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# NUMERICAL MINIMIZATION OF EIGENMODES OF A MEMBRANE WITH RESPECT TO THE DOMAIN

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**Abstract**. In this paper we introduce a numerical approach adapted to the minimization of the eigenmodes of a membrane with respect to the domain. This method is based on the combination of the Level Set method of S. Osher and J. A. Sethian with the relaxed approach. This algorithm enables both changing the topology and working on a fixed regular grid.

**Résumé**. Nous proposons dans cet article une approche numérique adaptée à la minimisation des modes propres d'une membrane par rapport au domaine. Cette méthode repose sur la combinaison de la méthode des lignes de niveaux de S. Osher et J. A. Sethian avec une approche de type relaxation. Elle présente entre autre intérêt celui d'autoriser des changements de topologie tout en travaillant sur un maillage fixe régulier.

**1991 Mathematics Subject Classification.**

14 mai 2003.

## 1. Statement and historical summary

Let $\Omega$ be a bounded open set of $\mathbb{R}^N$ and $H_0^1(\Omega)$ the Sobolev space defined as the closure of $C^\infty$ functions with compact support in $\Omega$ with respect to the norm

$$\|u\|_{H^1} := \left( \int_\Omega u(x)^2\,dx + \int_\Omega |\nabla u(x)|^2\,dx \right)^{1/2}.$$

The Laplace-Dirichlet operator on $\Omega$ being a self-adjoint operator with compact inverse, there is a sequence of positive eigenvalues which tends to $+\infty$ and an associated sequence of eigenvectors that will be labelled $0 < \lambda_1(\Omega) \leq \lambda_2(\Omega) \leq \lambda_3(\Omega) \leq \ldots$ and $u_1, u_2, u_3, \ldots$. Thus, those sequences solve the problems

$$\left\{ \begin{array}{ll} -\Delta u_k = \lambda_k(\Omega) u_k & \text{in } \Omega, \\ u_k = 0 & \text{on } \partial\Omega. \end{array} \right. \tag{1}$$

*Keywords and phrases:* shape optimization, eigenvalue, level set,relaxation

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In any case, the eigenfunctions will be supposed to satisfy the condition

$$\int_\Omega u_k(x)^2\,dx = 1\ . \tag{2}$$

The eigenfunctions constitute an Hilbert base of $L^2(\Omega)$. As a consequence of the hypo-analyticity of the laplacien they are analytic in $\Omega$. Moreover, we shall use the following variational formulation :

$$\lambda_1(\Omega) = \inf_{v\in H_0^1(\Omega), v\neq 0} \frac{\int_\Omega |\nabla v(x)|^2\,dx}{\int_\Omega v(x)^2\,dx}\ . \tag{3}$$

This infimum is reached when $v = u_1$, the first eigenfunction of the Laplace-Dirichlet operator.

The investigations of the relations between the eigenmodes of the domain $\Omega$, and specifically the isoperimetric inequalities where they are involved, are an important issue both in the field of partial differential equations and in differential geometry. In the beginning of the century, the first result obtained in this topic was the proof of the Rayleigh conjecture by Faber [7] and Krahn [13]. They proved that the ball minimizes the first eigenvalue of the Laplace-Dirichlet operator under a volume constraint. The domain which minimizes $\lambda_2$ always with a volume constraint is the union of two identical balls. This result was often attributed to P. Szegö as observed by G. Pòlya in [18], but it seems that this result was already published in a paper of E. Krahn (see [14]). For $k \geq 3$ fixed, the question to identify the open set of the plane which minimizes $\lambda_k$ amongst the sets of given area remains open today (see [9] for a survey on this subject).

In 1973, Troesch proposes in [23] some numerical values of resonant frequencies for several types of convex membranes of $\mathbb{R}^2$. The following observations stands out from this experiments :

- the convex open set which minimizes the second eigenvalue of the Laplace-Dirichlet operator under convexity and volume constraint displays two flat sections on its boundary,
- the stadium (see 1) .i.e. the convex hull of two tangent identical disks looks very close from the optimal set.

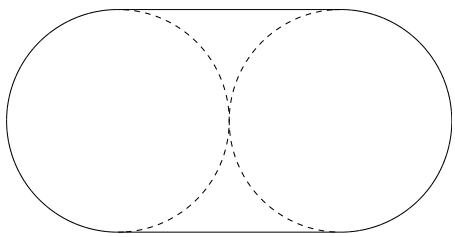

FIG. 1. One stadium

In fact, as reported in [10] and [11], the stadium does not minimizes $\lambda_2$ under convexity and volume constraints. The aim of this work is to present a new technique enabling to approximate numerically the solutions of such optimizations problems. More precisely we shall be interested in the two following problems :

$$\min\{\lambda_2(\Omega), \Omega \subset \mathbb{R}^2,\ \Omega \text{ convex}, |\Omega| = 1\} \tag{4}$$

$$\min\{\lambda_k(\Omega), \Omega \subset \mathbb{R}^2,\ |\Omega| = 1\} \text{ for } k \geq 3. \tag{5}$$

The method we are presenting combines two approaches that were generated in the last twenty years, respectively the homogeneization method and the level set methods. We shall start with a short description of the three main numerical methods in shape optimization, namely the boundary variation, homogeneization and the level set methods. For each of them we shall underline the drawbacks when applying those techniques to minimize the eigenmodes of the Laplace operator. In consequence we shall have to develop a new process.

In conclusion we shall report numerical results. On one side we improve the values published in [24]and on the other side we propose a geometrical description of the ten first optimal sets.

## 2. Boundary variation, derivative with respect to the domain

The boundary variation method is the first we used for our study of the eigenvalues. This allowed to obtain satisfying results when minimizing $\lambda_2$ under volume and convexity constraints. This approach turns out to be ineffective to investigate directly the minimization under volume constraint only but unlike the technique that will be introduced it does not require to modify the cost function.

### 2.1. General presentation

Many optimization methods are founded on the used of the so called first order optimality conditions (as the gradient and quasi-Newton methods). These techniques are descent methods which we try to converge to a local minimum of the cost function of the cost function. The question which arises in shape optimization is the deficiency of a natural derivative in a space of shapes. To bypass this difficulty we introduce for each vector field $V$ (so called *deformation field*) of $\mathbb{R}^N$, the derivative (if it does exist) of $J$ in the direction $V$ by

$$d\left(J\left(\Omega\right)\right)\left(\Omega,V\right)=\lim_{t\to 0}\frac{J\left(\Omega_t\right)-J\left(\Omega_0\right)}{t},$$

where

$$\Omega_t=\left\{\left(Id+tV\right)\left(x\right),\ x\in\Omega\right\}$$

A function $J$ which displays an extremum in $\Omega$ satisfies the relation

$$d\left(J\left(\Omega\right)\right)\left(\Omega,V\right)=0 \tag{6}$$

for all vector field $V$.

**Remark 1.** *Under some regularity assumptions it can been proved (see* [21]*) that*

$$\exists h_\Omega\in\mathcal{D}'(\partial\Omega)\ \textit{such that}\ \forall V\in\mathcal{D}(\mathbb{R}^N)^N,\ d\left(J\left(\Omega\right)\right)\left(\Omega,V\right)=<h_\Omega,V.n>_{\mathcal{D}'(\partial\Omega)\times\mathcal{D}(\Omega)}\ .$$

*This result expresses that the boundary derivative only takes into account the normal component of the deformation field.*

We shall now specify the relation (6) in the frame of eigenvalue problems. More precisely let us report the following Hadamard's theorem :

**Theorem 2.** ***(Hadamard)*** *Let $\Omega$ be an open set of class $C^2$ and $\lambda_k(\Omega)$ be its $k^{th}$ eigenvalue of the Laplace-Dirichlet operator. We assume that $\lambda_k(\Omega)$ is a simple eigenvalue. Then the function $t \mapsto \lambda_k(\Omega_t)$ has a derivative at $t = 0$, which is given by*

$$d\left(\lambda_k\left(\Omega\right)\right)\left(\Omega, V\right) = -\int_{\partial\Omega} \left(\frac{\partial u_k}{\partial n}\right)^2 V.n d\sigma \tag{7}$$

*where $u_k$ is the $k^{th}$ eigenfunction associated to $\lambda_k(\Omega)$, normalized by $\int_\Omega u_k^2(x)dx = 1$.*

A complete proof of this result is reported in [21] or in [12]. Here we will only present a formal proof of the relation (7). Let $u_k(t, x)$ and $\lambda_k(\Omega_t)$ be the $k^{th}$ eigenfunction and eigenvalue of the open set $\Omega_t$. We suppose that the derivative with respect to the real parameter $t$ exists. Deriving formally at $t = 0$ the relation

$$-\triangle u_k = \lambda_k(\Omega_t) u_k, \tag{8}$$

we obtain

$$-\triangle u_k' = \lambda_k(\Omega) u_k' + \lambda_k'(\Omega) u_k \text{ in } \Omega. \tag{9}$$

And the boundary condition

$$u_k(t, \left(Id + tV\right)(x)) = 0 \text{ for all } x \in \partial\Omega$$

becomes

$$u_k' + \sum_{i=1}^{N} \frac{\partial u_k}{\partial x_i} \frac{d\left(Id + tV\right)_i}{dt} = u_k' + \sum_{i=1}^{N} \frac{\partial u_k}{\partial x_i} V_i = 0.$$

Then

$$u_k' = -\frac{\partial u_k}{\partial n} V.n$$

since $\nabla u_k = \frac{\partial u_k}{\partial n}.n$ ($u_k$ equals zero on the boundary). Multiplying (9) by $u_k$ and integrating on $\Omega$ we obtain

$$-\int_\Omega u_k \triangle u_k' dx = \lambda_k(\Omega) \int_\Omega u_k u_k' dx + \lambda_k'(\Omega).$$

Hence by Green's formula,

$$-\int_{\partial\Omega} \frac{\partial u_k'}{\partial n} u_k d\sigma + \int_{\partial\Omega} \frac{\partial u_k}{\partial n} u_k' d\sigma - \int_\Omega u_k' \triangle u_k dx = \lambda_k(\Omega) \int_\Omega u_k u_k' dx + \lambda_k'(\Omega). \tag{10}$$

Moreover multiplying (8) at $t = 0$ by $u_k'$ and integrating on $\Omega$, we get the new relation

$$-\int_\Omega u_k' \triangle u_k dx = \lambda_k(\Omega) \int_\Omega u_k u_k' dx. \tag{11}$$

From (**??**), (10), (11), we conclude that

$$d\left(J\left(\Omega\right)\right)\left(\Omega, V\right) = \lambda_k'(\Omega) = -\int_{\partial\Omega} \left(\frac{\partial u_k}{\partial n}\right)^2 V.n d\sigma.$$

**Remark 3.** *According to the formula* $\lambda_k(l\Omega) = \frac{\lambda_k(\Omega)}{l^2}$ *(for* $l > 0$*), we can neglect the volume constraint in the problems (4) and (5). We have :*

$$\min_{|\Omega|=1,\ \Omega\ open} \lambda_k(\Omega) = \min_{\Omega\ open} |\Omega|\, \lambda_k(\Omega).$$

*To estimate the derivative of* $|\Omega|\, \lambda_k(\Omega)$ *,we have to compute the derivative of the volume with respect to a boundary variation. Such a result is easily obtained (see as an example* [12]*) going back to a fix domain by Green's formula. Then*

$$d(|\Omega|)(\Omega, V) = \int_{\partial\Omega} V.n d\sigma. \tag{12}$$

Thanks to this derivative we are able to describe the different steps of a boundary optimization algorithm designed to minimize the $k^{th}$ eigenvalue under volume constraint :

- **Discretization of the boundary :** let $\Omega_0$ be a given initial shape ; $n$ *control points* on $\partial\Omega_0$ are selected and referred to as $P_i$ $i = 1, ..., n$ .

- **Computation of the shape derivative :** let $n(P_i)$ be the exterior normal vector at the point $P_i$. Thanks to (7) and (12), for each $i = 1, ..., n$, we compute the derivative $d_i$ of the function of one variable

$$t \mapsto \lim_{t\to 0} \frac{J(\Omega_t^i) - J(\Omega_0)}{t}$$

  where

$$\begin{cases} J(\Omega) = |\Omega|\, \lambda_k(\Omega), \\ \Omega_t^i = \left\{(Id + tV_i)(x),\ x \in \mathbb{R}^N\right\}, \end{cases}$$

  for $V_i = v_i(x)n(x)$ where $v_i$ is a smooth real valued function equal to 1 at $P_i$ and zero on all other control points. Therefore we make use of the approach reported for example in [17] about the problem of electromagnetic shaping.

- **Exit criterium :** if the norm of the vector $(d_i)_{i=1,...,n}$ is small enough we stop the algorithm.

- **Boundary variation :** Each control point is moved along the exterior normal vector according to the opposite of the derivative $d_i$. Precisely, $P_i$ is translated from a vector $\alpha d_i n(P_i)$ for an adaptive choice of $\alpha > 0$. In this context the quasi-Newton method can also be carried out (see [17]).

- **Evaluation of the cost function :** After having verified that the new control points $(P_i)_{i=1,...,n}$ still generates a non crossing polygon, the value of the cost function for this new shape is evaluated. Then the descent step $\alpha$ is adjusted exactly the same way as in the finite dimension case for a classical gradient algorithm. An iteration takes end by going back to the step of derivative computation.

This method has been carried out in numerous situations different from the minimization of eigenvalues. However, it displays three major drawbacks that are respectively :

– no change of the topology,

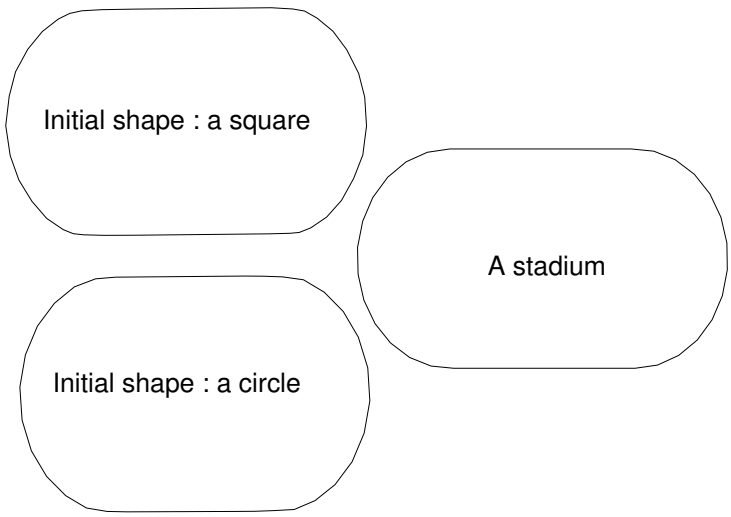


FIG. 2. Optimization obtained for several initial shapes

– the risk to get a local minimum,
– a major computational cost due to the remeshing process for each iteration.

However this method was shown to be efficient for the investigation of the second eigenvalue under convexity and volume constraints.

## 2.2. **Application to the problem of the stadium**

Let's go back to the problem (4). In [11] we could prove that the shape optimization problem

$$\min_{\Omega \text{ "convex", } |\Omega|=1} \lambda_2(\Omega) = \min_{\Omega \text{ "convex"}} \lambda_2(\Omega)\,|\Omega|$$

is well posed and that the stadium does not realize this minimum. Moreover we have shown that the optimal set has exactly two parallel segments on its boundary.

Here we wish to report a computational description of the optimal shape using the boundary variation method. The only difficulty we still have to deal with is taking into account the convexity constraint. Therefore the boundary variation algorithm is applied to the penalized function

$$\overline{J}(\Omega) := \lambda_2(\Omega)\,|\Omega| + p(|Co(\Omega)| - |\Omega|)^2$$

where $Co(\Omega)$ is the convex hull of $\Omega$ and $p$ is a positive penalization number. In figures 2 and 3 we present the results obtained with two distinct initial shapes. We observe the close similarity between a stadium and an optimal shape both from the geometrical viewpoint and their eigenvalues (see the table below where $j_0 \simeq 2.4048256$ and $j_1 \simeq 3.8317060$ are respectively the first zeros of the Bessel functions $J_0$ and $J_1$).

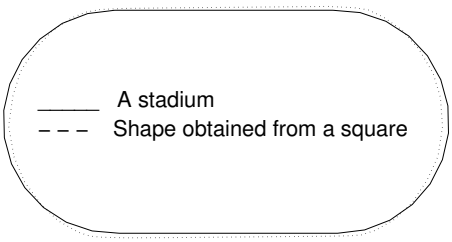


FIG. 3. A test of superposition

| Domain | Exact value | Computed value |
|---|---|---|
| Square | $5\pi^2$ | 49.348 |
| Ball | $j_1^2\pi$ | 46.124 |
| Best rectangle | $4\pi^2$ | 39.478 |
| Best ellipse | | 39.317 |
| The stadium | | 38.001 |
| The best convex domain[1] | | 37.980 |
| Two balls[2] | $2j_0^2\pi$ | 36.336 |

## 2.3. **Remarks and precisions**

In the previous computations, the boundary was discretized by 40 control points. We use classical finite element method to approximate the eigenvalues of the laplacian. Since we need a good approximation of the gradient of the eigenfunction on the boundary, we used a $P2$ finite element method with about 4000 triangles. In the cases of the disc or of the square, convergence takes place in less than 40 iterations.

Alternatively, we attempted to change the penalization approach for a projection method. At each iteration the new open set was replaced by its convex hull. Definitely this process was rather unefficient. The main reason is the following : taking the convex hull introduces segment on the boundary. Now, in many cases, the boundary variation method would like to push the nodes on the segment inside : then taking again the convex hull will restore the initial domain. At this point the method is stationary.

[1]that we have obtained with our algorithm.
[2]best domain (without constraint)

# 3. The relaxed approach

## 3.1. Introduction

We are now presenting a relaxed method by homogeneization which was very successful during this last years both for obtaining existence results in shape optimization and developing numerical algorithms.

Numerous optimization shape problems have no solution in the natural space of investigation (typically the open subsets of a compact domain). As weak solutions have been introduced in the field of partial differential equations, we are led to enlarge the space of shapes in order to obtain the existence of an optimal generalized shape. This process was applied namely in elastic structures (see [1], [3], [4]), where the expected domain is deduced from the homogenized solution after a screening technique.

## 3.2. The relaxed approach for eigenmodes

Let $D$ be a bounded regular open set of $\mathbb{R}^N$ and $\Omega$ an open set of $D$. Let $k \in \mathbb{N}^*$, we consider $u_\Omega \in H_0^1(\Omega)$, the $k^{\text{th}}$ eigenvalue of the laplacien. Therefore $u_\Omega$ is a solution of

$$\begin{cases} -\Delta u_\Omega = \lambda_k(\Omega) u_\Omega \text{ in } \Omega, \\ u_\Omega = 0 \text{ on } \partial\Omega. \end{cases}$$

We are interested in the shape optimization problem

$$\min\{\lambda_k(\Omega), \Omega \subset \mathbb{R}^2,\ |\Omega| = 1\} \text{ for } k \geq 3.$$

Apart from $k = 1$ and 2, it is not yet established whether such a problem has an open set solution. An important result in this field was obtained by Buttazzo and Dal Maso (see [5]) who proved that a quasi open set solution of (5) does exist.

Hereunder we will introduce a relaxed approached adapted to the problem (5). For each $\mu$ a non-negative Borel measure on $D$ we consider the new eigenvalue problem as follows :

$$\begin{cases} -\Delta u + \mu u = \lambda(\mu) u \text{ in } D, \\ u = 0 \text{ on } \partial D. \end{cases} \tag{13}$$

By analogy with the classical weak solution of partial differential equations we shall define a solution $(\lambda, u)$ of (13) as a set which satisfies

$$\begin{cases} \forall v \in V_\mu^0,\ \int_D \nabla u(x) \nabla v(x) dx + \int_D u(x) v(x) d\mu = \lambda(\mu) \int_D u(x) v(x) dx \\ \text{with } \lambda \in \mathbb{R} \text{ and } u \in V_\mu^0, \end{cases} \tag{14}$$

where $V_\mu^0 = H_0^1(D) \cap L_\mu^2(D)$ and $L_\mu^2(D)$ is the set of measurable functions whose square value admit a finite integral with respect to $\mu$. It is easily shown that for each $\mu \in \mathcal{M}_0(D)$ (the set of non-negative Borel measure on $D$) there are sequences of eigenvectors and eigenvalues satisfying (14).Let us now develop how to associate for each open set an element of $\mathcal{M}_0(D)$. Therefore we consider the capacity measure on $D$ defined for all $E \in \mathcal{P}(D)$ by :

$$cap_D(E) = \inf \left\{ \int_D |\nabla u|^2 (x)\, dx :\ u \in C_0^\infty(D),\ u \geq 1 \text{ in a neighborhood of } E \right\}. \tag{15}$$

For each $\Omega$, we introduce $\mu_\Omega \in \mathcal{M}_0(D)$ by

$$\mu_\Omega(B) = \begin{cases} 0 \text{ if } cap_D(B\backslash\Omega) = 0, \\ +\infty \text{ if } cap_D(B\backslash\Omega) > 0 \end{cases} \tag{16}$$

for each borelian set $B$ of $D$. So a natural injection of the open sets of $D$ into $\mathcal{M}_0(D)$ is built. We still have to define the volume of an element of $\mathcal{M}_0(D)$. Let us define for each $\mu \in \mathcal{M}_0(D)$

$$|\mu| = |\{x \in D,\ w_\mu(x) > 0\}| \tag{17}$$

where $w_\mu$ is the weak solution of

$$\begin{cases} -\Delta w_\mu + \mu w_\mu = 1 \text{ in } D, \\ w_\mu \in H_0^1(D). \end{cases}$$

**Remark 4.** *The expected equality $|\mu_\Omega| = |\Omega|$ is deduced from the maximum principle and the definition (16). Indeed the function $w_{\mu_\Omega}$ is formally zero outside $\Omega$ and sub-harmonic inside.*

The following theorem can thus be established (see [8] or [12]) :

**Theorem 5.** ***(Dal Maso-Mosco)*** *The set $\{\mu \in \mathcal{M}_0(D),\ |\mu| \leq 1\}$ is the completion for the $\gamma$-convergence of the family of open sets of $D$ whose volume are less than 1. Moreover, $\mathcal{M}_0(D)$ is compact for this topology.*

Having proven the continuity of the eigenvalues with respect to the $\gamma$-convergence, we prove that the following problem

$$\inf_{\mu \in \mathcal{M}_0(D),\ |\mu| \leq 1} \lambda_k(\mu) \tag{18}$$

does admit a solution.

### 3.3. Numerical approximation

In this chapter we shall show how the relaxed formulation (13) is well suited for finding an optimal shape. This method however exhibits an important practical difficulty : in such a context the volume of a measure appears to be non differentiable under its variation. A first application of this method for the minimization of eigenvalues was reported in [8]. In this paper the author bypasses this difficulty by a smoothing process. Unfortunately such a method requires the introduction in the algorithm of parameters whose adjustment proved to be delicate. For this reason we did not apply this method directly and preferred a mixed method that will be described in the following paragraph.

Let $(T_h)_{h=1,\ldots,l}$ be a triangulation of $D$. We aim to approximate the measure solution of (18) by a sequence of absolute continuous measures with respect to the Lebesgue measure whose densities are constant on each $T_h$. Having applied the algorithm we expect a measure of type (16), i.e. a zero or *"infinity"* value measure on each triangle. Let $(m_h)_{h=1,\ldots,l} \in \mathbb{R}_+^l$ be the measure whose density is given by

$$\sum_{h=1}^{l} m_h \chi_{T_h}$$

where $\chi_{T_h}$ is characteristic function of the triangle $T_h$. Let us recall from [8] the following (this formula can be obtained exactly as in section 2)

**Proposition 1.** *Let $\mu \in \mathcal{M}_0(D)$ with density $(m_h)_{h=1,\dots,l}$ and $k \in \mathbb{N}^*$. Let $\lambda_k(\mu)$, the $k^{th}$ eigenvalue of the operator $-\Delta + \mu I$ be a simple eigenvalue and $u_k$ its eigenfunction. Then*

$$\frac{\partial \lambda_k}{\partial m_h} = \int_{T_h} (u_k)^2(x)dx$$

Having computed this derivative, the aim is to apply a classical gradient algorithm to the sequence $(m_h)_{h=1,\dots,l}$ (exactly the same way we optimized the positions of control points here above). As it has been mentioned, the volume of a measure $\mu$ is not differentiable with respect to $m_h$. This problem is cancelled when the level set method is carried out as follows. Moreover, this level set method allows us to consider measure which takes only the values 0 and "infinity" (i.e. a large value $M$, see below).

## 4. The level set method and the relaxed formulation

### 4.1. Introduction

The method that will be reported is adapted from a paper of S. Osher and F. Santosa on problems of not degenerated densities (see [15]). A level set method exhibits several advantages. Particulary it does not imply any topological restrictions and enables working on a fixed regular mesh. Before detailing the different steps, let us recall the principle of a level set method. Let $\Omega \subset D$ be an initial shape. The first step will be to parametrize $\Omega$ by a $\Phi$ function so called *level set function* that must satisfy

$$\begin{cases} \Phi(x) < 0 \text{ if } x \in \Omega, \\ \Phi(x) > 0 \text{ if } x \in D \backslash \overline{\Omega}, \\ \Phi(x) = 0 \text{ if } x \in \partial\Omega. \end{cases}$$

Next, as suggested in [16], such a function will be initialized with the signed distance which is

$$\begin{cases} \Phi(x) = \text{dist}(x, \partial\Omega) \text{ if } x \in \Omega, \\ \Phi(x) = -\text{dist}(x, \partial\Omega) \text{ if } x \in D \backslash \Omega. \end{cases}$$

Let us observe that the constructed distance is generally not easily computed and often requires a specific attention (see [19]). As far as we are concerned, the initial shapes have been estimated thanks to genetic techniques (see 4.4). We choose an approximate signed-distance function which is constant on each triangle of the mesh. Its value in the triangle $T$ is computed by evaluating the distance between the center of mass of $T$ and the other centers of mass.

Once $\Phi$ defined, let the level set 0 (i.e. $\partial\Omega$) fluctuate with time under the vector field $vn$ (where $v$ is a real value function). In other words, if $x(t)$ describes the evolution of a point on $\partial\Omega$ under such a transformation, it has to verify

$$\Phi(t, x(t)) = 0 \text{ for all } t.$$

Differentiating this expression, we obtain

$$\frac{\partial \Phi}{\partial t}(t, x(t)) + \frac{dx}{dt}(t)\nabla_x \Phi(t, (t, x(t)) = \frac{\partial \Phi}{\partial t}(t, x(t)) + v(x(t))n(x(t)).\nabla_x \Phi(t, x(t)) = 0. \tag{19}$$

Now the normal to a level set in a non stationary point is given by

$$n = \frac{\nabla_x \Phi}{|\nabla_x \Phi|}(t, x(t)).$$

Hence according to (19),

$$\frac{\partial \Phi}{\partial t}(t, x(t)) + v(x(t)) \left|\nabla_x \Phi\right| (t, x(t)) = 0. \tag{20}$$

In order to compute the evolution of $\Phi$, we thus have to solve a Hamilton-Jacobi's equation. It has to be mentioned that the computation we have presented only concerns the level set 0. But usually the vector field $vn$ has a natural extension on a $D$,thus we solve the equation (20) in the whole set $D$.

A major difficulty that we did not yet mentioned and which will be treated in the next paragraph is the computation of a good speed field $vn$ for the shape optimization problem under investigation. For this purpose, a very natural approach has been introduced in [2] : consisting in choosing the vector field as the field obtained by boundary variation.

Before going into details, let us summarize the different steps of the level set optimization :

1. initialization of $\Phi$ by the signed distance,
2. computation of the speed field and checking an exit criterion,
3. propagation of the level set solving Hamilton-Jacobi's equation,
4. evaluation of the cost function,
5. redefinition of $\Phi$ and adjustment of the time step,
6. eventually, reinitialization of $\Phi$ with the signed distance. Back to step 2.

### 4.2. **Computation of the derivative**

We here report the computation of the variation of the eigenvalue with respect to $\Phi$. Here again, we only justify formally the derivative. We refer to [12] and [21] for a rigorous proof.

Let $\mu$ be a density on $D$ having only 0 or $M$ values. We notice that for $M$ large enough, the value of $\lambda_k(\mu)$ is closed of $\lambda_k(\Omega)$ where $\Omega = \{x \in D : \ \mu(x) = 0\}$ . We have to estimate the derivative of $\lambda_k(\mu)$ with respect to a variation of its level set subjected to a normal vector field $vn$. Let $k \in \mathbb{N}^*$ and $u_\mu$ the solution of

$$\left\{ \begin{array}{c} -\Delta u_\mu + \mu u_\mu = \lambda_k(\mu) u_\mu \text{ in } D, \\ u_\mu = 0 \text{ on } \partial D \end{array} \right. \tag{21}$$

according to the preceding paragraph. Let $u_{\mu,t}$ verifying

$$\left\{ \begin{array}{c} -\Delta u_{\mu,t} + \mu_t u_{\mu,t} = \lambda_k(\mu_t) u_{\mu,t} \text{ in } D, \\ u_{\mu,t} = 0 \text{ on } \partial D \end{array} \right. \tag{22}$$

for each $x \in D$,

$$\mu_t(x) = \left\{ \begin{array}{l} 0 \text{ for } x \in \{(Id + tvn)(y) : \ \mu(y) = 0\} \\ M \text{ otherwise.} \end{array} \right.$$

The derivability of $\lambda_k(\mu_t)$ and $u_{\mu,t}$ with respect to $vn$ can be proved as in [21] and [12]. Once this derivability admitted, let us establish the formula of the derivative.

After multiplying (22) by a test function of $C_0^\infty(D)$ and by integrating it using Green's formula we obtain the weak formulation

$$\int_D \nabla u_{\mu,t} \nabla w \, dx + \int_D u_{\mu,t} w \, d\mu_t = \lambda_k(\mu_t) \int_D u_{\mu,t} w \, dx \tag{23}$$

for each $w \in C_0^\infty(D)$. Hence

$$\int_D \nabla u_{\mu,t} \nabla w \, dx + M \int_{\omega_t} u_{\mu,t} w \, dx = \lambda_k(\mu_t) \int_D u_{\mu,t} w \, dx$$

where

$$\omega_t = \{x \in D : \ \mu_t(x) = M\}.$$

Deriving this latter identity with respect to $t$ and using classical identities, we have

$$\int_D \nabla u'_\mu \nabla w \, dx + M(\int_\omega u'_\mu w \, dx - \int_{\partial\omega} u_\mu w v \, dx) = \lambda'_k(\mu) \int_D u_\mu w \, dx + \lambda_k(\mu) \int_D u'_\mu w \, dx. \tag{24}$$

In order to eliminate $u'_\mu$ we evaluate (23) when $t = 0$ and $w = u'_\mu$ and we deduce

$$\int_D \nabla u_\mu \nabla u'_\mu \, dx + \int_D u_\mu u'_\mu \, d\mu = \lambda_k(\mu) \int_D u_\mu u'_\mu \, dx. \tag{25}$$

Combining (24) with $w = u_\mu$ and (25), we get

$$\lambda'_k(\mu) \int_D u_\mu^2 \, dx = -M \int_{\partial\omega} u_\mu^2 v \, dx$$

so

$$\lambda'_k(\mu) = -\frac{M \int_{\partial\omega} u_\mu^2 v \, dx}{\int_D u_\mu^2 \, dx} = -M \int_{\partial\omega} u_\mu^2 v \, dx$$

for a normalized function $u_\mu$.

Exactly in the same way as under boundary variation, the derivative of the measure's volume can be evaluated as :

$$|\mu|' = \int_{\partial\omega} v(x) \, dx.$$

### 4.3. **Computing a numerical solution of Hamilton-Jacobi's equation**

Now, the problem is to determine amongst the weak solutions the one that corresponds to the physical state under investigation. Introducing the concept of viscosity solution, in 1983 M. G. Crandall and P. L. Lions (see [6]), brought a satisfying answer to the global existence problem for Hamilton-Jacobi's equations. We shall not go into technical details for the definition of such a weak solution (we refer to [6] or [19]).

Our description will be limited to an algorithm reported in [16] designed to approached the weak viscosity solution of our problem.

4.3.1. *The numerical scheme of S. Osher and J. A. Sethian*

Let us consider the first order Cauchy's system :

$$\begin{cases} \frac{\partial \Phi}{\partial t}(t,x) - F(x)\,|\nabla \Phi(t,x)| = 0 \text{ in } \mathbb{R}_+ \times D, \\ \Phi(0,x) = u_0(x) \text{ in } \Omega, \end{cases}$$

where $\Omega$ is a bounded rectangle of $\mathbb{R}^N$ and $u_0$ and $v$ are given functions. Hereunder we shall use the classical notations for finite difference schemes on regular meshes of points indexed by $i, j$. Let us define $\Phi$ at the initial time by $\Phi(0,x) = u_0(x)$.Then the evolution of $\Phi$ after one time step $\Delta t$ is given by

$$\Phi_{ij}^{n+1} = \Phi_{ij}^{n} - \Delta t(\max(F_{ij},0)\nabla^+\Phi + \min(F_{ij},0)\nabla^-\Phi)$$

where

$$\nabla^+\Phi = \left[\max(D_{ij}^{-x}\Phi,0)^2 + \min(D_{ij}^{+x}\Phi,0)^2 + \max(D_{ij}^{-y}\Phi,0)^2 + \min(D_{ij}^{+y}\Phi,0)^2\right]^{1/2}$$

and

$$\nabla^-\Phi = \left[\max(D_{ij}^{+x}\Phi,0)^2 + \min(D_{ij}^{-x}\Phi,0)^2 + \max(D_{ij}^{+y}\Phi,0)^2 + \min(D_{ij}^{-y}\Phi,0)^2\right]^{1/2},$$

where

$$D_{ij}^{+x}\Phi = \frac{\Phi_{i+1,j} - \Phi_{i,j}}{\Delta x}$$

for a space step equal to $\Delta x$. The quantities $D_{ij}^{-x}\Phi$, $D_{ij}^{+y}\Phi$ and $D_{ij}^{-y}\Phi$ are easily deduced. Finally, to define completely our problem we add the following boundary condition :

$$\frac{\partial \nabla \Phi(t,x)}{\partial n} = 0 \text{ on } \partial\Omega.$$

## 4.4. **Initialization of the method**

In this kind of minimization problem, the choice of the initial shape can be very important. For example, if we use this level set method for $\lambda_3$ starting from an ellipse with eccentricity greater than $\frac{\sqrt{3}}{2}$, the method converges to the union of 3 identical discs which is a local minimum. Therefore, it seems important to start not too far of the global minimum. It is the reason why we have chosen to use a genetic algorithm as a preprocessor. This genetic algorithm is inspired from the ideas developed by M. Schoenauer see e.g. [20]. We refer to [22] for more details.

## 4.5. **Handling non simple eigenvalues**

One difficulty we met applying our method was the emergence of multiple eigenvalues. In such estimation, eigenvalues are well known to loose their derivability with respect to a boundary variation as well as in the context of the relaxed approach. From a numerical viewpoint, an oscillatory behavior has been observed in the case of multiple eigenvalues.

In order to slow down the convergence to such a shape, we modify our algorithm the following way. Every time we estimated the eigenvalue $\lambda_m$ too close from $\lambda_{m-1}$ (i.e. $|\Omega|\,(\lambda_m - \lambda_{m-1}) \leq 1$), we modified the descent direction by favoring the minimization of $\lambda_m + \lambda_{m-1}$ as compared with $\lambda_m$. In other words we replaced the vector field $-u_m^2 n$ by $-(\frac{u_m^2+u_{m-1}^2}{2})n$.

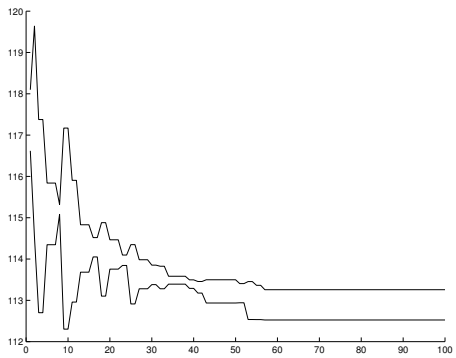


FIG. 4. Evolution of $\lambda_3$ and $\lambda_4$

In figure 4, the evolution of $\lambda_4$ et $\lambda_3$ is presented step by step during the minimization of $\lambda_4$. A posteriori, it can be observed that all the computed optimal shapes in fact generate at least double eigenvalues. The author believes that this is a generic property (but was unable to prove it) :

**Open problem :** let $\Omega_k^*$ the minimizer of the problem (5), then $\lambda_{k-1}(\Omega_k^*) = \lambda_k(\Omega_k^*)$ (let us recall from [10] that we always have $\lambda_{k+1}(\Omega_k^*) \neq \lambda_k(\Omega_k^*)$.

### 4.6. **The volume constraint**

In order to preserve the volume of $\mu$ during iterations, we used the Lagrange's multiplier technic reported in [15], which consists in applying the optimization algorithm to the function

$$L(\Phi, \nu) = \lambda_k(\mu_\Phi) + \nu \left|\mu_\Phi\right|,$$

where $\mu_\Phi$ is the measure density associated to $\Phi$ by the relation

$$\mu_\Phi(x) = \begin{cases} 0 \text{ if } \Phi(x) > 0, \\ M \text{ otherwise.} \end{cases}$$

where $M$ is a fixed positive constant. According to the derivative computed in the paragraph 4.2, the level set function $\Phi$ satisfies the Hamilton-Jacobi's equation

$$\frac{\partial \Phi}{\partial t}(t,x) - (-M u_k^2(t,x) + \nu) \left|\nabla \Phi(t,x)\right| = 0 \text{ in } \mathbb{R}_+ \times \Omega$$

where $u_k(t,.)$ is the eigenfunction associated to $\lambda_k(\mu_{\Phi(t,.)})$. As suggested by S. Osher and F. Santosa in [15], at each iteration we adapted our Lagrange multiplier $\nu$ to preserve the volume constraint.

### 4.7. **Parameters of the numerical experiments**

In all our experiments on eigenvalues, we used a regular mesh of size $\Delta x = \Delta y = 1/80$. The system (22) is solved by a classic $P1$ finite element method. The parameter $M$ (the maximum of the density $\mu$) was fixed to the value 800. Usually the algorithm found the minimum in less than 100 iterations (see figure 4.5).

## 5. Results and conclusion

The shapes obtained following the combination of the genetic algorithm reported in [20] and the topological optimization by the mixed relaxation/level set method are presented on figures 6, 7, 8, 9 and 10. The quality of the results generated with the two first eigenvalues is particulary satisfying.
For $\lambda_3$, it has been conjecture by G. Szegö (see [24]) that the minimum should be a disc. It is in accordance with our numerical results.
For $\lambda_4$, the conjecture states that the minimum is the union of 2 balls whose radii are in the ratio of $\frac{j_{0,1}}{j_{1,1}}$ (where $j_{0,1}$ and $j_{1,1}$ are the first zeros of the Bessel function $J_0$ and $J_1$). Our method seems to confirm this conjecture.

G. Szegö asked the question to know whether the minimum was always to be chosen amongst discs or union of discs. The unexpected result obtained for the minimization of $\lambda_5$ has to be pointed out. As reported by S. A. Wolf and J. B. Keller in [24], union of balls do not account for the only optimal shapes. It is noteworthy that this occurs as soon as the $5^{th}$ eigenvalue (the values reported in table 5 being in fact upper bounds).

On the other hand , let us point out the limit of the process. As already mentioned, the major difficulty we met is the non differentiability of multiple eigenvalues. The strategy proposed in the paragraph 4.5 requires to choose arbitrarily the moment when the vector field has to be modified.

For $\lambda_7$, this approach did not allow to identify the shape presented in table 5 (see the shape obtained by our method in the figure 9). Indeed we obtain it using the following theorem of S. A. Wolf et J. B. Keller : let $k \in \mathbb{N}^*$, and for $j = 1, ..., k$ we define $\lambda_j^* = \min_{|\Omega|=1} \lambda_j(\Omega) = \lambda_j(\Omega_j^*)$.

**Theorem 6.** ***(Wolf and Keller)*** *Let $\Omega_k^*$ be a non convex open set of $\mathbb{R}^N$, which minimizes $\lambda_k$ among open sets of volume 1. Then :*

$$(\lambda_k(\Omega_k^*))^{N/2} = \min_{1 \leq j \leq (k-1)/2} ((\lambda_j^*)^{N/2} + (\lambda_{k-j}^*)^{N/2})$$

*and*

$$\Omega_k^* = \left[ \left( \frac{\lambda_i^*}{\lambda_k^*} \right)^{1/2} \Omega_i^* \right] \cup \left[ \left( \frac{\lambda_{k-i}^*}{\lambda_k^*} \right)^{1/2} \Omega_{k-i}^* \right].$$

So once known the $k$ first optimal domains, the non convex optimal domain minimizing $\lambda_{k+1}$ can be determined. This recursive procedure enabled to identify the shape figured in table 5 that has a smaller $\lambda_7$ that the one deduced from our algorithm.

Nevertheless, it seems that the combination of the Level Set method of S. Osher and J. A. Sethian with the relaxed approach is quite promising for problems with Dirichlet's boundary condition. The author believes that this approach can be applied to a wide variety of shape optimization problems.

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| No | Optimal union of discs | | Computed shapes | |
|---|---|---|---|---|
| 3 | | 46.125 | | 46.125 |
| 4 | | 64.293 | | 64.293 |
| 5 | | 82.462 | | **78.47** |
| 6 | | 92.250 | | **88.96** |
| 7 | | 110.42 | | **107.47** |
| 8 | | 127.88 | | **119.9** |
| 9 | | 138.37 | | **133.52** |
| 10 | | 154.62 | | **143.45** |

FIG. 5. Best-known shapes

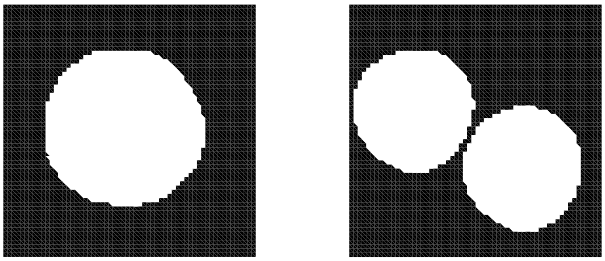

FIG. 6. $\lambda_1$ (left) and $\lambda_2$ (right)


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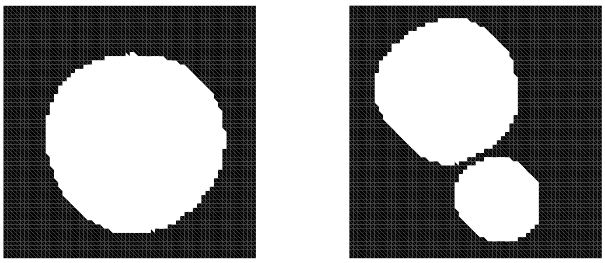

FIG. 7. $\lambda_3$ (left) and $\lambda_4$ (right)

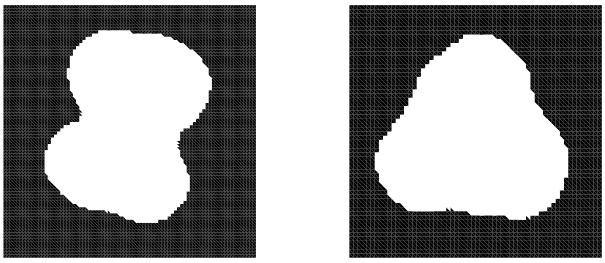

FIG. 8. $\lambda_5$ (left) and $\lambda_6$ (right)

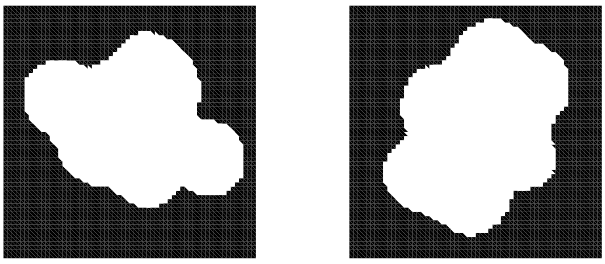

FIG. 9. $\lambda_7$ (left) and $\lambda_8$ (right)

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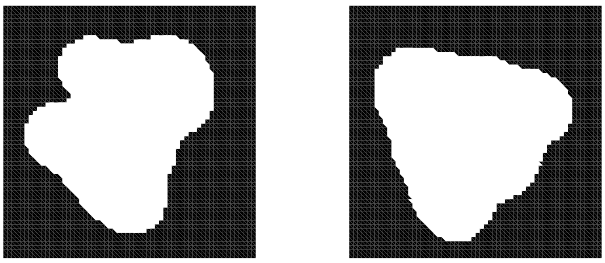

FIG. 10. $\lambda_9$ (left) and $\lambda_{10}$ (right)


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