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Adaptive level set evolution starting with a constant function

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ABSTRACT

In this paper, we propose a novel level set evolution model in a partial differential equation (PDE) formulation. According to the governing PDE, the evolution of level set function is controlled by two forces, an adaptive driving force and a total variation (TV)-based regularizing force that smoothes the level set function. Due to the adaptive driving force, the evolving level set function can adaptively move up or down in accordance with image information as the evolution proceeds forward in time. As a result, the level set function can be simply initialized to a constant function rather than the widely-used signed distance function or piecewise constant function in existing level set evolution models. Our model completely eliminates the needs of initial contours as well as re-initialization, and so avoids the problems resulted from contours initialization and re-initialization. In addition, the evolution PDE can be solved numerically via a simple explicit finite difference scheme with a significantly larger time step. The proposed model is fast enough for near real-time segmentation applications while still retaining enough accuracy; in general, only a few iterations are needed to obtain segmentation results accurately.

1. Introduction

Image segmentation has been, and still is, an important area in image analysis and computer vision. For a given image, its goal is to separate the image domain into a set of regions which are visually distinct and uniform with respect to certain properties, such as grey level, texture or color.

To perform the image segmentation task, many successful techniques including geometric active contour models using the level set method [1] have been presented. The use of level set method allows us to handle complex geometry and even changing topology, without the need of user-interaction [2]. According to the way representing the surface, applications of the level set method in image segmentation can be typically divided into two classes: the standard level set method [3–9] and the piecewise constant level set method [10–12]. The surface is represented by the zero level set of a Lipschitz function in the standard level set method, while it is represented by discontinuities of a piecewise constant function in the piecewise constant level set framework. In this paper, we confine our discussion to the standard level set method for image segmentation.

In the standard level set method [1,3,4], the level set function can develop shocks, very sharp and/or flat shape during the evolution, which makes further computation highly inaccurate. To avoid these problems, a common numerical scheme is to initialize the level set function to a signed distance function to initial contour before the evolution, and then to periodically re-initialize the level set function to be a signed distance function to the evolving curve during the evolution. Indeed, such initialization and re-initialization are crucial and cannot be avoided in the standard level set method. From the practical viewpoints, however, such initialization and re-initialization are fraught with its own problems, such as the determinations

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of where to define the initial contour and when and how to re-initialize the level set function to a signed distance function [13]. Recently, Li et al. [6,7] proposed a novel level set evolution model without re-initialization. It employed a deviation penalization energy to preserve the level set function close to a signed distance function, thus not only the re-initialization is entirely eliminated, but also the level set function can be initialized to a piecewise constant function. Unfortunately, the initial piecewise constant function also needs to be computed from an initial contour or several initial contours that partition the image domain into different regions; it still has the problems resulted from contours initialization, such as how and where to define the initial contours.

In this paper, we propose a new level set evolution model in a partial differential equation (PDE) formulation. According to the evolution PDE, the level set evolution is controlled by two forces, an adaptive driving force that ensures the level set function to adaptively move up or down according to image information, a TV (total variation)-based regularizing force that smoothes the level set function. Due to the adaptive driving force, the level set function can be simply initialized as a positive constant function rather than the signed distance function [3–5,9] or piecewise constant function [6–8], and thus completely eliminates the need for initial contours. Also, re-initialization is not necessary in our model because the level set function is no longer required to keep as a signed distance function. In addition, a simple explicit finite difference scheme with a significantly larger time step can be used for solving the evolution PDE numerically.

The remainder of this paper is organized as follows. Section 2 reviews an early geometric active contour [3], the level set evolution model without re-initialization [6,7] and their limitations. The idea of TV-based regularization [14] in image restoration is also summarized in this section. The proposed model is introduced in Section 3. Section 4 presents numerical algorithm and validates our model by experiments, followed by the discussions of the parameters in Section 5. This paper is summarized in Section 6.

2. Related works

2.1. An early geometric active contour

An early geometric active contour, introduced by Caselles et al. [3], is based on the theory of curve evolution [15] and the level set method [1].

Let $\phi(x,y,t)$ be a Lipschitz function whose zero level set defines the evolving curve $C(x,y,t)$. In the level set method, the evolution of the curve $\phi(x,y,t)$ along its normal direction with speed F is implicitly defined via the following nonlinear evolution PDE [1]:

$$\frac{\partial \phi}{\partial t} = F|\nabla \phi|, \tag{1}$$

with initial condition $\phi(x,y,0) = \phi_0(x,y)$, where $\phi_0(x,y)$ is the initial level set function corresponding to the initial curve $C(x,y,0)$.

A particular case is the motion derived by mean curvature, where $F = \mathrm{div}\,(\nabla\phi/|\nabla\phi|) = \kappa$ is the curvature of the level curve of ϕ passing though (x,y). The Eq. (1) becomes

$$\frac{\partial \phi}{\partial t} = \kappa|\nabla \phi|. \tag{2}$$

An early geometric active contour model [3] based on Eq. (2) is given by the following evolution equation:

$$\begin{cases} \frac{\partial \phi}{\partial t} = g(|\nabla I)(\kappa + v)|\nabla \phi| \\ \phi(x,y,0) = \phi_0(x,y), \end{cases} \tag{3}$$

where $v \geqslant 0$ is a constant serving as a balloon force, and $g(|\nabla I|) = \frac{1}{1+|\nabla(G_\sigma * I)|^2}$ is an edge-stopping function derived from the image. In Eq. (3), the term $g(|\nabla I|)(\kappa + v)$ determines the overall evolution speed of level sets of $\phi(x,y,t)$ along their normal direction. The use of curvature κ has the effect of smoothing the contour, while the use of v has the effect of shrinking or expanding contour at a constant speed. The speed of contour evolution is coupled with the image data through a multiplicative stopping term $g(|\nabla I|)$.

In implementing this model via the level set method, it is necessary to initialize the level set function to a signed distance function and periodically reshape it during the evolution by the following PDE [16]:

$$\frac{\partial \phi}{\partial t} = sign(\hat{\phi})(1 - |\nabla \phi|), \tag{4}$$

with initial condition $\phi(x,y,0) = \hat{\phi}(x,y)$, where $\hat{\phi}$ is the function to be re-initialized, and $sign(\cdot)$ is the signum function. When the steady state of the initial value problem (4) is reached, ϕ will be a distance function having the same zero level set as $\hat{\phi}$. This is commonly known as the re-initialization procedure in the level set formwork.

The initial signed distance function is defined by an initial contour or several initial contours in the image domain. Naturally, the problem of contour initialization arises; we have to choose suitable initial contours to correctly detect objects of interest in a given image, e.g., the initial contour must encircle all the objects to be detected or several contours must be used. In addition, the re-initialization is crucial and cannot be avoided as a numerical remedy for evolution. However, from the practical viewpoints, it can be quite complicated and expensive, and is fraught with its own problems, such as when and how to reinitialize.

2.2. Level set evolution model without re-initialization

To the problem of re-initialization, Li et al. [6,7] recently introduced a variational level set formulation that eliminates the need for re-initialization. In their model, the level set evolution PDE is directly derived from the minimization of an energy functional of level set functions.

They first proposed an energy functional, then minimized it using the standard steepest descent method and obtained the level set evolution equation as follows [6]:

$$\frac{\partial \phi}{\partial t} = \mu\left(\Delta\phi - \mathrm{div}\left(\frac{\nabla\phi}{|\nabla\phi|}\right)\right) + \lambda\delta(\phi)\mathrm{div}\left(g\frac{\nabla\phi}{|\nabla\phi|}\right) + v g\delta(\phi), \tag{5}$$

where μ, $\lambda > 0$ and v are constants. The first term penalizes the deviation of ϕ from a signed distance function and plays a key role in this model. The second and third term, corresponding to the gradient flows of the weighted length of the zero level set of ϕ and the weighted area of the region $\{(x,y)|\phi(x,y) < 0\}$, respectively, would smoothly drive the motion of the zero level set toward the desired edges. Dirac function $\delta(\cdot)$ is the derivative of one-dimensional Heaviside function.

Due to the penalizing term, the level set function driven by Eq. (5) is naturally and automatically kept as an approximate signed distance function during the evolution; therefore, the re-initialization procedure is completely eliminated. Also, this level set evolution model has two main advantages over the traditional level set formulations. First, it can be implemented using a simple finite difference scheme instead of the complex upwind scheme as in traditional level set formulations, and a significantly larger time step can be used for solving Eq. (5) numerically. Second, the level set function ϕ (x,y,t) can be efficiently initialized by a piecewise constant function as follows:

$$\phi_0(x,y) = \begin{cases} -\rho, & (x,y) \in R, \\ 0, & (x,y) \in \partial R, \\ +\rho, & (x,y) \in \Omega - \overline{R}, \end{cases} \tag{6}$$

where R is an arbitrary open region in the image domain Ω, $\rho > 0$ is a constant.

However, the initial level set function in Eq. (6) is in fact defined by the initial contour ∂R. Such initialization of level set function still cannot avoid the problem of contour initialization, such as where and how to initialize the contours.

2.3. TV-based regularization

TV-based regularization was first introduced by Rudin et al. in their pioneering work [14] for image restoration problems. Over the years, the ROF model has been extended to many other image restoration tasks, and has been modified in a variety of ways to improve its performance [17–19]. The revolutionary aspect of this model is its regularization term that allows for discontinuities but at the same time disfavors oscillations. The ROF model was originally formulated in [14] for gray image in the following form:

$$\min_{\frac{1}{2}\int_\Omega (u-u_0)^2 dxdy = \sigma^2} \int_\Omega |\nabla u| dxdy. \tag{7}$$

Here, the function $u_0(x,y) : \Omega \to \mathbb{R}$ represents the given observed image, which is assumed to be corrupted by Gaussian noise of variance σ^2. The objective functional in Eq. (7) is the total variation (TV) of the function $u(x,y)$.

Regardless of the fitting constrain, the minimization of the objective functional can be achieved by solving the following Euler–Lagrange equation:

$$\frac{\partial u}{\partial t} = \mathrm{div}\left(\frac{\nabla u}{|\nabla u|}\right), u|_{t=0} = u_0, \tag{8}$$

where div $(\nabla u/|\nabla u|)$ actually denotes the mean curvature of the level set of u passing through (x,y). The effects of Eq. (8) in noise removal can be explained as follows. The level curves in the neighborhoods of noise points on the image have high curvatures. The level curves of the viscosity solution to Eq. (8) shrink with the speed of the mean curvature and eventually disappear. Consequently, the level curves with very high curvature (noise) disappear much faster than those with relatively lower curvatures. As the time t proceeds, the result of TV regularization tends to be a smoother version of the original image u_0.

3. The proposed model

As we all know, the level set evolution models in [3–9] are highly sensitive to the contour initialization in terms of segmentation result and iteration number. One important reason is that their level set functions must be initialized to either a signed distance function [3–5,9] or a piecewise constant function [6–8], all of which are computed from an initial contour or several initial contours in image domain. In this section, we propose a novel level set evolution model, in which the level set function can be simply initialized to a positive constant function, and so completely eliminates the need for initial contours.

With our model, the evolution of the level set function ϕ is controlled by two forces, an adaptive driving force F_{adp} and a TV-based regularizing force F_{reg}, according to the following governing equation:

$$\frac{\partial\phi}{\partial t} = \alpha F_{adp} + \beta F_{reg}, \tag{9}$$

where both α and β are positive constants controlling the weight of the corresponding term. The details of these forces are discussed in the subsequent subsections.

3.1. Adaptive driving force

For a given image $I : \Omega \to \mathbb{R}$ and a Lipschitz function $\phi(x,y) : \Omega \to \mathbb{R}$, where $\Omega \subset \mathbb{R}^2$ is the image domain, we define the adaptive driving force as

$$f(I,c_1,c_2) = sign\Big(I(x,y) - \frac{c_1 + c_2}{2}\Big), \tag{10}$$

with

$$\begin{cases} c_1(\phi) = \frac{\int_\Omega I(x,y)H(\phi(x,y))dxdy}{\int_\Omega H(\phi(x,y))dxdy}, & \text{if } \int_\Omega H(\phi(x,y))dxdy \neq 0, \\ c_2(\phi) = \frac{\int_\Omega I(x,y)H(-\phi(x,y))dxdy}{\int_\Omega H(-\phi(x,y))dxdy}, & \text{if } \int_\Omega H(-\phi(x,y))dxdy \neq 0, \end{cases} \tag{11}$$

where $H(\cdot)$ is the one-dimensional Heaviside function. For the corresponding "degenerate" cases (i.e. $\int_\Omega H(\phi(x,y))dxdy = 0$, or $\int_\Omega H(-\phi(x,y))dxdy = 0$), the values of both $c_1(\phi)$ and $c_2(\phi)$ are initialized to the intensity average of the whole image. For the "nondegenerate" cases, $c_1(\phi)$ and $c_2(\phi)$ are in fact the intensity averages of $I(x,y)$ in the regions $\{(x,y)|\phi > 0\}$ and $\{(x,y)|\phi < 0\}$, respectively.

The interesting property of $f(I,c_1,c_2)$ can easily be seen from a simple example as follows. We assume that the image $I(x,y)$ is formed by two regions R_0 and R_1 of piecewise constant intensities with different values 200 and 100, respectively, where the region R_0 represents the object (square) (see Fig. 1); mathematically,

$$I(x,y) = \begin{cases} 200, (x,y) \in R_0, \\ 100, (x,y) \in R_1. \end{cases} \tag{12}$$

Besides, the function $\phi(x,y)$ is defined by a closed curve C (e.g., the signed distance function to the curve C), having the following properties:

$$\begin{cases} \phi(x,y) > 0, & (x,y) \in in(C), \\ \phi(x,y) = 0, & (x,y) \in C, \\ \phi(x,y) < 0, & (x,y) \in out(C), \end{cases} \tag{13}$$

where $in(C)$ and $out(C)$ stand for the "inside" and "outside" regions divided by the curve C, respectively.

For all possible cases in the position of the curve, it is easily obtained that

$$100 < \frac{c_1 + c_2}{2} < 200. \tag{14}$$

This can be seen easily as illustrated in Fig. 1, where four cases may happen according to the position of the curve C. If the curve C is outside the object (Fig. 1(a)), then $100 < c_1 < 200$, $c_2 = 100$; thus $100 < \frac{c_1+c_2}{2} < 150 < 200$. If the curve C is inside the object (Fig. 1(b)), then $c_1 = 200$, $100 < c_2 < 200$; thus $100 < 150 < \frac{c_1+c_2}{2} < 200$. If the curve C is partially inside and outside the object (Fig. 1(c)), then $100 < c_1 < 200$, $100 < c_2 < 200$; thus $100 < \frac{c_1+c_2}{2} < 200$. Finally, if the curve C is just on the boundary of the object (Fig. 1(d)), then $c_1 = 200$, $c_2 = 100$; thus $100 < \frac{c_1+c_2}{2} = 150 < 200$.

It follows from Eq. (14) that the values of $f(I,c_1,c_2)$ are positive and negative inside and outside the object (square), respectively, for all possible cases in the position of the curve; also see Fig. 1(e)–(h).

In order to stop the zero level set evolution just on the desired edges, we multiply $f(I,c_1,c_2)$ by the following edge-stopping function:

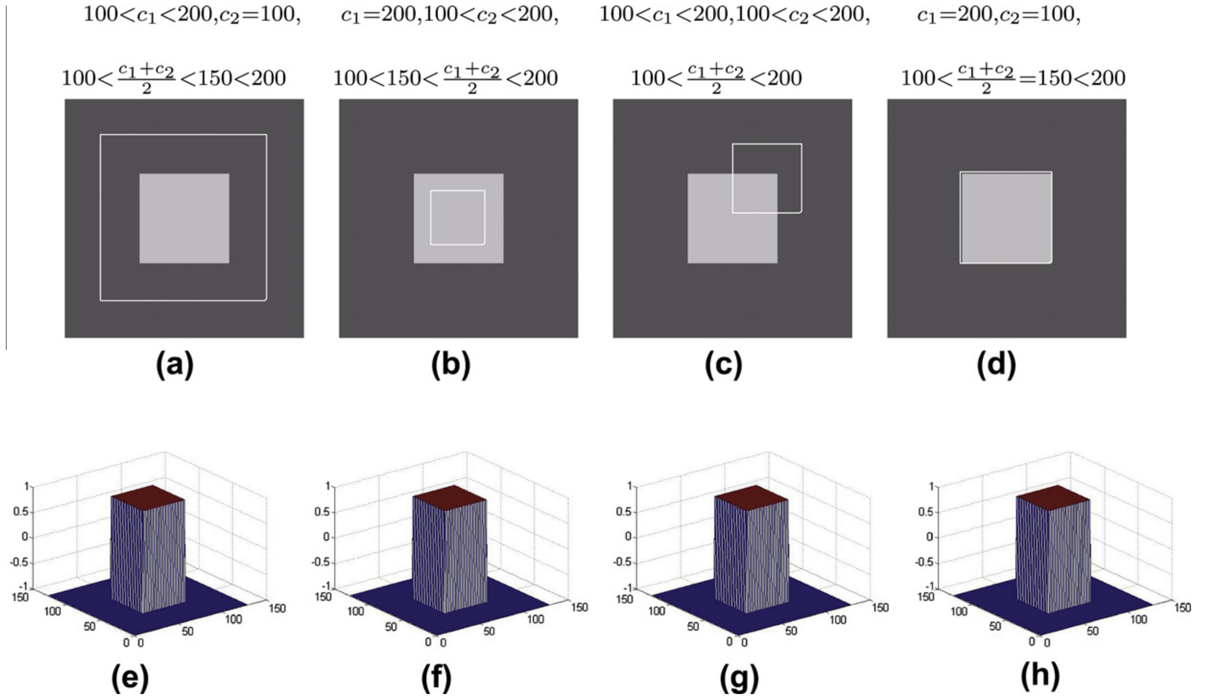

Fig. 1. For all possible cases in the position of the curve, Eq. (14) holds, and values of $f(I,c_1,c_2)$ are positive and negative inside and outside the object (square), respectively. (a)–(d): A binary image together with four distinct locations of the curve; (e)–(f): 3-D plots of the corresponding $f(I,c_1,c_2)$.

$$g(|\nabla I_\sigma|) = \exp\left(-\frac{|\nabla I_\sigma|}{20}\right), \tag{15}$$

where $I_\sigma = G_\sigma * I$, G_σ is a Gaussian kernel with standard deviation σ. The adaptive driving force F_{adp} is thus defined as

$$F_{adp}(I,\phi) = g(|\nabla I_\sigma|) sign\left(I(x,y) - \frac{c_1 + c_2}{2}\right), \tag{16}$$

where c_1 and c_1 are given in Eq. (11).

The level set function ϕ driven by the following evolution PDE:

$$\frac{\partial \phi}{\partial t} = F_{adp}(I,\phi) \tag{17}$$

can adaptively move up or down according to image information due to the sign variability of the force $F_{adp}(I,\phi)$. This can be clearly seen from a simple experiment for a binary image as shown in Fig. 2, in which the level set function ϕevolves according to Eq. (17), starting with a constant function shown in Fig. 2(b). Fig. 2(c) shows the 3-D plot of the level set function after

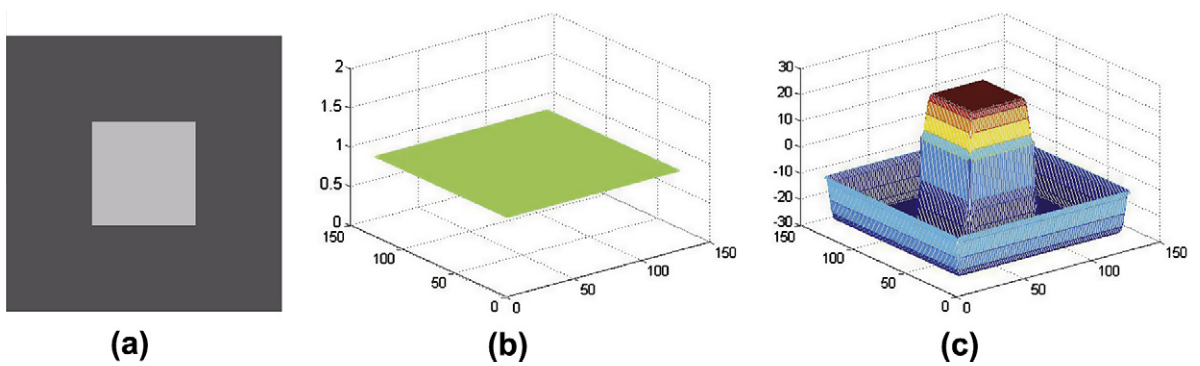

Fig. 2. Contribution of the adaptive driving force F_{adp} to level set evolution. (a) A binary image (128×128); (b) 3-D plot of initial constant level set function; (c) Level set function after one iteration using Eq. (17).

only one iteration. It is obviously seen that the level set function ϕ really moves up inside the object (square) and moves down outside the object. This can be easily explained as follows: the positive (negative) values of $F_{adp}(I,\phi)$ makes the level set function move up (down) inside (outside) the object. The property that the level set function adaptively moves up or down according to image information allows the level set function to be initialized to a constant function. We will discuss this further in next subsection.

3.2. TV-based regularizing force

As the existing level set evolution models, for our model it is also necessary to add a regularizing term into the evolution equation to smooth the level set function ϕ. About regularization, most of models focus on penalizing the length of contours (zero level set), in the spirit of the Mumford–Shah functional [20]. For our model, we pursue the idea of TV regularization [14] in image denoising and define the following regularizing force for the level set function ϕ:

$$F_{reg} = g(|\nabla I_\sigma|)\mathrm{div}\left(\frac{\nabla\phi}{|\nabla\phi|}\right), \tag{18}$$

where g is still defined by Eq. (15).

TV-based regularization was proposed for removing noise while preserving the edges in an image. In applied mathematics, TV-based models and analysis appear in more classical applications such as elasticity and fluid dynamics. Due to the ROF model [14], this notion has now become central also in image processing [21]. Here, we develop its idea and apply it to regularize the evolving level set function.

With the above defined forces F_{adp} and F_{reg}, our model is expressed as

$$\frac{\partial\phi}{\partial t} = g(|\nabla I_\sigma|)\left(\alpha f(I,c_1,c_2) + \beta\mathrm{div}\left(\frac{\nabla\phi}{|\nabla\phi|}\right)\right) \tag{19}$$

with the initial and Neumann boundary conditions:

$$\phi(x,y,0) = \phi_0(x,y) \text{ in } \Omega, \quad \frac{\partial\phi}{\partial n} = 0 \text{ on } \partial\Omega$$

The effect of the regularizing force in Eq. (19) is illustrated in Fig. 3. The first row shows the test image, which was obtained by adding Gaussian noise with standard deviation 0.01 to a binary image shown in Fig. 2(a). The level set function evolves according to Eq. (19) starting with a constant function shown in Fig. 2(b). In our experiment, we fix $\alpha = 5$, and then take $\beta = 0$ and $\beta = 10$, respectively. The segmentation results and the 3-D plots of the corresponding level set function are shown in the last two rows of Fig. 3. It is obvious that in the case of $\beta = 0$, the level set function ϕ oscillates disorderly in the background due to noise, while in the case of $\beta = 10$, as expected, the level set function ϕ tends to keep "smooth", whose zero level set becomes the contour that separates the object from the background.

It should be noted that if the image is noise-free, the effect of this regularization term can be neglected. However, real images are always disturbed by noise more or less, so the regularization term must be included in our model.

3.3. Constant initialization of level set function

In our formulation, the adaptive driving force F_{adp} can drive the level set function to move up or down automatically according to image information, and thus the resulting level evolution can be implemented by taking a constant initialization scheme. Let Ω be the image domain, the initial level set function is defined as:

$$\phi_0(x,y) = \rho, \quad (x,y) \in \Omega \tag{20}$$

where ρ is a positive constant.

The constant function in Eq. (20) is computed from the entire image domain, while both signed distance and piecewise constant level set functions are computed from an initial contour or several initial contours in image domain. Such constant initialization of level set function is not only computationally efficient, but also extremely convenient in practice; we need not to consider the problems such as where and how to initialize the contours and only need to choose an arbitrary positive constant as the initial level set function.

It should be pointed out that for many models, such as in [6,7], the level set function cannot be simply initialized to a positive constant because the level set function starting with such initial function does not change during evolution at all. This means that such models must have the initial contours. For our model, we can choose a positive constant as the initial level set function. Although there is no zero level set for such positive constant function, the initial level set function can move up or down according to the image information adaptively as the evolution proceeds forward in time. As a result, the zero level set can be generated automatically and finally converges to the object edges.

4. Numerical algorithm and results

4.1. Numerical algorithm

Eq. (19) is implemented via a simple explicit finite difference scheme; the temporal and spatial partial derivatives are approximated by the forward and central differences, respectively. There are several possible choices for the discretization

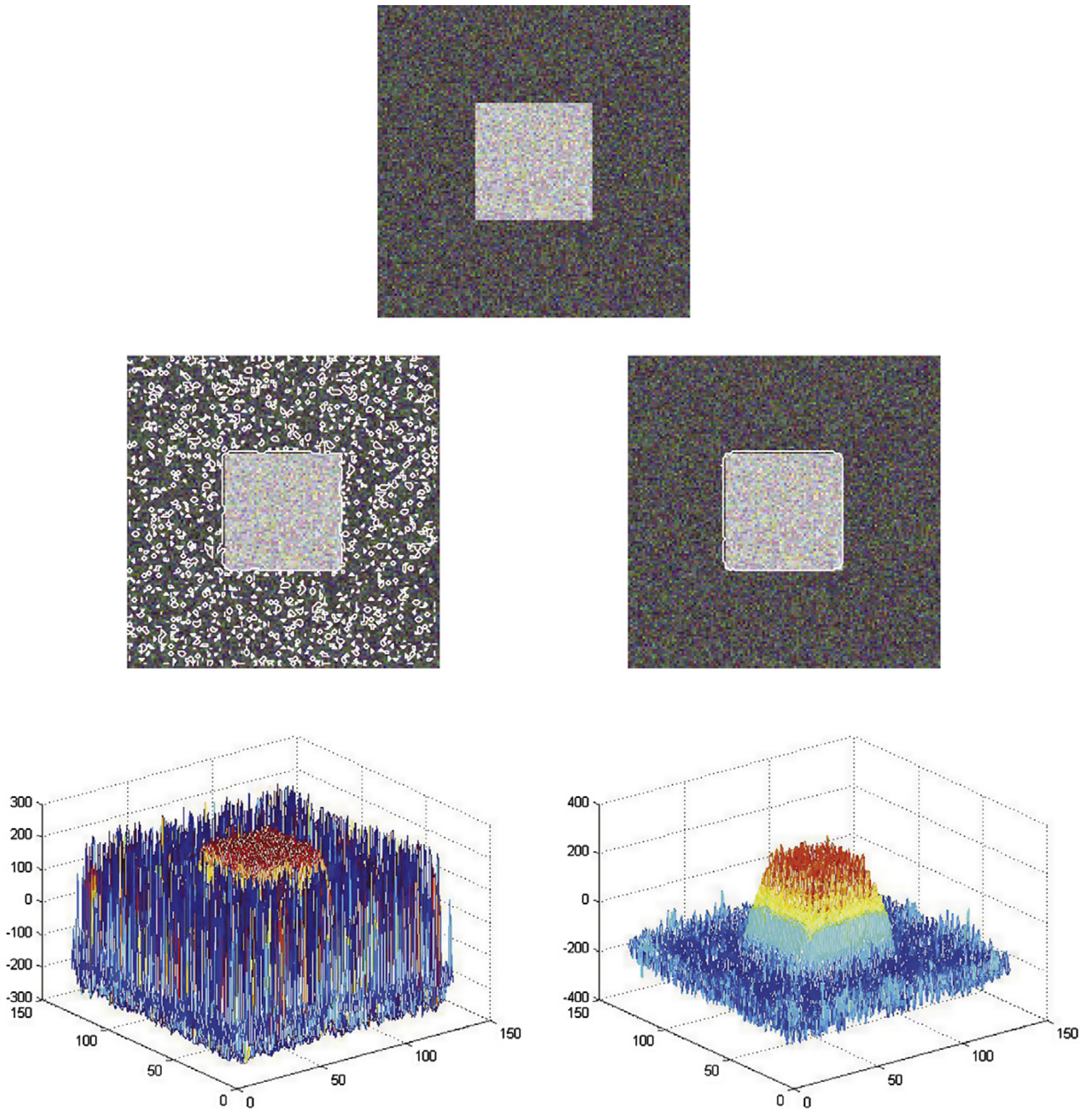

Fig. 3. Contribution of the regularizing force F_{reg} to level set evolution. Upper row: Noisy image (128 × 128) with Gaussian noise of standard deviation 0.01; Middle row: Final results after 10 iterations using Eq. (19) with $\alpha = 5$, but $\beta = 0$ and $\beta = 10$, respectively, starting with the constant function $\phi = 1$; Lower row: 3-D plots of the corresponding level set functions.

of the TV term div $(\nabla\phi/|\nabla\phi|)$; see [14,22,23] for example. The work in [22] shows that the central difference is suitable for the discretization of the TV term. Thus, we argue that the central difference scheme should be also suitable for the TV-based regularizing force in the proposed model; indeed, we will see that this numerical scheme does remain numerically stable for a large time step. Besides, in order to avoid the issue of singularity at $|\nabla\phi| = 0$, $|\nabla\phi|$ is replaced with

$$|\nabla\phi|_\varepsilon = (|\nabla\phi|^2 + \varepsilon^2)^{1/2}, \quad 0 < \varepsilon \ll 1, \tag{21}$$

which is a smooth approximation of $|\nabla\phi|$ [24].

Before starting with algorithm, we recall the standard notations for the finite difference scheme. Let h and Δt be the space and time steps, respectively, $(x_i, y_j) = (ih, jh)$ $(1 \leqslant i \leqslant M, 1 \leqslant j \leqslant N)$ be the grid points where $M \times N$ corresponds to the image size. Let $\phi_{i,j}^n = \phi(x_i, y_j, n\Delta t)$ $(n \geqslant 0)$ be an approximation of $\phi(x, y, t)$ and $I_{i,j}$ denotes the value of the image I at the grid point (x_i, y_j). The partial derivatives $\partial\phi/\partial x$ and $\partial\phi/\partial y$ are approximated by the following central differences, respectively:

$$\Delta_x^0\phi_{i,j}^n = \frac{\phi_{i+1,j}^n - \phi_{i-1,j}^n}{2h}, \quad \Delta_y^0\phi_{i,j}^n = \frac{\phi_{i,j+1}^n - \phi_{i,j-1}^n}{2h}. \tag{22}$$

Similarly for $\partial I_\sigma/\partial x$ and $\partial I_\sigma/\partial y$.

With the above notations, the numerical approximation to Eq. (19) is given by the following discrete equation:

$$\frac{\phi_{i,j}^{n+1} - \phi_{i,j}^n}{\Delta t} = g(|\nabla I_\sigma|_{i,j})\left(\alpha f\left(I_{i,j}, c_1\left(\phi_{i,j}^n\right), c_2\left(\phi_{i,j}^n\right)\right) + \beta K_{i,j}^n\right), \tag{23}$$

where

$$|\nabla I_\sigma|_{i,j} = \sqrt{\left(\Delta_x^0(I_\sigma)_{i,j}\right)^2 + \left(\Delta_y^0(I_\sigma)_{i,j}\right)^2},$$

$$f(I_{i,j}, c_1(\phi_{i,j}^n), c_2(\phi_{i,j}^n)) = sign\left(I_{i,j} - \frac{c_1(\phi_{i,j}^n) + c_2(\phi_{i,j}^n)}{2}\right),$$

$$K_{i,j}^n = \Delta_x^0\left(\frac{\Delta_x^0\phi_{i,j}^n}{\sqrt{\left(\Delta_x^0\phi_{i,j}^n\right)^2 + \left(\Delta_y^0\phi_{i,j}^n\right)^2 + \varepsilon^2}}\right) + \Delta_y^0\left(\frac{\Delta_y^0\phi_{i,j}^n}{\sqrt{\left(\Delta_x^0\phi_{i,j}^n\right)^2 + \left(\Delta_y^0\phi_{i,j}^n\right)^2 + \varepsilon^2}}\right),$$

for $i = 1,\ldots,M$, $j = 1,\ldots,N$ and with the boundary conditions obtained by reflection as

$$\phi_{-1,j}^n = \phi_{1,j}^n, \quad \phi_{M+1,j}^n = \phi_{M-1,j}^n, \quad \phi_{i,-1}^n = \phi_{i,1}^n, \quad \phi_{i,N+1}^n = \phi_{i,N-1}^n.$$

The numerical algorithm is as follows:

1. Initialize the level set function $\phi_{i,j}^0 = \phi_0(x_i, y_j)$ and set $n = 0$.
2. Solve the discrete equation Eq. (23), to obtain $\phi_{i,j}^{n+1}$.
3. Check whether the evolution has converged. If not, set $n = n + 1$ and return to step 2.

Note that re-initialization is not necessary in our model because the level set function is no longer required to keep as a signed distance function.

4.2. Experimental results

This subsection shows the results of the proposed model for both synthetic and real images. The level set function $\phi(x,y,t)$ is simply initialized to a constant function $\phi_0(x,y) = 1$ for all experiments. Besides, unless otherwise specified, we use the following parameters: $\sigma = 1.5$, $\alpha = 5$, $\beta = 1$, $h = 1$ and $\Delta t = 5$.

The first experiment shows that our model allows for the constant initialization of the level set function and thus can avoid the problems resulted from contour initialization. We apply our model to a real hand image shown in Fig. 4(a). As shown in Fig. 4(b), our model successfully extracts the object (hand) after only one iteration. The corresponding final level set function is shown in Fig. 4(c).

The second experiment shows that our model has capability to detect multiple objects or objects with interior holes. Two sample images are plotted in the first row of Fig. 5, which are a cell image with certain objects crossing over the image

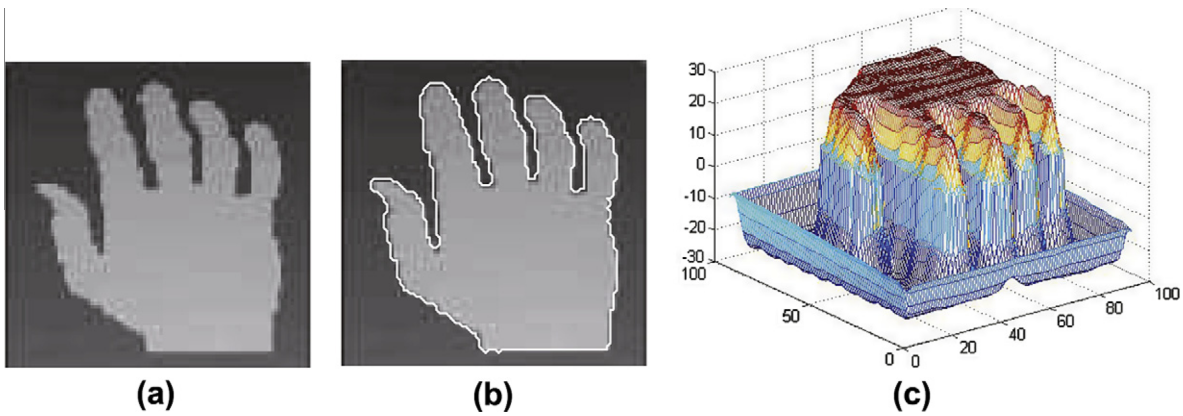

Fig. 4. Segmentation of a real image, starting with the constant function $\phi = 1$. (a) Original image (93 × 93); (b) Final result after one iteration; (c) 3-D plot of the corresponding level set function.

boundary and a wrench image with interior holes in each object. Several well-known models, such as [4] and [6], fail to segment such images starting with only one initial curve, because it is impossible to choose an initial curve surround all the edges of objects. By contrast, our model successfully detects all the objects (8 iterations) and/or the interior and exterior edges (5 iterations); see the middle row of Fig. 5.

The third experiment shows that the proposed model can better handle images with boundary concavities or fake boundaries. The first row of Fig. 6 presents such images, a synthetic image with boundary concavities and a real plane image with distinct shadow. The proposed model exactly captures the boundary concavity and the plane after only 1 and 7 iterations, respectively.

The final experiment shows that our model is robust to noise. Fig. 7 shows the results obtained with the applications of our model to two images with high level noise. The first row of Fig. 7 presents the original images. For noisy images, we increase the value of β to improve the model's performance; $\beta = 5$ for the first image and $\beta = 10$ for the second one. The segmentation results and the corresponding level set functions are plotted in the second and third rows, respectively. We can see from the second row that our model still extracted the objects, even if the images are heavily contaminated by noise.

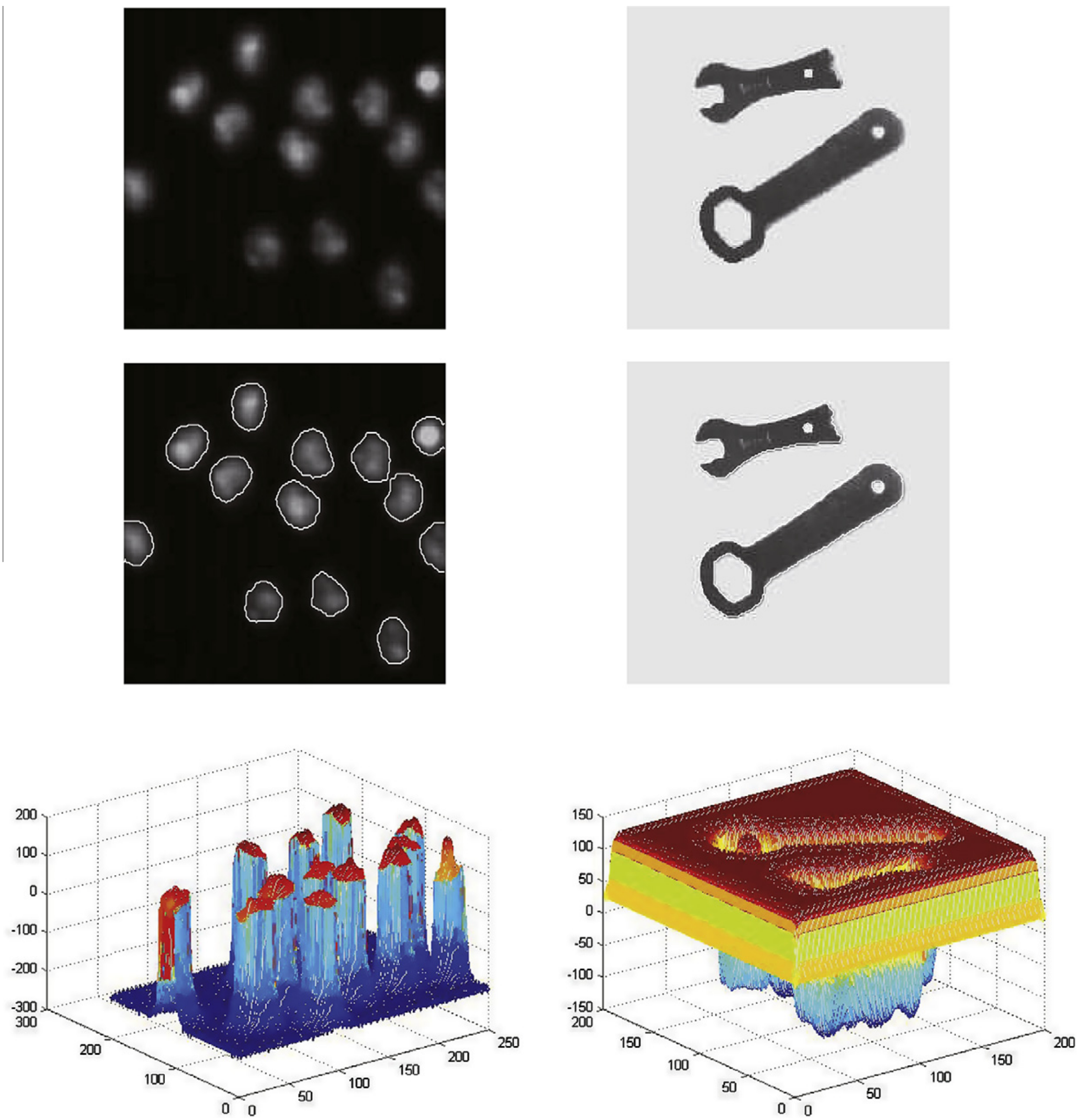

Fig. 5. Segmentation for images with multiple objects or interior holes (Cell: 250 × 203, Wrench: 200 × 200). Upper row: Original images; Middle row: Final results after 8 and 5 iterations, respectively, starting with the constant function $\phi = 1$; Lower row: 3-D plots of the corresponding level set functions.

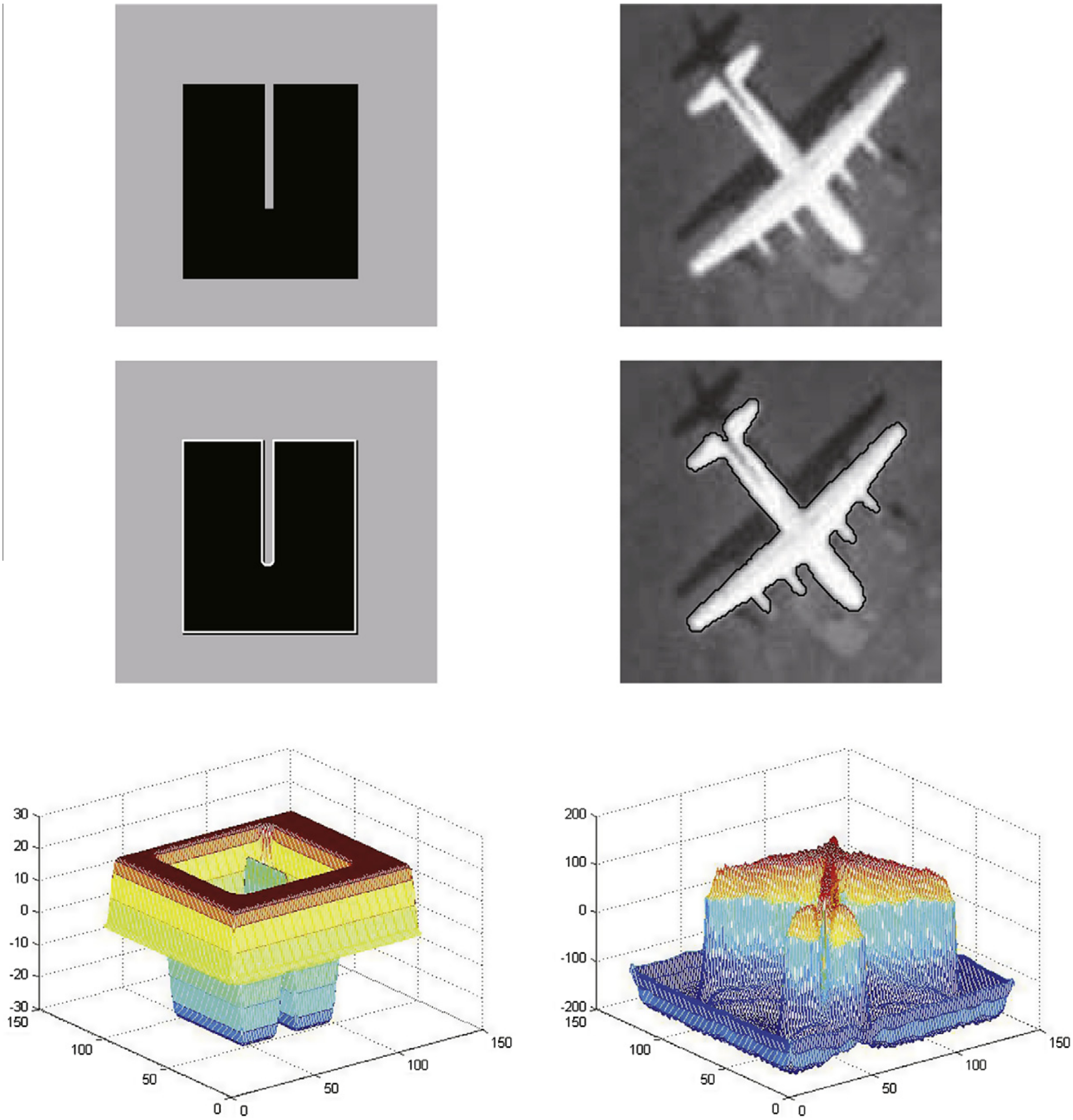

Fig. 6. Segmentation for images with boundary concavities (114 × 101) or fake boundaries (135 × 125). Upper row: Original images; Middle row: Final results after 1 and 7 iterations, respectively, starting with the constant function $\phi = 1$; Lower row: 3-D plots of the corresponding level set functions.

5. Discussions

5.1. About the time step

The proposed model allows the use of a larger time step in the numerical implementation. The larger time step can speed up the evolution, but may cause error in the boundary location if the time step is too large. If the time step is too small, the algorithm clearly takes too much iteration to reach convergence. There should be a tradeoff between iteration numbers and accuracy in boundary location. Our observations from experiments are that we can typically choose the time step in the range between 1 and 10 for the used explicit finite difference method.

5.2. About the parameter ρ

As discussed in Section 3.3, our model allows the level set function $\phi$to be simply initialized to a constant function $\phi_0(x,y) = \rho > 0$. Our observation from experiments is that as the ρ value increases, the rate of convergence of the zero level

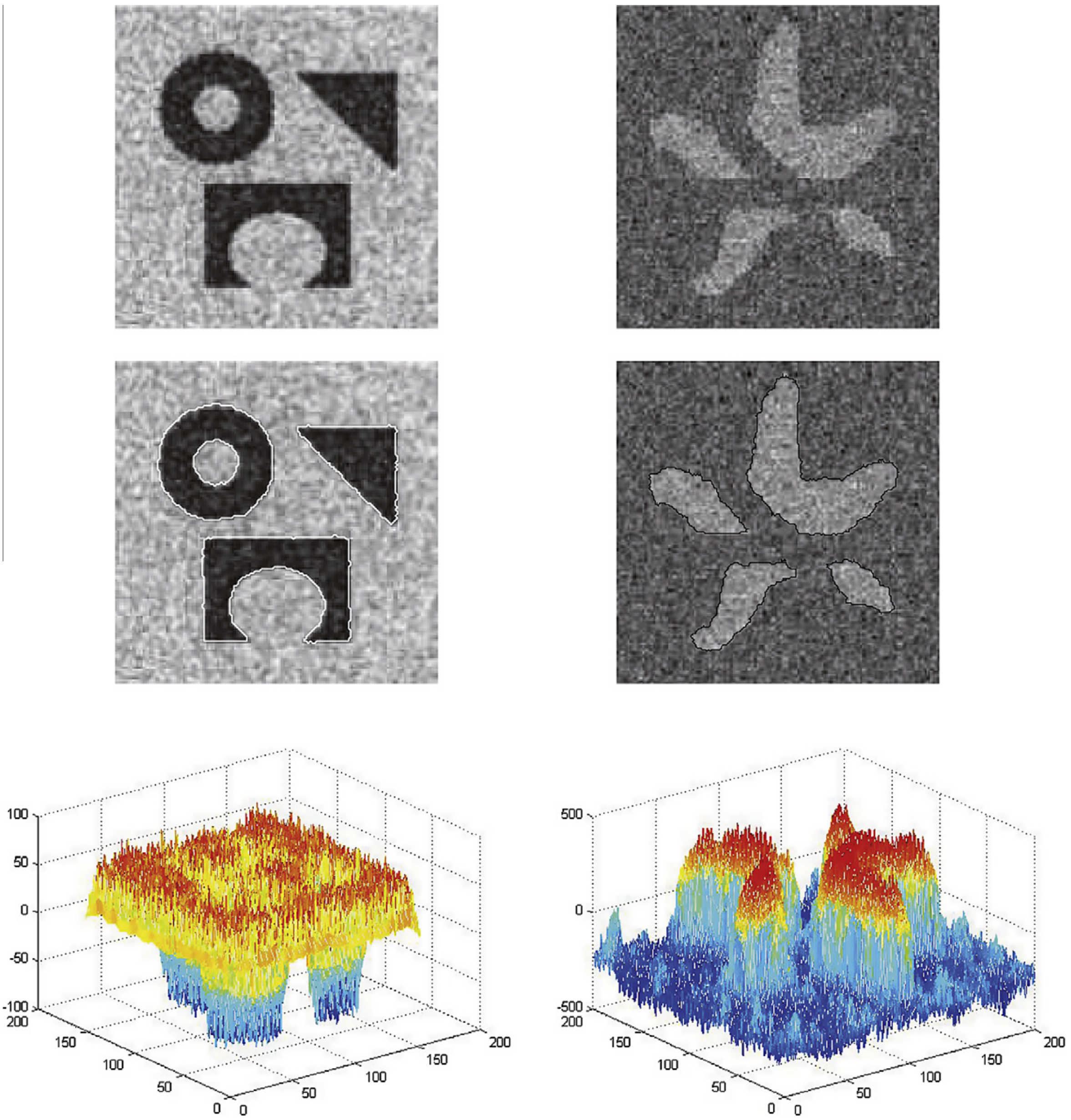

Fig. 7. Applications of our model to two images with high level noise. Upper row: Original images (152 × 152 and 200 × 200); Middle row: Final results after 3 and 18 iterations, respectively, starting with the constant function $\phi = 1$; Lower row: 3-D plots of the corresponding level set functions.

set to edge slows down gradually. Therefore, the constant ρ should be chosen as a smaller value to obtain a faster convergence. In applications, we suggest that the constant ρshould be chosen in the range between 0 and 5 for most images.

5.3. About the parameter α and β

In Eq. (9) or (19), αF_{adp} is an external force which controls the deformation of the level set function to drive the motion of the zero level set (contour) toward the desired object edges, while βF_{reg} is a regularized force which controls the smoothness of the contour to penalize complicated boundaries of regions and avoid the occurrence of isolated small regions (e.g., noise points) in final segmentation. Therefore, the parameter α mainly controls the deformation of the level set function, while the parameter β is mostly responsible for robustness to noise.

The motion of the level set function is mainly controlled by the external force αF_{adp}; thus a larger α value can speed up the curve evolution, while a smaller α value makes the evolution curve take too much iteration to reach convergence. However, if

α value is too large, the robustness to noise may be reduced, especially for images with high noise; if α is too small, the segmentation process may become unstable. By experiments, we find that the alpha equal to 5 (i.e., $\alpha = 5$) is the best choice for most images in terms of the rate of convergence, stability and the robustness to noise. This is a reason why we choose $\alpha = 5$ as the default value of the proposed model.

Real images are usually corrupted by noise more or less, thus the regularized force must be used for smoothing the zero level set (contour), i.e., $\beta > 0$. Our observation from experiments is as follows: if β is too small, the robustness to noise may be reduced; if β is too large, the segmentation process may become unstable. In applications, the β value should be selected according to the noise level: for images with small amounts of noise, we can choose β in the range between 0 and 5; for images with large amount of noise, we need to choose β in the range between 5 and 10.

6. Conclusion

In this paper, we propose a PDE-based level set evolution model, in which the evolution PDE has two terms, an adaptive driving term and a TV-based regularization term. The level set function can be simply initialized to a constant function that eliminates the need for initial contours, and so completely avoids the problems resulted from contours initialization. The re-initialization of the level set function is also not necessary in the proposed model. A simple explicit finite difference scheme with a significantly larger time step is used for solving the evolution PDE numerically. The proposed model has been successfully applied to many synthesized and real images with promising results.

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