_+ \in \mathbb{R}$

Continuity:

$$\lim_{t \to 0^-} z_-(t) = \lim_{t \to 0^-} \frac{K_- - 1 + t^2/2 + O(t^4)}{t} = \begin{cases} +\infty & \text{if } K_- < 1 \\ 0 & \text{if } K_- = 1 \\ -\infty & \text{if } K_- > 1 \end{cases}$$

$$\lim_{t \to 0^+} z_+(t) = \lim_{t \to 0^+} \frac{K_+ - 1 + t^2/2 + O(t^4)}{t} = \begin{cases} -\infty & \text{if } K_+ < 1 \\ 0 & \text{if } K_+ = 1 \\ +\infty & \text{if } K_+ > 1 \end{cases}$$

Connection of solutions: example #1

- on $]-\infty,0[$: $\quad z_-(t) = -\dfrac{\cos t}{t} + \dfrac{K_-}{t} \qquad K_- \in \mathbb{R}$
- on $]0,+\infty[$: $\quad z_+(t) = -\dfrac{\cos t}{t} + \dfrac{K_+}{t} \qquad K_+ \in \mathbb{R}$

Continuity:

$$\lim_{t\to 0^-} z_-(t) = \lim_{t\to 0^-} \frac{K_- - 1 + t^2/2 + O(t^4)}{t} = \begin{cases} +\infty & \text{if } K_- < 1 \\ 0 & \text{if } K_- = 1 \\ -\infty & \text{if } K_- > 1 \end{cases}$$

$$\lim_{t\to 0^+} z_+(t) = \lim_{t\to 0^+} \frac{K_+ - 1 + t^2/2 + O(t^4)}{t} = \begin{cases} -\infty & \text{if } K_+ < 1 \\ 0 & \text{if } K_+ = 1 \\ +\infty & \text{if } K_+ > 1 \end{cases}$$

The only possible choice for a continuous connection at $t = 0$ is thus $K_- = K_+ = 1$.

$$\text{Let } \; z^*(t) = \begin{cases} \dfrac{1-\cos t}{t} & \text{if } t \neq 0 \\ 0 & \text{if } t = 0 \end{cases}$$

Connection of solutions: example #1

- on $]-\infty, 0[$: $\quad z_-(t) = -\dfrac{\cos t}{t} + \dfrac{K_-}{t} \qquad K_- \in \mathbb{R}$
- on $]0, +\infty[$: $\quad z_+(t) = -\dfrac{\cos t}{t} + \dfrac{K_+}{t} \qquad K_+ \in \mathbb{R}$

Continuity:

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$$\text{Let } \; z^*(t) = \begin{cases} \dfrac{1-\cos t}{t} & \text{if } t \neq 0 \\ 0 & \text{if } t = 0 \end{cases}$$

$\longrightarrow$ **Remaining point: is z^* differentiable at $t = 0$?**

Connection of solutions: example #1

$$z^*(t) = \begin{cases} \dfrac{1-\cos t}{t} & \text{if } t \neq 0 \\ 0 & \text{if } t = 0 \end{cases}$$

Differentiability: $\quad z^{*\prime}(t) = \dfrac{\sin t}{t} + \dfrac{\cos t - 1}{t^2}$ for $t \neq 0$.

In the vicinity of 0: $z^{*\prime}(t) = \dfrac{t + O(t^3)}{t} + \dfrac{(1 - t^2/2 + O(t^4)) - 1}{t^2} = 1 - \dfrac{1}{2} + O(t^2) = \dfrac{1}{2} + O(t^2)$.

So $z^{*\prime}(0) = \dfrac{1}{2}$. $\quad z^*$ is thus differentiable in 0, and its derivative is continuous.

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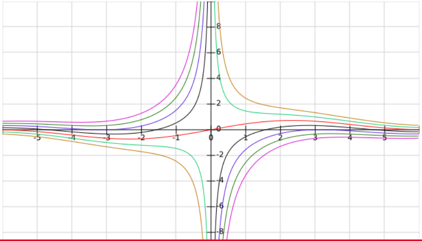
So $z^{*\prime}(0) = \dfrac{1}{2}$. z^* is thus differentiable in 0, and its derivative is continuous.

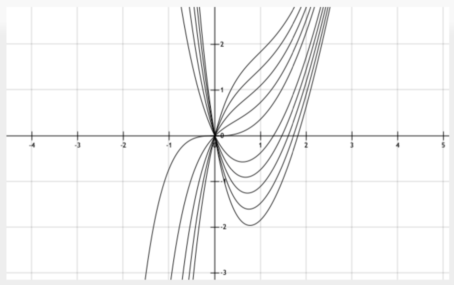
z^* is the unique $\mathcal{C}^1$ solution of (E) on $\mathbb{R}$.



Connection of solutions: example #2

$$(E) \qquad y'(x) + \frac{x-1}{x}\, y(x) = x^2$$

- on $]-\infty, 0[$: $\quad y_-(x) = x^2 - x + K_-\, x\, e^{-x} \qquad K_- \in \mathbb{R}$
- on $]0, +\infty[$: $\quad y_+(x) = x^2 - x + K_+\, x\, e^{-x} \qquad K_+ \in \mathbb{R}$

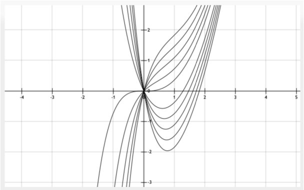


Given those solutions, what are the solutions of (E) on $\mathbb{R}$? i.e. what are the possible continuous and differentiable connections at $x = 0$ between a function y_- and a function y_+ ?

Connection of solutions: example #2

The solutions of (E) on $\mathbb{R}^*$ are the functions

$$y(x) = \begin{cases} x^2 - x + K_- \, x \, e^{-x} & \text{on }]-\infty, 0[\\ x^2 - x + K_+ \, x \, e^{-x} & \text{on }]0, +\infty[\end{cases}$$



Continuity: $\lim_{x \to 0^-} y(x) = 0$ and $\lim_{x \to 0^+} y(x) = 0$. Thus any branch of solution on $]-\infty, 0[$ is continuously connected to any branch of solution on $]0, +\infty[$.

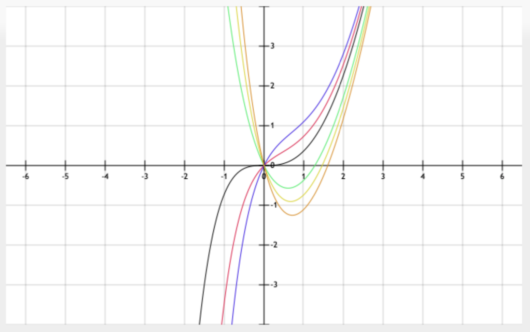
Differentiability: $(x^2 - x + K \, x \, e^{-x})' = 2x - 1 + K\,(1 - x)\, e^{-x}$. Thus $\lim_{x \to 0^-} y'(x) = -1 + K_-$ and $\lim_{x \to 0^+} y(x) = -1 + K_+$.

Thus a branch of solution on $]-\infty, 0[$ connects smoothly to a branch of solution on $]0, +\infty[$ iff $K_- = K_+$.

Connection of solutions: example #2

$$(E) \qquad y'(x) + \frac{x-1}{x}\, y(x) = x^2$$

The solutions of (E) on $\mathbb{R}$ are the functions $y(x) = x^2 - x + K\, x\, e^{-x}$, $\quad K \in \mathbb{R}$



And what for first-order nonlinear differential equations?

And what for first-order nonlinear differential equations?

- No fully general method
- A simple method for separable differential equations
- Some methods, on a case-by-case basis (often by a change of unknown function), for some particular equations

Separable differential equations

DefinitionA first-order differential equation is separable iff it can be written as
$y'(t) = g(y(t))\, f(t)$.

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Computation method (schematically)

$$y'(t) = g(y(t))\, f(t) \iff \frac{y'(t)}{g(y(t))} = f(t)$$

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y'(t) = g(y(t))\, f(t) \quad &\iff \quad \frac{y'(t)}{g(y(t))} = f(t) \\
&\iff \quad \int \frac{y'(t)}{g(y(t))} = \int f(t) \\
&\iff \quad H(y(t)) = F(t) + C \quad \text{where } H \text{ is a primitive of } 1/g \\
&\qquad\qquad F \text{ a primitive of } f, \text{ and } C \in \mathbb{R}
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\Longleftrightarrow \quad & y(t) = H^{-1}(F(t) + C) \quad C \in \mathbb{R}
\end{aligned}
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Definition A first-order differential equation is separable iff it can be written as $y'(t) = g(y(t))\, f(t)$.

Are the following equations separable?

1. $y'(t) - y^3(t)\ \sin 2t = 0$ **YES - NO**
2. $y'(t) - y^3(t)\ \sin 2t + 2t + 1 = 0$ **YES - NO**
3. $y'(t) - y^3(t)\ \sin 2t + (2t + 1)\, y(t) = 0$ **YES - NO**
4. $y'(t) - y^3(t)\ \sin 2t - \sin 2t + 1 + y^3(t) = 0$ **YES - NO**
5. $e^{z(x)}\, z'(x) + \ln(x)\, z^2(x) = 0$ **YES - NO**
6. $t\, u'(t) - u(t) + e^{u'(t)} = 0$ **YES - NO**
7. $f'(t) - f(t) = t\, f^2(t)$ **YES - NO**
8. $x\, y'(x) - \sin(x)\ \cos(y^2(x)) = 0$ **YES - NO**

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4. $y'(t) - y^3(t)\,\sin 2t - \sin 2t + 1 + y^3(t) = 0$ **YES** - **NO**

   $y'(t) = (y^3(t) + 1)\,(\sin(2t) - 1) \qquad g(X) = X^3 + 1 \quad f(t) = \sin(2t) - 1$

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$$y'(x) = \frac{\sin x}{x}\, \cos(y^2(x)) \qquad g(X) = \cos(X^2) \quad f(t) = \frac{\sin t}{t}$$

Separable differential equations: example #1

$$e^{-t}\, y'(t) - t\, y^2(t) = 0$$

Separable differential equations: example #1

$$e^{-t}\, y'(t) - t\, y^2(t) = 0$$

Let assume that $y(t)$ does not cancel. Then $\dfrac{y'(t)}{y^2(t)} = t\, e^t$, which leads to:

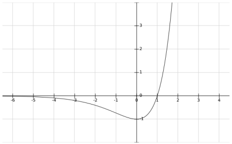
$$\int \frac{y'(t)}{y^2(t)}\, dt = \int t\, e^t\, dt$$

$$\iff \quad -\frac{1}{y(t)} = (t-1)\, e^t + C \qquad \text{(by IBP)}$$

$$\iff \quad y(t) = \frac{-1}{(t-1)\, e^t + C} \qquad C \in \mathbb{R}$$

We see a posteriori that y does indeed not cancel. Its definition domain depends on the integration constant C.

For $C > 1$: $\mathcal{D}_y = \mathbb{R}$
For $C = 1$: $\mathcal{D}_y = \mathbb{R}^*$
For $0 < C < 1$: $\mathcal{D}_y = \mathbb{R}$ except 2 forbidden values
For $C \leq 0$: $\mathcal{D}_y = \mathbb{R}$ except 1 forbidden value



$(t-1)\, e^t$

Separable differential equations: example #1

$$y(t) = \frac{-1}{(t-1)\,e^t + C} \qquad C \in \mathbb{R}$$

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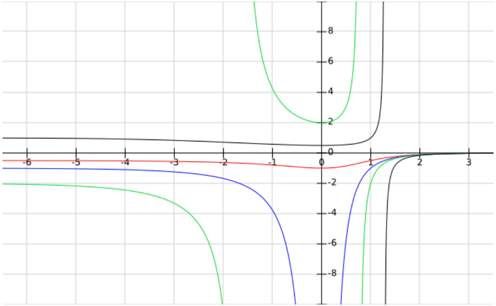

$$y(t) = \frac{-1}{(t-1)\,e^t + 2}$$

$$y(t) = \frac{-1}{(t-1)\,e^t + 1}$$

$$y(t) = \frac{-1}{(t-1)\,e^t + 0.5}$$

$$y(t) = \frac{-1}{(t-1)\,e^t - 1}$$

Separable differential equations: example #2

$$(E) \qquad x\,y' + \frac{3}{y} = 0$$

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On $I =]-\infty, 0[$ or $]0, +\infty[$, we have: $y\,y' = -\dfrac{3}{x}$, which leads by integration to: $\dfrac{1}{2}y^2(x) = -3\ln|x| + C$,

hence $y(x) = \pm\sqrt{-6\ln|x| + K} \quad (K = 2C \in \mathbb{R})$

Separable differential equations: example #2

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hence $y(x) = \pm\sqrt{-6\ln|x| + K} \quad (K = 2C \in \mathbb{R})$

The domain of y is given by the condition $-6\ln|x| + K \geq 0$, i.e. $|x| \leq e^{K/6}$.

Separable differential equations: example #2

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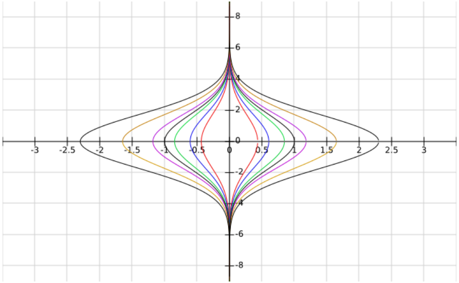
The domain of y is given by the condition $-6\ln|x| + K \geq 0$, i.e. $|x| \leq e^{K/6}$.

In summary:

$\forall K \in \mathbb{R}$,

$$\begin{cases} y_{K,-}(x) = -\sqrt{-6\ln|x| + K} \\ y_{K,+}(x) = \sqrt{-6\ln|x| + K} \end{cases}$$

are solutions of (E) on $[-e^{K/6}, 0[\cup]0, e^{K/6}]$.



Second-order linear differential equations with constant coefficients

Second-order linear ODEs with constant coefficients

$$(E) \qquad au''(x) + bu'(x) + cu(x) = f(x) \qquad \text{with } a, b, c \in \mathbb{R} \text{ and } a \neq 0$$

Principle of superposition $\mathcal{S} = u_p + \mathcal{S}_0$, where $u_p(x)$ is a particular solution of (E) and $\mathcal{S}_0$ is the set of solutions of the associated homogeneous equation (E_0).

Solutions of (E_0) Let $P(X) = aX^2 + bX + c$ the *characteristic polynomial* associated to (E_0), and $\Delta = b^2 - 4ac$ its discriminant. Then:

- if $\Delta > 0$, $u_0(x) = Ae^{r_1 x} + Be^{r_2 x}$ $A, B \in \mathbb{R}$, where $r_1 = \frac{-b-\sqrt{\Delta}}{2a}$ and $r_2 = \frac{-b+\sqrt{\Delta}}{2a}$ are the two real roots of P.
- if $\Delta = 0$, $u_0(x) = (Ax + B)e^{rx}$ $A, B \in \mathbb{R}$, where $r = \frac{-b}{2a}$ is the unique root of P.
- if $\Delta < 0$, $u_0(x) = (A\cos\alpha x + B\sin\alpha x)e^{\beta x}$ $A, B \in \mathbb{R}$, where $\alpha = \frac{\sqrt{-\Delta}}{2a}$ and $\beta = \frac{-b}{2a}$.

Particular solution of (E) (similar to first-order equations) A particular solution u_p can be obtained either by analogy with the right-hand side (if simple), or by the method of variation of constants (i.e. replacing the constants A and B, or only one of them, by a function of x).