

Graph Cut Optimization for the Piecewise Constant Level Set Method Applied to Multiphase Image Segmentation*

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Abstract. The piecewise constant level set method (PCLSM) has recently emerged as a variant of the level set method for variational interphase problems. Traditionally, the Euler-Lagrange equations are solved by some iterative numerical method for PDEs. Normally the speed is slow. In this work, we focus on the piecewise constant level set method (PCLSM) applied to the multiphase Mumford-Shah model for image segmentation. Instead of solving the Euler-Lagrange equations of the resulting minimization problem, we propose an efficient combinatorial optimization technique, based on graph cuts. Because of a simplification of the length term in the energy induced by the PCLSM, the minimization problem is not NP hard. Numerical experiments on image segmentation demonstrate that the new approach is very superior in terms of efficiency, while maintaining the same quality.

1 Introduction

The level set method [1, 2] is a powerful tool for interphase problems. It has numerous applications in computer vision, fluid dynamics and inverse problems. The interphase is implicitly represented by a higher dimensional level set function. Originally, the signed distance functions were used as level set functions. Later the work of [3, 4, 5] introduced piecewise constant level set functions, representing the interphases by discontinuities. This has certain advantages, such as ability to represent several interphases by one single level set function. This method will be referred to as the piecewise constant level set method (PCLSM)

In computer vision, the level set method has been applied with great success to image segmentation. Of particular importance is the Mumford-Shah model [6],

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which is an established image segmentation model. In [7, 8], Chan and Vese proposed a numerical realization of this model based on traditional level set functions. In [3, 4, 5], piecewise constant level set functions were proposed. Both approaches lead to a system of nonlinear PDEs that needs to be solved numerically. They both have the drawback of expensive computation.

This work aims to significantly reduce the computational cost of the piecewise constant level set method for the multiphase Mumford-Shah model. The length term is often simplified in the energy induced when representing this model by piecewise constant level set functions. We will show that this simplification makes it possible to efficiently compute global minimizers via graph cuts, when the mean image intensity value in each phase is known. Graph cuts is a well-known technique in image analysis and computer vision [9, 10, 11, 12]. Usually, NP-hard multilabeling problems are approached by constructing algorithms for finding approximate suboptimal solutions, such as alpha expansion [12]. We instead do the approximation in the model, and then compute the exact solution of the approximate model. The graph used for optimization is constructed as in [13, 14, 15], except for some small modifications. Finally, for unknown mean intensity values, an iterative algorithm is presented, which we believe will have large practical value because of the strong efficiency.

In case of two phases, some work on graph cut optimization for the Mumford-Shah model has been made in [16, 17]. Also a multiphase approach based on graph cuts has recently been presented in [18]. The process is started by splitting the image into two regions, by solving the two-phase Mumford-Shah model to optimality. In the next step, each new region is splitted in two by solving the two-phase Mumford-Shah model within each region. The process is repeated until the intensity variation within each region falls below a predefined threshold. The limitation of this approach is that the possibility of a region to evolve has been ignored. For instance, the optimal interphase for two regions may not be a subset of the optimal interphases for three regions. An experiment in Section 4 will clarify this.

The paper is organized as follows: Section 2 gives a brief overview of the piecewise constant level set method and the Mumford-Shah model. Section 3 presents the new integer optimization approach, while numerical experiments are presented in Section 4.

2 Image Segmentation and the PCLSM

2.1 The Mumford-Shah Model

The Mumford-Shah model [6] is an established image segmentation model with a wide range of applications. Let u_0 be the input image. In the most common variant with closed contours, one seeks a partition $\{\Omega_i\}_{i=1}^n$ of the image domain Ω , and an approximation image u which minimizes the functional

$$E(u, \Gamma_i) = \int_{\Omega} (u - u_0)^2 dx + \mu \int_{\Omega \setminus \cup_i \Gamma_i} |\nabla u|^2 dx + \sum_{i=1}^n \nu \int_{\Gamma_i} ds, \quad (1)$$

where $\{\Gamma_i\}_{i=1}^n$ denotes the interphases between the regions $\{\Omega_i\}_{i=1}^n$. Often u is assumed to be constant within each phase, in which case the second term disappears and one ends up with the simpler version

$$E(\{c_i\}, \Gamma_i) = \int_{\Omega} (u - u^0)^2 dx + \sum_{i=1}^n \nu \int_{\Gamma_i} ds, \quad (2)$$

where $u = \sum_{i=1}^n c_i \Psi_i$, and Ψ_i is the characteristic function of Ω_i . As a numerical realization, Chan and Vese [7,8] proposed to represent the above functional with level set functions, and solve the resulting gradient descent equations numerically. In order to represent n phases, $\log_2(n)$ level set functions were required. For any $n > 2$ the length term had to be simplified.

2.2 Piecewise Constant Level Set Functions

In [3, 4, 19], instead the piecewise constant level set method was proposed, and applied to the Mumford-Shah model. This approach has certain benefits, such as the ability to represent any number of phases with one single level set function. Let $\{\Omega_i\}_{i=1}^n$ be a partition of the domain Ω into n regions. Any such partition can be described by a piecewise constant level set function ϕ as follows

$$\phi = i \quad \text{in } \Omega_i \quad \text{for } i = 1, 2, \dots, n. \quad (3)$$

Note that all interphases are represented by discontinuities in ϕ . The Mumford-Shah functional can now be written in terms of ϕ

$$E(\mathbf{c}, \phi) = \int_{\Omega} (u - u^0)^2 dx + \frac{\nu}{2} \sum_{i=1}^n \int_{\Omega} |\nabla \psi_i| dx. \quad (4)$$

where $u = \sum_{i=1}^n c_i \psi_i$, and ψ_i is the characteristic function of Ω_i . It can be derived from the level set function by

$$\psi_i = \frac{1}{\alpha_i} \prod_{j=1, j \neq i} (\phi - j) \quad \text{with } \alpha_i = \prod_{k=1, k \neq i} (i - k). \quad (5)$$

The length term can be approximated by the total variation of the level set function itself, especially when the number of phases is not too large

$$E(\mathbf{c}, \phi) = \sum_{i=1}^n \int_{\Omega} (u - u^0)^2 dx + \nu \int_{\Omega} |\nabla \phi| dx, \quad (6)$$

see for instance [5] for a justification. Most often this approximation is preferred, since it is computationally easier. Such a simplification of the length term has also been made in [20, 21] among others for multiphase image segmentation. In this work we will consider (6).

There are some variants of the total variation regularization term. The commonly used version is the isotropic total variation: $TV_2(\phi) = \int_{\Omega} |\nabla\phi|_2 dx = \int_{\Omega} \sqrt{|\phi_{x_1}|^2 + |\phi_{x_2}|^2} dx$. In order to simplify computation, often a simpler version is used: $TV_1(\phi) = \int_{\Omega} |\nabla\phi|_1 dx = \int_{\Omega} |\phi_{x_1}| + |\phi_{x_2}| dx$. However, since TV_1 is not isotropic, regularization will be stronger in certain directions. A more isotropic version based on the 1-norm can be obtained by splitting TV_1 using the original gradient operator, and one rotated counterclockwise $\pi/4$ radians: $TV_{1, \frac{\pi}{4}}(\phi) = \frac{1}{2} \int_{\Omega} \{|\nabla\phi(x)|_1 + |R_{\frac{\pi}{4}}\nabla\phi(x)|_1\} dx$, where $R_{\frac{\pi}{4}}\nabla$ is the gradient in the rotated coordinate system. It is also possible to create even more isotropic versions by considering more such rotations.

Previous attempts to minimize (6) have been made by continuous optimization. In order to force a solution taking only integer values, the following constraint was imposed

$$K(\phi) = \prod_{i=1}^n (\phi - i) = 0 \quad (7)$$

The constrained optimization problem (6) and (7) could be solved by the augmented lagrangian method as in [3, 4, 19]. Some attempts to speed up the computation can be found in [5].

In the next section we propose to solve the minimization problem by graph cuts. We start by discretizing the variational problem (6) on a grid \mathcal{P} of mesh size $\delta = 1$. For each $p = (i, j) \in \mathcal{P}$, define the neighborhood systems

$$N_4(p) = \{(i \pm 1, j), (i, j \pm 1)\}, \text{ and } N_8(p) = \{(i \pm 1, j), (i, j \pm 1), (i \pm 1, j \pm 1)\}.$$

The modification of the definition for boundary points is clear. The discrete energy function can now be written compactly

$$E_d(\mathbf{c}, \phi) = \sum_{p \in \mathcal{P}} \delta^2 (u_p - u_p^0)^2 + \nu \sum_{p \in \mathcal{P}} \sum_{q \in \mathcal{N}_k(p)} \frac{1}{2} w_{pq} |\phi_p - \phi_q|, \quad (8)$$

where $k = 4$ for TV_1 and $k = 8$ for $TV_{1, \frac{\pi}{4}}$. The weights w_{pq} are given by $w_{pq} = \frac{4\delta^2}{k||p-q||_2}$. In case of two phases, similar weights can also be derived by using the Cauchy-Crofton formula [22]. Note that each term is being counted twice in the last summation. This is compensated by multiplication by the factor $\frac{1}{2}$.

3 Integer Optimization for PCLSM and Mumford-Shah

Instead of imposing constraints to force an integer solution by continuous optimization, we instead propose the much more natural approach of using integer optimization to minimize (6). We will show that the discretized functional (8) can be minimized by graph cuts in case the values \mathbf{c} are known in Section 3.2. Finally, in Section 3.3 an algorithm is designed to minimize with respect to both \mathbf{c} and ϕ .

3.1 Background on Graph Cuts and Terminology

Min-cut is a well known optimization problem. Due to a duality theorem by Ford and Fulkerson [23], there are several fast algorithms for this problem. Graph cuts is a reference to such algorithms, and was introduced as a computer vision tool by Greig et. al. [9] in connection with markov random fields [24].

A graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a set of vertices \mathcal{V} and a set of directed edges \mathcal{E} . We let (a, b) denote the directed edge going from vertex a to vertex b , and let $c(a, b)$ denote the cost (weight) on this edge. In the graph cut scenario there are two distinguished vertices in \mathcal{V} , called the source $\{s\}$ and the sink $\{t\}$. A cut on \mathcal{G} is a partitioning of the vertices \mathcal{V} into two disjoint and connected (through edges) sets $(\mathcal{V}_s, \mathcal{V}_t)$ such that $s \in \mathcal{V}_s$ and $t \in \mathcal{V}_t$. For each cut, the set of severed edges \mathcal{C} is uniquely defined as

$$\mathcal{C} = \{(a, b) \mid a \in \mathcal{V}_s, b \in \mathcal{V}_t \text{ and } (a, b) \in \mathcal{E}\}. \quad (9)$$

We say that the cut severs the edge e if e is contained in \mathcal{C} . From now on, we refer to the cut as the set of severed edges \mathcal{C} . The cost of the cut is defined as

$$|\mathcal{C}| = \sum_{e \in \mathcal{C}} c(e). \quad (10)$$

We are interested in finding the cut of minimum cost on \mathcal{G} , from now on called the minimum cut. The duality theorem by Ford and Fulkerson [23] states this is equivalent to finding the maximum flow from $\{s\}$ to $\{t\}$, where the edge weights are bounds on the maximum amount of flow that can be pushed through the edges. Cuts of minimum cost can thus be computed very efficiently by max-flow algorithms such as Ford-Fulkerson [23]. See [10] for a detailed discussion about implementation.

3.2 Graph Cuts for the Multiphase PCLSM

For fixed values \mathbf{c} , we will show that the minimizer of (8) can be obtained by finding the minimum cut over an appropriate graph, i.e. we will construct a graph \mathcal{G} such that

$$\min_{\mathcal{C} \text{ cut on } \mathcal{G}} |\mathcal{C}| = \min_{\phi} E_d(\mathbf{c}, \phi) + \sigma, \quad (11)$$

where σ is a constant that will be specified later. Note that the minimizer ϕ is not influenced by this constant.

Some work on graph cuts for the two phase Mumford-Shah model can be found in [17, 16]. Unfortunately, the extension to more than two phases is NP hard [25]. The usual graph cut approach to optimization problems of several labels, is to use some sort of approximation method, such as the alpha expansion [12]. Since we have already made an approximation in the model (6), we will show that graph cuts can be used to find the exact minimum. The idea is to introduce an extra dimension to take care of several phases. We construct the graph in a similar way as Ishikawa [13, 14], except for some small technical differences:

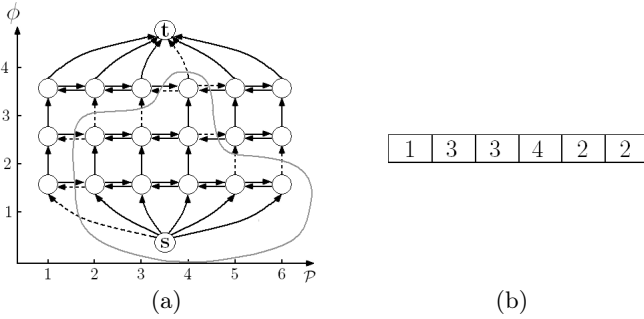


Fig. 1. (a) The graph corresponding to a 1D signal of 6 grid points used for 4 phase segmentation. Edges in \mathcal{E}_D are depicted as vertical arrows and edges in \mathcal{E}_R are depicted as horizontal arrows. The gray curve is used to visualize the cut, vertices in the interior to the curve belongs to \mathcal{V}_s , vertices in the exterior to the curve belongs to \mathcal{V}_t . Edges in \mathcal{C} are depicted as dotted arrows. Figure (b) shows the values of ϕ at each grid point corresponding to the cut in (a), they are determined from definition 1.

our graph consists of one less layer of vertices and edges, and is a generalization from the binary construction of Greig et. al. [9]. We also avoid edges of infinite capacity. When the number of phases is small, this will have a little effect on the efficiency.

For each grid point $p \in \mathcal{P}$, we associate $(n-1)$ vertices in the graph \mathcal{G} , denoted $v_{p,\ell}, \ell = 1, \dots, n-1$. The set of vertices \mathcal{V} is formally defined

$$\mathcal{V} = \{v_{p,\ell} \mid p \in \mathcal{P}, \ell \in \{1, \dots, n-1\}\} \cup \{s\} \cup \{t\}. \quad (12)$$

An illustration in case of a 1D image where $\mathcal{P} = \{1, 2, \dots, 6\}$, is shown in Figure 1. For ease of visualization, no 2D cases are shown.

The edges are arranged in two groups, \mathcal{E}_D and \mathcal{E}_R . The first group \mathcal{E}_D corresponds to the data term in (8). It is defined as

$$\mathcal{E}_D = \cup_{p \in \mathcal{P}} \mathcal{E}_p, \quad (13)$$

where for each $p \in \mathcal{P}$ the edge set \mathcal{E}_p is defined as

$$\mathcal{E}_p = (s, v_{p,1}) \cup_{\ell=1}^{n-2} (v_{p,\ell}, v_{p,\ell+1}) \cup (v_{p,n-1}, t). \quad (14)$$

The edges in \mathcal{E}_D are illustrated as the vertical arrows in Figure 1.

The second group of edges \mathcal{E}_R corresponds to the regularization term in (8). These are illustrated as the horizontal arrows in Figure 1, i.e.

$$\mathcal{E}_R = \{(v_{p,\ell}, v_{q,\ell}) \mid p \in \mathcal{P}, q \in \mathcal{N}_k(p), \ell \in \{1, \dots, n-1\}\}. \quad (15)$$

We say that a cut is admissible if it severs exactly one edge in \mathcal{E}_p for each $p \in \mathcal{P}$.

We can now establish the relationship between a cut on \mathcal{G} and a level set function ϕ .

Definition 1. Let $\mathcal{C} \subset \mathcal{E}$ be an admissible cut on \mathcal{G} . For any grid point $p \in \mathcal{P}$, the corresponding level set function ϕ is defined as

$$\phi_p = \begin{cases} 1 & \text{if } (s, v_{p,1}) \in \mathcal{C}, \\ \ell + 1 & \text{if } (v_{p,\ell}, v_{p,\ell+1}) \in \mathcal{C}, \\ n & \text{if } (v_{p,n-1}, t) \in \mathcal{C}. \end{cases} \quad (16)$$

Note that ϕ is single valued by the admissible cut requirement. We can now define the edge costs (weights) such that the relationship (11) is satisfied. We start by edges in \mathcal{E}_D , i.e. the data edges

$$\begin{aligned} c(s, v_{p,1}) &= \delta^2 |u_p^0 - c_1|^2 + \frac{\sigma}{|\mathcal{P}|} \quad \forall p \in \mathcal{P}, \\ c(v_{p,\ell}, v_{p,\ell+1}) &= \delta^2 |u_p^0 - c_{\ell+1}|^2 + \frac{\sigma}{|\mathcal{P}|} \quad \forall p \in \mathcal{P}, \quad \forall \ell \in \{1, \dots, n-2\}, \\ c(v_{p,n}, t) &= \delta^2 |u_p^0 - c_n|^2 + \frac{\sigma}{|\mathcal{P}|} \quad \forall p \in \mathcal{P}. \end{aligned} \quad (17)$$

The costs (weights) for the regularization edges \mathcal{E}_R are defined by

$$c(v_{p,\ell}, v_{q,\ell}) = \nu w_{pq}, \quad \forall p \in \mathcal{P}, \quad \forall q \in \mathcal{N}_k(p), \quad \forall \ell \in \{1, \dots, n-1\}. \quad (18)$$

By choosing σ as any positive value, the cut of minimum cost will be admissible, which implies that its corresponding level set function is single valued.

Theorem 1. Let \mathcal{C} be a minimum cut on \mathcal{G} , then \mathcal{C} is admissible if $\sigma > 0$

Proof. Suppose \mathcal{C} is a minimum cut on \mathcal{G} and for some $p \in \mathcal{P}$ several edges in \mathcal{E}_p belongs to \mathcal{C} . That is, there exists a set of indices L_p such that $(v_{p,\ell}, v_{p,\ell+1}) \in \mathcal{C} \quad \forall \ell \in L_p$. Define the cut \mathcal{C}^* s.t. for each $p \in \mathcal{P}$, $v_{p,\ell} \in \mathcal{V}_s^*$ if $\ell \leq \max L_p$, else $v_{p,\ell} \in \mathcal{V}_t^*$. Then $\mathcal{C}^* \cap \mathcal{E}_D \subset \mathcal{C} \cap \mathcal{E}_D$. Since $\sigma > 0$, no edges have zero weight, therefore $|\mathcal{C}^* \cap \mathcal{E}_D| < |\mathcal{C} \cap \mathcal{E}_D|$. Furthermore, $|\mathcal{C}^* \cap \mathcal{E}_R|_{\text{cardinality}} \leq |\mathcal{C} \cap \mathcal{E}_R|_{\text{cardinality}}$. For TV_1 , the weights on all edges in \mathcal{E}_R are equal. Therefore $|\mathcal{C}^*| < |\mathcal{C}|$, which is a contradiction. The same contradiction can also be derived for $TV_{1, \frac{\sigma}{4}}$.

To summarize, for any piecewise constant level set function ϕ taking values in $\{1, 2, \dots, n\}$, there exists a unique admissible cut on \mathcal{G} . Moreover, the function ϕ and its corresponding cut satisfies

$$|\mathcal{C}| = E_d(\mathbf{c}, \phi). \quad (19)$$

Thus, a function ϕ corresponding to a minimum cut, is a minimizer of the functional (8), i.e. it solves the segmentation problem. Note that in case $n = 2$, the extra dimension breaks down, and the graph becomes identical to that of Greig et. al. [9] for binary problems.

It is also possible to exactly minimize (8) as in [26], by solving a sequence of binary optimization problems via graph cuts. This approach is likely to be faster when n is very large, and is a power of 2. However, for image segmentation n is relatively small, and we expect the presented approach to be faster.

3.3 Algorithm for Minimizing the Mumford-Shah Functional

The algorithm presented in the last section minimizes $E_d(\mathbf{c}, \phi)$ with respect to ϕ for a fixed \mathbf{c} . Vice versa, for a fixed ϕ the values \mathbf{c} minimizing $E_d(\mathbf{c}, \phi)$ are given by the average intensity in each region

$$c_i = \frac{\int_{\Omega} u^0(x) \psi_i(x) dx}{\int_{\Omega} \psi_i(x) dx} \quad i = 1, 2, \dots, n, \quad (20)$$

or in discrete form

$$c_i = \frac{\sum_{p \in \mathcal{P}} u_p^0 \psi_{i,p}}{\sum_{p \in \mathcal{P}} \psi_{i,p}} \quad i = 1, 2, \dots, n. \quad (21)$$

We want an algorithm to minimize both with respect to ϕ and \mathbf{c} . This is achieved by combining the two above results in the following iterative descent algorithm

Algorithm 1.

Estimate initial values \mathbf{c}^0 , set $l = 0$
 while($\|\mathbf{c}^l - \mathbf{c}^{l-1}\| > \text{tol}$)

1. Use graph cuts to estimate ϕ from

$$\phi = \arg \min_{\tilde{\phi}} E_d(\mathbf{c}^l, \tilde{\phi}). \quad (22)$$

2. Update \mathbf{c}^{l+1} according to equation (21).
3. Update $l \leftarrow l + 1$.

Note that no initialization of the level set function is required. Only the values \mathbf{c}^0 need to be initialized, which can be achieved very efficiently by the isodata algorithm [27]. Note that algorithm 1 has an exact termination criterion, as tol can be set to zero. In all our experiments, convergence was reached in 4-12 iterations. It must be noted that this algorithm is no longer guaranteed to find the global minima. Theoretically it may get trapped in a local minima close to the initial values \mathbf{c}^0 . However, in practice it is usually rather insensitive to initialization.

4 Numerical Experiments

In this section we validate our new optimization method by numerical experiments. The results are compared with the original gradient descent approach [3] for minimizing (4) (note: not the variant with simplified length term). The implementation of both these methods is made in C++. Comparisons are made both with respect to quality and computation time on an intel 2.19 GHz laptop. The list of computation times is shown in Table 1. The test images are shown in Figure 2. In all results, the estimated phases are depicted as a bright region. The results of experiment 1 and 2 are depicted in Figure 3 (a) - (d). We observe that graph cut

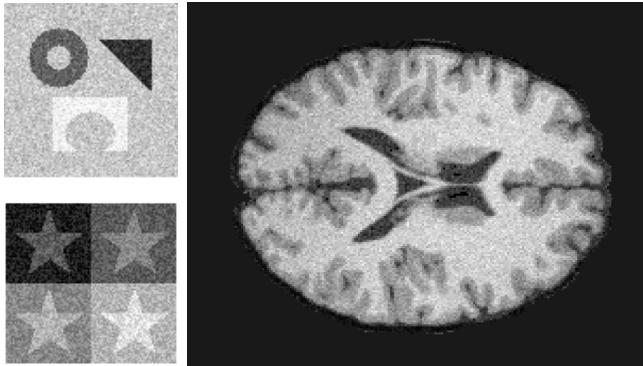


Fig. 2. Test images

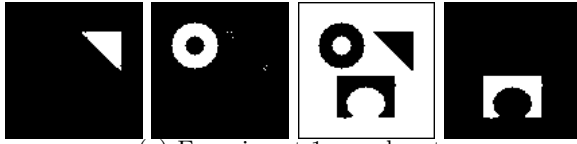
Table 1. Computation times in seconds for gradient descent vs graph cut optimization

	Size	Number of Phases	Gradient descent	Graph Cut
Experiment1	100x100	4	50.3	0.120
Experiment2	100x100	5	70.0	0.179
Experiment3	92x98	5	55.4	0.165
Brain	933x736	4	5401	25.22

optimization solve the multiphase problem with at least as good quality as gradient descent. In experiment 3 Figure 3(e)(f), the number of regions is assumed to be unknown. The optimal number of regions can be estimated by using more phases than necessary in the minimization problem. As we can see, this results in some empty phases, while the remaining phases capture the correct regions.

We have also tested the method on a synthetic brain MRI image. The noise level is 7%, and non-uniformity of the RF-puls is of 20 % (see <http://www.bic.mni.mcgill.ca/brainweb/> for details). We want to extract four tissue classes from the image: region 1; background, region 2; cerebrospinal fluid, region 3; gray matter and region 4; white matter. This is achieved by minimizing the Mumford-Shah model with 4 phases. In Figure 4 we compare the results of graph cuts, gradient descent and the exact results. The background phase is not shown. Again, we observe that graph cut results are very good, while the computation time is dramatically reduced compared to gradient descent (c.f. Table 1, brain).

Finally, in Figure 5 we show an example which demonstrates the limitation of the multiphase approach presented in [18], described in the related work section. For the chosen parameter ν , the global minimum consists of three phases, which we are able to detect by applying our multiphase algorithm with 4 phases, Figure 5(b) top. The result of the first step of the algorithm presented in [18] is shown in Figure 5(b) bottom, which is the global minimum of the two phase Mumford-Shah functional. Clearly, no further splitting of these regions can result in the correct three phases, since the interphase from the first step is not allowed to evolve.



(a) Experiment 1: graph cut



(b) Experiment 1: gradient descent



(c) Experiment 2: graph cut



(d) Experiment 2: gradient descent



(e) Experiment 3: graph cut



(f) Experiment 3: gradient descent

Fig. 3. (a) and (b) Experiment 1, from left to right: phase 1 - phase 4. (c) and (d) Experiment 2, from left to right: phase 1 - phase 5. (e) and (f) Experiment 3, from left to right: phase 1 - phase 5.

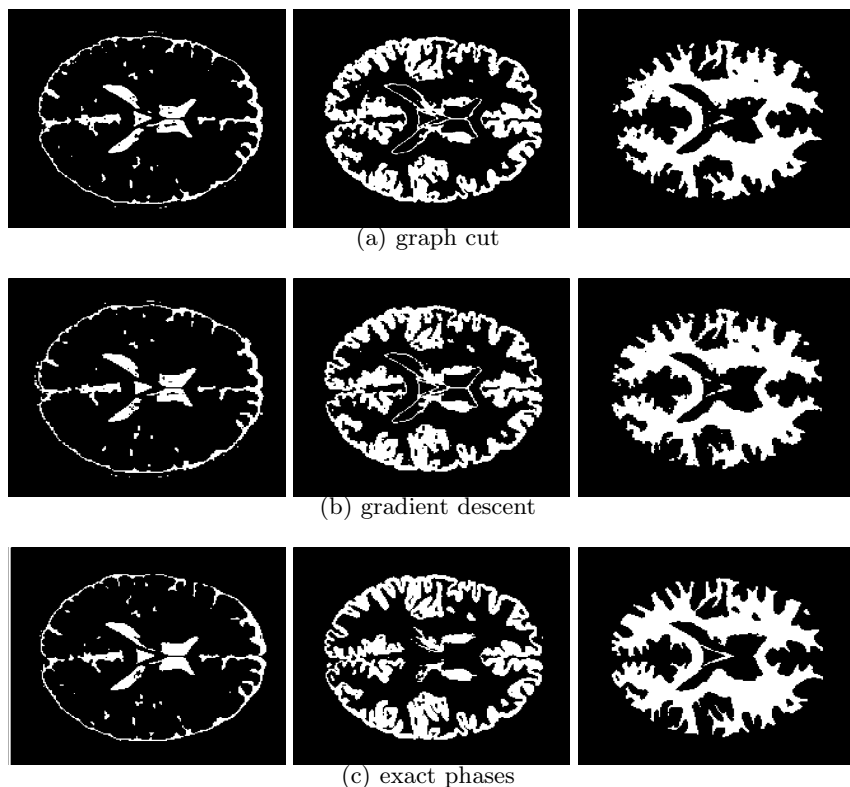


Fig. 4. From left to right: phase 1 - phase 3. (a) Graph cut, (b) gradient descent, (c) exact phases.

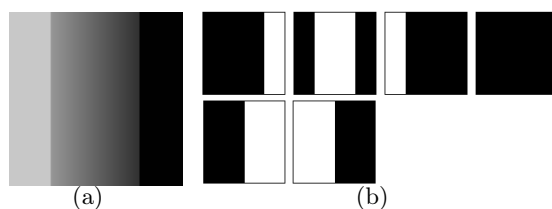


Fig. 5. (a) Input image. (b) Top: Our approach, from left to right phase 1 - phase 4. Bottom: First step of approach reported in [18], from left to right phase 1 - phase 2.

5 Summary

We have presented an algorithm for efficiently minimizing the energy induced by the piecewise constant level set representation of the multiphase Mumford-Shah functional. This minimization method is based on graph cuts. Numerical

experiments demonstrated the method is very superior in efficiency compared to the previous PDE based approach, while maintaining the same quality of results.

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