Partial differential operators

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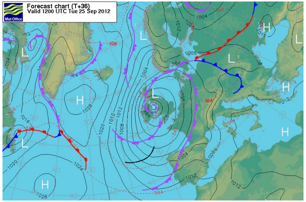
$$u : \begin{array}{rcl} \mathbb{R}^n & \longrightarrow & \mathbb{R} \\ \mathbf{x} = (x_1, \dots, x_n) & \longrightarrow & u(x_1, \dots, x_n) \end{array}$$

- Gâteaux derivative $$\frac{\partial u}{\partial \mathbf{d}}(\mathbf{x}) = \lim_{\alpha \to 0} \frac{u(\mathbf{x} + \alpha \mathbf{d}) - u(\mathbf{x})}{\alpha}$$

R. Gâteaux
(1889-1914)



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- **Gradient** $$\nabla u(\mathbf{x}) = \begin{pmatrix} \frac{\partial u}{\partial x_1}(\mathbf{x}) \\ \vdots \\ \frac{\partial u}{\partial x_n}(\mathbf{x}) \end{pmatrix}$$

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- **Gradient** $$\nabla u(\mathbf{x}) = \begin{pmatrix} \dfrac{\partial u}{\partial x_1}(\mathbf{x}) \\ \vdots \\ \dfrac{\partial u}{\partial x_n}(\mathbf{x}) \end{pmatrix}$$

$$\boxed{\frac{\partial u}{\partial \mathbf{d}}(\mathbf{x}) = \nabla u(\mathbf{x}).\ \mathbf{d}}$$

Partial differential operators: Jacobian

$$\mathbf{u}: \quad \mathbb{R}^n \longrightarrow \mathbb{R}^p$$

$$\mathbf{x} = (x_1, \ldots, x_n) \longrightarrow \mathbf{u}(x_1, \ldots, x_n) = \begin{pmatrix} u_1(x_1, \ldots, x_n) \\ \vdots \\ u_p(x_1, \ldots, x_n) \end{pmatrix}$$

Jacobian $$J(\mathbf{u})(\mathbf{x}) = \begin{pmatrix} \frac{\partial u_1}{\partial x_1}(\mathbf{x}) & \ldots & \frac{\partial u_1}{\partial x_n}(\mathbf{x}) \\ \vdots & & \vdots \\ \frac{\partial u_p}{\partial x_1}(\mathbf{x}) & \ldots & \frac{\partial u_p}{\partial x_n}(\mathbf{x}) \end{pmatrix}$$

Partial differential operators: Jacobian

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Jacobian $\qquad J(\mathbf{u})(\mathbf{x}) = \begin{pmatrix} \frac{\partial u_1}{\partial x_1}(\mathbf{x}) & \ldots & \frac{\partial u_1}{\partial x_n}(\mathbf{x}) \\ \vdots & & \vdots \\ \frac{\partial u_p}{\partial x_1}(\mathbf{x}) & \ldots & \frac{\partial u_p}{\partial x_n}(\mathbf{x}) \end{pmatrix}$

Exercise Let $F(x, y) = \begin{pmatrix} x^2 + y^2 \\ 2xy \end{pmatrix}$. Compute the Jacobian of F.

Exercise Let a 2D vector field $\mathbf{U}(x, y) = \begin{pmatrix} u(x, y) \\ v(x, y) \end{pmatrix}$ where u and v are given regular functions. Let $F(x, y) = \begin{pmatrix} \mathbf{U}(x, y) \cdot \nabla u(x, y) \\ \mathbf{U}(x, y) \cdot \nabla v(x, y) \end{pmatrix}$. What is the Jacobian of F? Can it be written in a more compact way? Can you make a parallel with usual derivation?

Reminder: Schwarz theorem

Let Ω an open subset of $\mathbb{R}^n$, and $\mathbf{a} \in \Omega$.
Let $f : \Omega \longrightarrow \mathbb{R}$.

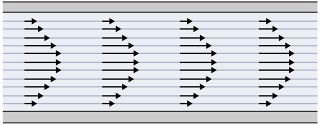
If f has continuous second partial derivatives on a neighborhood of $\mathbf{a}$, then

$$\forall i, j \in \{1, 2, \ldots, n\}, \qquad \frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{a}) = \frac{\partial^2 f}{\partial x_j \partial x_i}(\mathbf{a})$$

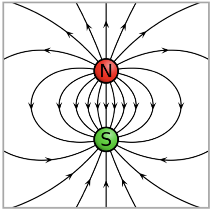
Partial differential operators: Divergence

$$\mathbf{u} : \quad \mathbb{R}^n \longrightarrow \mathbb{R}^n$$
$$\mathbf{x} = (x_1, \ldots, x_n) \longrightarrow \mathbf{u}(x_1, \ldots, x_n) = \begin{pmatrix} u_1(x_1, \ldots, x_n) \\ \vdots \\ u_n(x_1, \ldots, x_n) \end{pmatrix}$$

Divergence $\quad \operatorname{div} \mathbf{u}(\mathbf{x}) = \sum_{i=1}^{n} \frac{\partial u_i}{\partial x_i}(\mathbf{x})$ $\quad$ Also denoted $\nabla.\mathbf{u}(\mathbf{x})$



laminar flow



magnetic poles

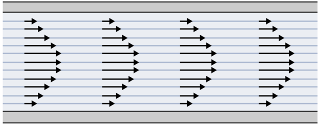
$$\operatorname{div} \; \mathbf{u} = 0$$

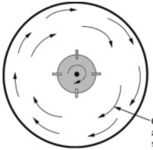
incompressible flow

Partial differential operators: Curl

$$\mathbf{u} : \begin{array}{rcl} \mathbb{R}^3 & \longrightarrow & \mathbb{R}^3 \\ \mathbf{x} = (x_1, x_2, x_3) & \longrightarrow & \mathbf{u}(x_1, x_2, x_3) = \begin{pmatrix} u_1(x_1, x_2, x_3) \\ u_2(x_1, x_2, x_3) \\ u_3(x_1, x_2, x_3) \end{pmatrix} \end{array}$$

Curl $\quad \operatorname{curl} \mathbf{u}(\mathbf{x}) = \begin{pmatrix} \dfrac{\partial u_3}{\partial x_2}(\mathbf{x}) - \dfrac{\partial u_2}{\partial x_3}(\mathbf{x}) \\ \dfrac{\partial u_1}{\partial x_3}(\mathbf{x}) - \dfrac{\partial u_3}{\partial x_1}(\mathbf{x}) \\ \dfrac{\partial u_2}{\partial x_1}(\mathbf{x}) - \dfrac{\partial u_1}{\partial x_2}(\mathbf{x}) \end{pmatrix}$ also denoted $\nabla \wedge \mathbf{u}(\mathbf{x})$





Hessian matrix

$$\begin{array}{rcl} u: \quad \mathbb{R}^n & \longrightarrow & \mathbb{R} \\ \mathbf{x} = (x_1, \ldots, x_n) & \longrightarrow & u(x_1, \ldots, x_n) \end{array}$$

$$\mathrm{Hess}(u)(\mathbf{x}) = \begin{pmatrix} \dfrac{\partial^2 u}{\partial x_1^2}(\mathbf{x}) & \dfrac{\partial^2 u}{\partial x_1 \partial x_2}(\mathbf{x}) & \ldots & \dfrac{\partial^2 u}{\partial x_1 \partial x_n}(\mathbf{x}) \\ \vdots & & & \vdots \\ \dfrac{\partial^2 u}{\partial x_n \partial x_1}(\mathbf{x}) & \dfrac{\partial^2 u}{\partial x_n \partial x_2}(\mathbf{x}) & \ldots & \dfrac{\partial^2 u}{\partial x_n^2}(\mathbf{x}) \end{pmatrix}$$

Schwarz theorem Let Ω an open subset of $\mathbb{R}^n$, and $\mathbf{a} \in \Omega$. Let $f : \Omega \longrightarrow \mathbb{R}$.

If f has continuous second partial derivatives on a neighborhood of $\mathbf{a}$, then $\mathrm{Hess}(u)(\mathbf{a})$ is symmetric.

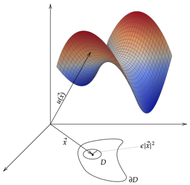
Partial differential operators: Laplacian

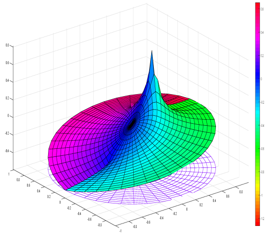
$$u : \begin{array}{rcl} \mathbb{R}^n & \longrightarrow & \mathbb{R} \\ \mathbf{x} = (x_1, \ldots, x_n) & \longrightarrow & u(x_1, \ldots, x_n) \end{array}$$

$$\mathbf{u} : \begin{array}{rcl} \mathbb{R}^n & \longrightarrow & \mathbb{R}^p \\ \mathbf{x} = (x_1, \ldots, x_n) & \longrightarrow & \mathbf{u}(x_1, \ldots, x_n) = \begin{pmatrix} u_1(x_1, \ldots, x_n) \\ \vdots \\ u_p(x_1, \ldots, x_n) \end{pmatrix} \end{array}$$

Laplacian $\quad \Delta u(\mathbf{x}) = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}(\mathbf{x}) = \mathrm{Tr}\,(\mathrm{Hess}(u)(\mathbf{x}))$

$$\Delta \mathbf{u} = \begin{pmatrix} \Delta u_1 \\ \vdots \\ \Delta u_p \end{pmatrix}$$





Harmonic functions: $\Delta u = 0$

Exercises

1. Let $u(x, y) = 2x^2y + y^3$. Compute ∇u and Δu.

2. For the same u, compute $\dfrac{\partial u}{\partial \mathbf{d}}$ for $\mathbf{d} = (1, -1)$.

3. Let $\mathbf{u}(x, y, z) = (xy^2 - z^2, x^3 - y^3, x^2 - 2z)$. Compute div $\mathbf{u}$.

1. Let $u(x, y) = 2x^2y + y^3$. Compute ∇u and Δu.

$$\nabla u = \begin{pmatrix} \dfrac{\partial u}{\partial x}(x, y) \\ \\ \dfrac{\partial u}{\partial y}(x, y) \end{pmatrix} = \begin{pmatrix} 4xy \\ \\ 2x^2 + 3y^2 \end{pmatrix}$$

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 4y + 6y = 10y$$

2. For the same u, compute $\dfrac{\partial u}{\partial \mathbf{d}}$ for $\mathbf{d} = (1, -1)$.

$$\frac{\partial u}{\partial \mathbf{d}} = \nabla u.\ \mathbf{d} = \begin{pmatrix} 4xy \\ 2x^2 + 3y^2 \end{pmatrix} . \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 4xy - 2x^2 - 3y^2$$

3. Let $\mathbf{u}(x,y,z) = (xy^2 - z^2, x^3 - y^3, x^2 - 2z)$. Compute div $\mathbf{u}$.

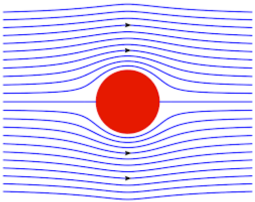
$$\begin{aligned}
\text{div } \mathbf{u} &= \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} + \frac{\partial u_3}{\partial z} \\
&= y^2 - 3y^2 - 2 \\
&= -2y^2 - 2
\end{aligned}$$

Exercises

1. Let $\varphi : \mathbb{R}^3 \to \mathbb{R}$. Compute $\mathrm{curl}(\nabla\varphi)$.

2. Let $\varphi : \mathbb{R}^n \to \mathbb{R}$. Compute $\mathrm{div}(\nabla\varphi)$.

3. Let $\psi : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$. $(u, v) = (\partial\psi/\partial y, -\partial\psi/\partial x)$ is the *vector field derived from the streamfunction* ψ. Prove that the vector field is everywhere tangent to the isolines of ψ. Compute the divergence of the vector field.



Exercise: spectrum of the Laplacian operator

Let $\Omega \subset \mathbb{R}^n$ a bounded domain, and consider the following eigenvalue problem:

$$\begin{cases} \Delta u(\mathbf{x}) = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}(x_1, \ldots, x_n) = \lambda\, u(x_1, \ldots, x_n) & \mathbf{x} \in \Omega \\ u(\mathbf{x}) = 0 \quad \text{on } \partial\Omega \end{cases}$$

1. Particular case $n = 1$: Let $\Omega = (0, L)$ and find the eigenvalues and eigenfunctions.
2. Generalization for any value of n:
   Prove that all eigenvalues are negative.
   Prove that eigenfunctions associated to different eigenvalues are orthogonal.

Exercise: spectrum of the Laplacian operator

1. 1-D case: $\Omega = (0, L)$. The eigenvalue problem reads $u''(x) = \lambda\, u(x) \quad x \in (0, L)$, with $u(0) = u(L) = 0$. $\lambda < 0$ and can be written $\lambda = -\omega^2$ (otherwise the only solution is $u = 0$). Hence $u''(x) + \omega^2 u(x) = 0$, which yields $u(x) = \alpha \sin \omega x + \beta \cos \omega x$. $u(0) = 0$ implies $\beta = 0$, while $u(L) = 0$ implies $\alpha \sin \omega L = 0$.

   Non zero solutions are then obtained for $\quad \omega_k = \dfrac{k\pi}{L} \quad$ and $\quad u_k(x) = \sin \dfrac{k\pi x}{L} \quad , k \in \mathbb{N}$

Exercise: spectrum of the Laplacian operator

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   Non zero solutions are then obtained for $\quad \omega_k = \dfrac{k\pi}{L} \quad$ and $\quad u_k(x) = \sin \dfrac{k\pi x}{L} \quad , k \in \mathbb{N}$
2. All eigenvalues are negative: $(\Delta u - \lambda u = 0) \implies \displaystyle\int_\Omega u \Delta u = -\int_\Omega \|\nabla u\|^2 = \lambda \int_\Omega u^2$.

   Hence $\qquad \lambda = -\dfrac{\int_\Omega \|\nabla u\|^2}{\int_\Omega u^2} \leq 0.$

   Eigenfunctions associated to different eigenvalues are orthogonal:
   Let u_k and u_l two eigenfunctions associated to two different eigenvalues $-\omega_k^2$ and $-\omega_l^2$.

$$\begin{cases} \Delta u_k + \omega_k^2 u_k = 0 \quad \implies \int_\Omega \Delta u_k\, u_l + \omega_k^2 \int_\Omega u_k\, u_l = -\int_\Omega \nabla u_k\, \nabla u_l + \omega_k^2 \int_\Omega u_k\, u_l = 0 \\ \Delta u_l + \omega_l^2 u_l = 0 \quad \implies \int_\Omega \Delta u_l\, u_k + \omega_l^2 \int_\Omega u_l\, u_k = -\int_\Omega \nabla u_l\, \nabla u_k + \omega_l^2 \int_\Omega u_l\, u_k = 0 \end{cases}$$

   Making the difference between those two equations yields $(\omega_k^2 - \omega_l^2) \displaystyle\int_\Omega u_l\, u_k = 0$, hence $\displaystyle\int_\Omega u_l\, u_k = 0$.

   Note that this also implies $\displaystyle\int_\Omega \nabla u_l\, \nabla u_k = 0$. u_k and u_l are orthogonal both in $L^2(\Omega)$ and in $H^1(\Omega)$.