Norms and Scalar Products

Norms

Let E a vector space.

Definition $\|.\| : E \to \mathbb{R}_+$ is a **norm** on E iff it satisfies:

(N1) $(\|x\| = 0) \Longrightarrow (x = 0)$

(N2) $\forall \lambda \in \mathbb{R}, \forall x \in E, \quad \|\lambda x\| = |\lambda| \, \|x\|$

(N3) $\forall x, y \in E, \quad \|x + y\| \leq \|x\| + \|y\|$ *(triangle inequality)*

A vector space equipped with a norm is called a **normed space**.

Vector norms: $E = \mathbb{R}^n$

$$\mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in E = \mathbb{R}^n$$

- $\|\mathbf{x}\|_p = \left(\sum_{i=1}^{n} x_i^p \right)^{1/p}$ for all $p \in \mathbb{N}$
- In particular: $\|\mathbf{x}\|_1 = \sum_{i=1}^{n} |x_i|$ $\quad \|\mathbf{x}\|_2 = \left(\sum_{i=1}^{n} x_i^2 \right)^{1/2} = (\mathbf{x}^T \mathbf{x})^{1/2}$ $\quad \|\mathbf{x}\|_\infty = \sup_i |x_i|$
- $\|.\|_2$ is the mostly used one, and is called **Euclidian norm**.

Vector norms: $E = \mathbb{R}^n$

Reminder The $n \times n$ matrix $\mathbf{M}$ is

- **positive** iff $\mathbf{x}^T\mathbf{M}\mathbf{x} = \sum_{i=1}^{n}\sum_{j=1}^{n} M_{ij}\, x_i x_j \geq 0 \qquad \forall \mathbf{x} \in \mathbb{R}^n$
- **definite** iff $\left(\mathbf{x}^T\mathbf{M}\mathbf{x} = 0\right) \Longrightarrow (\mathbf{x} = \mathbf{0})$

Matrix-induced vector norms For any $n \times n$ symmetric positive definite matrix $\mathbf{A}$, one can define its associated norm

$$\|\mathbf{x}\|_{\mathbf{A}} = \left(\sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij}\, x_i x_j\right)^{1/2} = (\mathbf{x}^T\mathbf{A}\mathbf{x})^{1/2}$$

The Euclidian norm thus corresponds to the particular case $\mathbf{A} = \mathbf{I}_n$ (the identity matrix).

Matrix norms: $E = \mathcal{M}_{n,p}(\mathbb{R})$

Let $\mathbf{A} \in \mathcal{M}_{n,p}(\mathbb{R})$.

$$\|\mathbf{A}\|_1 = \sup_{\mathbf{x}\in\mathbb{R}^p} \frac{\|\mathbf{Mx}\|_1}{\|\mathbf{x}\|_1} = \max_{j=1,\dots,p} \left(\sum_{i=1}^{n} |A_{ij}| \right)$$

$$\|\mathbf{A}\|_\infty = \sup_{\mathbf{x}\in\mathbb{R}^p} \frac{\|\mathbf{Mx}\|_\infty}{\|\mathbf{x}\|_\infty} = \max_{i=1,\dots,n} \left(\sum_{j=1}^{p} |A_{ij}| \right)$$

$$\|\mathbf{A}\|_F = \left(\sum_{i=1}^{n} \sum_{j=1}^{p} A_{ij}^2 \right)^{1/2}$$ *(Frobenius norm)*

Scalar products

Let E a vector space.

Definition Any bilinear symmetric positive definite form (i.e. a real-valued function) is called a scalar product in E.

$< .,. >: E \times E \to \mathbb{R}$ is thus a scalar product in E iff it satisfies:

(S1) $\forall\, x, y \in E, \quad < x, y > = < y, x >$

(S2) $\forall\, x_1, x_2, y \in E, \quad < x_1 + x_2, y > = < x_1, y > + < x_2, y >$

(S3) $\forall\, x, y \in E, \forall\, \lambda \in \mathbb{R}, \quad < \lambda x, y > = \lambda < x, y >$

(S4) $\forall\, x \in E, x \neq 0, \quad < x, x > > 0$

A scalar product generates a definition of **orthogonality**.

Given a scalar product, one can define its **induced norm**: $\|x\| = \sqrt{< x, x >}$

- A vector space equipped with a scalar product is called a **prehilbertian space**. In particular, it is thus a normed space for the induced norm.
- A prehilbertian space of finite dimension is called an **Euclidian space**.

Usual scalar products for vectors and matrices

- In $E = \mathbb{R}^n$: $\quad < \mathbf{x}, \mathbf{y} >= \sum_{i=1}^{n} x_i\, y_i = \mathbf{x}^T \mathbf{y}$ is called the **Euclidian scalar product** (because its induced norm is the usual Euclidian norm $\|.\|_2$).

- In $E = \mathbb{R}^n$: More generally, for any $n \times n$ symmetric positive definite matrix $\mathbf{A}$, one can define its associated scalar product $\quad < \mathbf{x}, \mathbf{y} >_{\mathbf{A}} = < \mathbf{x}, \mathbf{A}\mathbf{y} > = \mathbf{x}^T \mathbf{A}\mathbf{y} = \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij}\, x_i y_j$

  Its induced norm is (of course) the already defined norm $\|.\|_{\mathbf{A}}$.

- In $E = \mathcal{M}_{n,p}(\mathbb{R})$: the usual scalar product is $\langle \mathbf{A}, \mathbf{B} \rangle = \sum_{i=1}^{n} \sum_{j=1}^{p} A_{ij} B_{ij} = \sum_{i=1}^{n} (\mathbf{A}\mathbf{B}^T)_{ii} = \mathrm{Tr}\left(\mathbf{A}\mathbf{B}^T\right)$

  Its induced norm is the Frobenius norm $\|\mathbf{A}\| = \left(\sum_{i=1}^{n} \sum_{j=1}^{p} A_{ij}^2 \right)^{1/2}$.

Norms and scalar products for functions

Definition a **function space** is a vector space which elements are functions.

$L^p(\Omega)$ spaces ($p \in [1, +\infty[$, $\Omega \in \mathbb{R}^n$)

$$L^p(\Omega) = \left\{ u : \Omega \to \mathbb{R}, \text{ measurable, such that } \int_\Omega |u|^p < \infty \right\}$$

$L^p(\Omega)$ norms

$$\|u\|_{L^p} = \left(\int_\Omega |u|^p \right)^{1/p} \quad , \; p \in [1, +\infty[, \qquad \text{and} \qquad \|u\|_{L^\infty} = \text{Sup}_\Omega |u|$$

Euclidian (or L^2) norm and associated scalar product

$$\|u\|_{L^2} = \left(\int_\Omega u^2(\mathbf{x}) \, d\mathbf{x} \right)^{1/2} \qquad (u, v)_{L^2} = \int_\Omega u(\mathbf{x}) \, v(\mathbf{x}) \, d\mathbf{x}$$

Norms and scalar products for functions

Exercise #1

1. Prove that the set of functions $\left\{ \frac{1}{\sqrt{2}}, \cos\frac{2\pi kx}{L}, \sin\frac{2\pi kx}{L}, k \geq 1 \right\}$ is orthonormal for the scalar product $< f, g >= \frac{2}{L} \int_0^L f(x)\, g(x)\, dx$
2. What is then the interpretation of the following identity

$$f(x) = a_0 + \sum_{k \geq 1} \left(a_k \cos\frac{2\pi kx}{L} + b_k \sin\frac{2\pi kx}{L} \right)$$

for any $f \in \mathcal{C}^1(0, L)$?

Norms and scalar products for functions

Exercise #2 For 2 functions f and g defined on $\mathbb{R}_+$, let $\langle f, g\rangle = \int_0^{+\infty} f(x)g(x)\,dx$.

1. Prove that $\langle f, g\rangle$ is a scalar product on $\mathbb{R}[X]$ (polynomials with real coefficients)
2. Let $L_n(x) = \frac{e^x}{n!}\frac{d^n}{dx^n}\left(x^n\, e^{-x}\right)$. Give the analytical expression of L_0, L_1, L_2, L_3.
3. Prove that $(L_n)_{n\geq 0}$ is an orthonormal family in $\mathbb{R}[X]$.
4. Prove that $\forall n \geq 0, \quad xL_n'' + (1-x)L_n'(x) + nL_n(x) = 0 \qquad \forall x \in \mathbb{R}$.
5. Prove that $\forall n \geq 1, \quad (n+1)L_{n+1}(x) + (x - 2n - 1)L_n(x) + nL_{n-1}(x) = 0$.

Those Laguerre polynomials are useful in particular in quantum physics.

Norms and scalar products for functions

Projection / decomposition on an orthogonal family of basis functions is a fundamental tool in many domains of mathematics and physics.

Common examples are Fourier series, or orthogonal polynomials (e.g. Legendre, Laguerre, Hermite, Chebyshev...).

Some vocabulary for spaces

- A vector space equipped with a norm is a **normed space**.
- A vector space equipped with a scalar product is a **prehilbertian space**. (It is thus a normed space, for the induced norm).
- A finite dimension prehilbertian space is an **Euclidian space**.

Reminder on Cauchy sequences

Let E a vector space, and $(x_n)_n$ a sequence in E.
$(x_n)_n$ is a Cauchy sequence iff $\forall \varepsilon > 0, \;\; \exists N \,/\, \forall p, q > N, \quad \|x_p - x_q\| < \varepsilon$

Property Every convergent sequence is a Cauchy sequence. The converse is false.

Definition A vector space is **complete** iff every Cauchy sequence is convergent.

- A complete normed space is a **Banach space**.
- A complete prehilbertian space is a **Hilbert space**.
  $L^p(\Omega)$ equipped with the L^p norm is a Banach space (i.e. is complete).
  $L^2(\Omega)$ equipped with the L^2 scalar product is a Hilbert space.