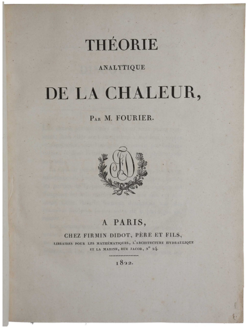
Fourier series and Fourier transforms



THÉORIE

ANALYTIQUE

DE LA CHALEUR,

PAR M. FOURIER.

A PARIS,

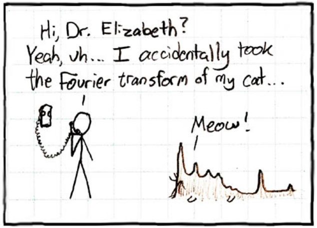
CHEZ FIRMIN DIDOT, PÈRE ET FILS,

LIBRAIRES POUR LES MATHÉMATIQUES, L'ARCHITECTURE HYDRAULIQUE

ET LA MARINE, RUE JACOB, N° 24.

1822.

J.B.J. Fourier (1768 - 1830)



Fourier series expansion

Let f an integrable and periodic function, with period L. One can then define

$$F(x) = a_0 + \sum_{k\geq 1}\left(a_k \cos\frac{2\pi kx}{L} + b_k \sin\frac{2\pi kx}{L}\right)$$

$$\text{with } a_0 = \frac{1}{L}\int_0^L f(x)\,dx\ , \quad a_k = \frac{2}{L}\int_0^L f(x)\,\cos\frac{2\pi kx}{L}\,dx\ , \quad b_k = \frac{2}{L}\int_0^L f(x)\,\sin\frac{2\pi kx}{L}\,dx$$

F is the so-called **Fourier series expansion** of f.

This expansion also reads $F(x) = \sum_{k=-\infty}^{+\infty} c_k\, e^{\frac{2i\pi kx}{L}}$ with $c_k = \frac{1}{L}\int_0^L f(x)\, e^{-\frac{2i\pi kx}{L}}\,dx$

Fourier series expansion

Odd and even functions

- If f is an even function, $b_k = 0 \quad \forall k \geq 1 \qquad$ (i.e. $c_k = c_{-k} \quad \forall k$)
- If f is an odd function, $a_k = 0 \quad \forall k \geq 0 \qquad$ (i.e. $c_k = -c_{-k} \quad \forall k$)

Pointwise convergence

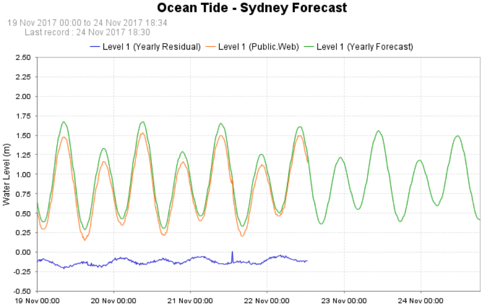
If f is $\mathcal{C}^1(0, L)$, then $F = f \qquad$ (note that some similar results exist which require less regularity for f)

Parseval's equality (conservation of energy)

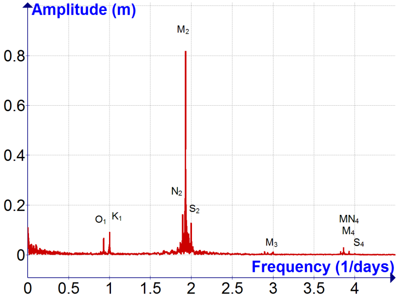
If $f \in \mathcal{L}^2(0, L)$, then $\quad \|f\|_{L^2}^2 = \frac{1}{L}\int_0^L (f(x))^2\, dx = a_0^2 + \frac{1}{2}\sum_{k=1}^{+\infty}(a_k^2 + b_k^2) = \sum_{k=-\infty}^{+\infty} |c_k|^2 = \|F\|_{L^2}^2$

(even without pointwise convergence)

Fourier analysis



time series



power spectrum

Exercise #1

1. Let the 1-D diffusion equation on a bounded domain

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) - \nu \dfrac{\partial^2 u}{\partial x^2}(x,t) = 0 & x \in (0,L), t > 0 \\ u(0,t) = u(L,t) = 0 \quad t > 0 \\ u(x,0) = u_0(x) \quad x \in (0,L) \end{cases}$$

   Solve this equation using a separation of variables technique.

2. Same question, replacing the boundary conditions $u(0,t) = u(L,t) = 0$ with $\dfrac{\partial u}{\partial x}(0,t) = \dfrac{\partial u}{\partial x}(L,t) = 0$.

Fourier transform

Let f integrable on $\mathbb{R}$.

The **Fourier transform** of f is $FT[f](\xi) = \widehat{f}(\xi) = \int_{\mathbb{R}} f(x)\, e^{-2i\pi\xi x}\, dx$

The **inverse Fourier transform** of $\widehat{f}$ is $FT^{-1}[\widehat{f}](x) = \int_{\mathbb{R}} \widehat{f}(\xi)\, e^{2i\pi\xi x}\, d\xi$

Some properties of the Fourier transform

Reciprocity If $f \in \mathcal{C}^1(\mathbb{R})$ and if $\widehat{f}$ is $L^1(\mathbb{R})$, then $FT^{-1}[\widehat{f}] = f$

Parseval's equality (conservation of energy) If $f \in \mathcal{L}^2(\mathbb{R})$, then $\displaystyle\int_{\mathbb{R}} |f(x)|^2\, dx = \int_{\mathbb{R}} \left|\widehat{f}(\xi)\right|^2\, d\xi$

Derivation $\widehat{f'}(\xi) = 2i\pi\xi\,\widehat{f}(\xi)$

Convolution $\widehat{f}\,\widehat{g} = \widehat{f * g}$ and $\widehat{f} * \widehat{g} = \widehat{fg}$

Reminder: convolution product $(a * b)(x) = \displaystyle\int_{\mathbb{R}} a(y)\, b(x-y)\, dy = \int_{\mathbb{R}} a(x-y)\, b(y)\, dy$

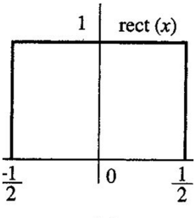
Some properties of the Fourier transform

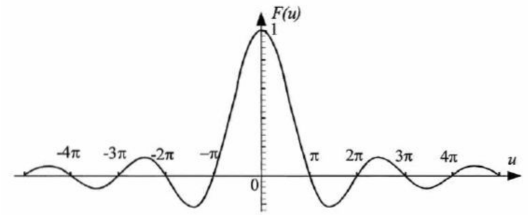
Translation If $g(x) = f(x - x_0)$, then $\widehat{g}(\xi) = e^{-2i\pi x_0 \xi}\,\widehat{f}(\xi)$

Gaussian functions

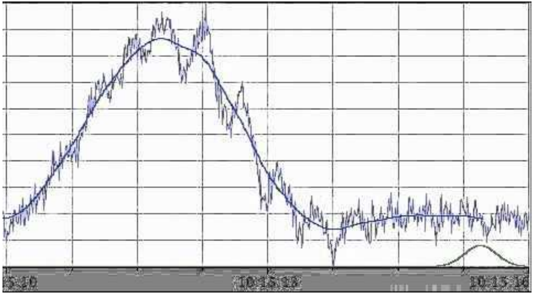
The Fourier transform of the Gaussian function $\exp(-\pi\alpha x^2)$ is the Gaussian function $\dfrac{1}{\sqrt{\alpha}}\exp\left(-\dfrac{\pi}{\alpha}\,\xi^2\right)$

Gate function The Fourier transform of the gate function $\Pi(x) = 1$ for $x \in (-1/2; 1/2)$ and 0 elsewhere is $\mathrm{sinc}(\pi\xi)$ where sinc is the **sine cardinal function** defined by $\mathrm{sinc}\, a = (\sin a)/a$.





Filtering, smoothing



Convolutions with a Gaussian function

Exercise #2

1. Let consider the diffusion equation $\dfrac{\partial u}{\partial t}(x,t) - \nu \dfrac{\partial^2 u}{\partial x^2}(x,t) = 0 \;\; (x \in \mathbb{R}, t > 0, \nu > 0)$ with the initial condition $u(x,0) = u_0(x)$. Solve this equation using a Fourier transform.

2. Same question with a source term: $\dfrac{\partial u}{\partial t}(x,t) - \nu \dfrac{\partial^2 u}{\partial x^2}(x,t) = f(x,t) \;\; (x \in \mathbb{R}, t > 0, \nu > 0)$ with the initial condition $u(x,0) = 0$.

3. Same question with an additional reaction term :
$$\frac{\partial u}{\partial t}(x,t) - \nu \, \frac{\partial^2 u}{\partial x^2}(x,t) + r\, u(x,t) = 0 \qquad (x \in \mathbb{R},\; t > 0, \nu > 0, r > 0)$$

Exercise #3

Let the diffusion equation:

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) - \nu\,\dfrac{\partial^2 u}{\partial x^2}(x,t) = 0 \qquad x > 0,\ t > 0 \\ u(0,t) = 0 \qquad t > 0 \\ u(x,0) = u_0(x) \qquad x > 0 \end{cases}$$

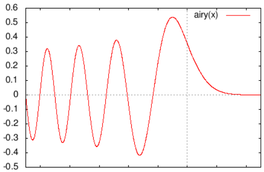
1. Solve this equation by extending u_0 on $\mathbb{R}_-$ into an odd function.
2. Solve again the equation when the boundary condition at $x = 0$ is replaced by $\dfrac{\partial u}{\partial x}(0,t) = 0$.

Exercise #4

The Airy equation, that appears for instance in biological fluid modeling, reads:

$$\begin{cases} \dfrac{\partial u}{\partial t} + k\,\dfrac{\partial^3 u}{\partial x^3} = 0 \qquad x \in \mathbb{R}, t > 0, \qquad k \in \mathbb{R} \text{ given} \\ u(x,0) = u_0(x) \qquad x \in \mathbb{R} \end{cases} \tag{1}$$

1. Using Fourier transform (assuming sufficient regularity), give the analytical expression of the solution $u(x,t)$ under the form of a convolution product.
2. Let the so-called Airy function:
   $$A(x) = \frac{1}{\pi}\int_0^{+\infty} \cos\left(sx + \frac{s^3}{3}\right)\, ds$$
   It can be shown that $\widehat{A}(\xi) = e^{\frac{i}{3}(2\pi\xi)^3}$ where $\widehat{\ }$ is the symbol of the Fourier transform.
   What does the analytical expression of $u(x,t)$ become, as a function of the Airy function A?



Hint: $\widehat{f(ax)} = \dfrac{1}{|a|}\,\widehat{f}\left(\dfrac{\xi}{a}\right)$