LOGICAL RULES ARE FRACTIONS

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FACT:

Logical rules are written as fractions

$$\frac{H}{C}$$

IN FACT:

Logical rules ARE fractions

$$\frac{C}{H}$$

I – FRACTIONS

P. Gabriel & M. Zisman (1967)

Categorical fractions

Given two categories **S**, **T** and a functor $\boxed{\mathbf{S} \xrightarrow{F} \mathbf{T}}$

a fraction $\frac{c}{h} : \mathcal{C} \to \mathcal{H}$ is ("essentially")

a cospan (h, c) in **S**

$$\begin{tikzcd} & \mathcal{H}' & \\ \mathcal{H} \arrow[ur, "h"] & & \mathcal{C} \arrow[ul, "c"'] \end{tikzcd}$$

such that $F(h)$ is invertible in **T**

$$\begin{tikzcd} & F(\mathcal{H}') \arrow[dl, shift left, "F(h)^{-1}"] & \\ F(\mathcal{H}) \arrow[ur, shift left, "F(h)"] & & F(\mathcal{C}) \arrow[ul, "F(c)"'] \arrow[ll, bend left, "F(h)^{-1} \circ F(c)"] \end{tikzcd}$$

A notation for **"both"** **S** and **T**:

$$\begin{tikzcd} & \mathcal{H}' \arrow[dl, shift left] & \\ \mathcal{H} \arrow[ur, "h"] & & \mathcal{C} \arrow[ul, "c"'] \arrow[ll, bend left, "\frac{c}{h}"] \end{tikzcd}$$

Localisation and reflection

A functor $F : \mathbf{S} \to \mathbf{T}$ is

- a localisation if it adds inverses for some morphisms in S.
- a reflector if $\mathbf{T}$ is a full subcategory of $\mathbf{S}$ and F is left adjoint to inclusion. Such an adjunction is called a reflection

$$\mathrm{Hom}_{\mathbf{S}}(S, T) \cong \mathrm{Hom}_{\mathbf{T}}(F(S), T)$$

$$\begin{tikzcd} \mathbf{S} \arrow[r, "F"'] \arrow[r, phantom, bend left=20, "\top"] & \mathbf{T} \arrow[l, bend right, "\supseteq_{\text{full}}"'] \end{tikzcd}$$

Theorem. Every reflector is a localisation.

A (usual) fraction: $\boxed{\frac{3}{4}}$

Integers:

$$\begin{tikzcd} & \mathbb{Z} & \\ \mathbb{Z} \arrow[ur, "\times 3"] & & \mathbb{Z} \arrow[ul, "\times 4"'] \end{tikzcd}$$

Rationals:

$$\begin{tikzcd} & \mathbb{Q} \arrow[dr, shift right, "\times \frac{1}{4}"'] & \\ \mathbb{Q} \arrow[ur, "\times 3"] \arrow[rr, bend right, "\times \frac{3}{4}"'] & & \mathbb{Q} \arrow[ul, shift right, "\times 4"'] \end{tikzcd}$$

“Both:”

$$\begin{tikzcd} & \mathbb{Z} \arrow[dr, shift right] & \\ \mathbb{Z} \arrow[ur, "\times 3"] \arrow[rr, bend right] & & \mathbb{Z} \arrow[ul, shift right, "\times 4"'] \end{tikzcd}$$

(Usual) fractions are categorical fractions

S $= \mathrm{Module}(\mathbb{Z})$ the category of modules over $\mathbb{Z}$

T $= \mathrm{Vect}(\mathbb{Q})$ the category of vector spaces over $\mathbb{Q}$

$\boxed{F : \mathrm{Module}(\mathbb{Z}) \to \mathrm{Vect}(\mathbb{Q})}$ is the extension of scalars:

$$F(V) = \mathbb{Q} \otimes V$$

FACT. A (usual) fraction is a categorical fraction wrt F

Ex. Then $F(\mathbb{Z}) = \mathbb{Q}$ and
the integer 4 non-invertible in $\mathbb{Z}$ becomes
the rational 4 invertible in $\mathbb{Q}$

A logical rule: $\boxed{\dfrac{p \quad p\Rightarrow q}{q}}$ (*Modus Ponens*)

Sets of formulas:

$$
\begin{tikzcd}
 & \{p,\ p\Rightarrow q,\ q\} & \\
\{p,\ p\Rightarrow q\} \arrow[ur, "\subseteq"] & & \{q\} \arrow[ul, "\subseteq"']
\end{tikzcd}
$$

Sets of generated theorems:

$$
\begin{tikzcd}
 & \{p,\ p\Rightarrow q,\ q\} \arrow[dl, shift left, "="] & \\
\{p,\ p\Rightarrow q,\ q\} \arrow[ur, shift left, "="] & & \{q\} \arrow[ul, "\subseteq"'] \arrow[ll, bend left, "\subseteq"]
\end{tikzcd}
$$

“Both:”

$$
\begin{tikzcd}
 & \{p,\ p\Rightarrow q,\ q\} \arrow[dl, shift left] & \\
\{p,\ p\Rightarrow q\} \arrow[ur, shift left, "\subseteq"] & & \{q\} \arrow[ul, "\subseteq"'] \arrow[ll, bend left]
\end{tikzcd}
$$

Logic, specifications, theories

INFORMALLY:

Given a *logic*, with its formulas and rules, we say that:

- a *specification* S is a family of formulas
- a *theory* T is a family of formulas which is closed under application of the rules

Logical rules are categorical fractions

INFORMALLY:

Let us assume the existence of:

- a category $\mathbf{S}$ of specifications
- a category $\mathbf{T}$ of theories
- and a generating functor $\boxed{F : \mathbf{S} \to \mathbf{T}}$
  such that $F(S)$ is the family of formulas (or theorems)
  deduced from the formulas (or axioms) in S

FACT. A logical rule is a categorical fraction wrt F

Ex. When *modus ponens* is a rule of the logic:
let $S = \{p,\ p \Rightarrow q\}$:
it is a specification that does not contain q
then $F(S) = \{p,\ p \Rightarrow q,\ q,\ \ldots\}$:
it is a theory that contains q

To sum up (I)

A LOGICAL RULE IS A FRACTION

fraction $\dfrac{n}{d}$ rule $\dfrac{\mathcal{H}}{\mathcal{C}}$

$$
\begin{tikzcd}
 & Z \arrow[dr, blue, shift right] & \\
Z \arrow[ur, "\times n"] & & Z \arrow[ul, "\times d"']
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
 & \mathcal{H} \cup \mathcal{C} \arrow[dl, blue, shift left] & \\
\mathcal{H} \arrow[ur, "h"] & & \mathcal{C} \arrow[ul, "c"']
\end{tikzcd}
$$

A logical rule $\boxed{\dfrac{\mathcal{H}}{\mathcal{C}}}$ IS a fraction $\boxed{\dfrac{c}{h}}$

"THE HYPOTHESIS BECOMES INVERTIBLE"

II – SKETCHES ("**E**squisses")

C. Ehresmann (1968)

In this talk:

SKETCH = LIMIT SKETCH

Sketches and their realisations

A sketch $\mathbf{E}$ is a *presentation* for a category with limits $\overline{\mathbf{E}}$
It is made of:

- objects,
- "morphisms" with only "some" identities and composition
- and "limits" with only "some" associated tuples

which become *actual* objects, morphisms and limits in $\overline{\mathbf{E}}$

A realisation R of $\mathbf{E}$ is a *set-valued model* of $\mathbf{E}$:
it maps each object, morphism and limit in $\mathbf{E}$
to a set, function and limit in **Set**

Equivalently, it is a limit-preserving functor $R : \overline{\mathbf{E}} \to \mathbf{Set}$

$\boxed{\mathrm{Real}(\mathbf{E})}$ denotes the category of realisations of $\mathbf{E}$

$\mathrm{Real}(\mathbf{E})$ is a kind of generalised presheaf

- A linear sketch $\mathbf{E}$ has only objects and morphisms (no limit) then $\mathrm{Real}(\mathbf{E}) = \mathrm{Func}(\overline{\mathbf{E}}, \mathbf{Set})$ is a presheaf category

  Ex. $\mathrm{Real}(\ V \underset{t}{\overset{s}{\leftleftarrows}} E\ )$

  is the category $\mathbf{Gr}$ of directed graphs

- In general, for a [limit] sketch $\mathbf{E}$, $\mathrm{Real}(\mathbf{E})$ is a locally presentable category

  Ex. $\mathrm{Real}(\ V \underset{t}{\overset{s}{\leftleftarrows}} E = V^2\ )$

  is the category $\mathbf{Gr}_1$ of directed graphs with exactly one edge $n \to p$ for each pair of vertices (n, p)

"Many" properties of presheaves
are still valid for locally presentable categories

“What is a logic?”

Yet another proposal:

A LOGIC IS A SKETCH

This is a very simple and very abstract algebraic proposal...

A logic with *modus ponens*

Syntactic entities:

formulas (Form) and theorems (Theo)

Each theorem is a formula

Formation rule:

$$(IM) \quad \frac{p,\ q : \mathrm{Form}}{p \Rightarrow q\ : \mathrm{Form}}$$

If p and q are formulas then $p \Rightarrow q$ is a formula

Deduction rule:

$$(MP) \quad \frac{[\, p,\ q,\ p \Rightarrow q : \mathrm{Form}\,] \quad p,\ p \Rightarrow q\ : \mathrm{Theo}}{q : \mathrm{Theo}}$$

If p and $p \Rightarrow q$ are theorems then q is a theorem

A sketch for syntactic entities

Syntactic entities:

formulas (Form) and theorems (Theo)

Each theorem is a formula

Sketch:

$$\boxed{\mathrm{Form} \leftarrowtail \mathrm{Theo}}$$

A realisation R of this sketch is:

- a set of formulas $R(\mathrm{Form})$
- a set of theorems $R(\mathrm{Theo})$
- with $R(\mathrm{Theo}) \subseteq R(\mathrm{Form})$

A sketch for the formation rule

Formation rule: (IM) $\dfrac{p,\ q : \mathrm{Form}}{p \Rightarrow q\ : \mathrm{Form}}$

If p and q are formulas then $p \Rightarrow q$ is a formula

$C_{IM} = \mathrm{Form}, \quad H_{IM} = \mathrm{Form}^2$

$$\begin{tikzcd}
H_{IM} \arrow[rr, "c_{IM}"] \arrow[dr, bend right] \arrow[dr, bend left] & & C_{IM} \arrow[dl] \\
& \mathrm{Form} &
\end{tikzcd}$$

A realisation R of this sketch is:

- a set of formulas $R(\mathrm{Form})$
- the sets $R(C_{IM}) = R(\mathrm{Form})$ and $R(H_{IM}) = R(\mathrm{Form})^2$
- and a function $R(c_{IM}) : R(H_{IM}) \to R(C_{IM})$
  denoted $c_{IM}(p, q) = p \Rightarrow q$

A sketch for the deduction rule

Deduction rule (simplified): $(MP)\quad \dfrac{p,\ p\Rightarrow q\ :\mathrm{Theo}}{q:\mathrm{Theo}}$

If p and $p\Rightarrow q$ are theorems then q is a theorem

$C_{MP}=\mathrm{Theo},\quad H_{MP}\approx\mathrm{Theo}^2$ (simplified!)

$$\mathbf{E_T} =
\begin{tikzcd}
 & & & H_{MP} \arrow[r, "c_{MP}"] \arrow[dlll, dotted] \arrow[dll, dotted] \arrow[dd, shift right] \arrow[dd, shift left] & C_{MP} \arrow[ddl] \\
H_{IM} \arrow[r, "c_{IM}"] \arrow[dr, shift right] \arrow[dr, shift left] & C_{IM} \arrow[d] & & & \\
 & \mathrm{Form} & & \mathrm{Theo} \arrow[ll, tail] &
\end{tikzcd}$$

A realisation of $\mathbf{E_T}$ is a theory: $\boxed{\mathrm{Real}(\mathbf{E_T})=\mathbf{T}}$

To sum up (II)

A LOGIC IS A SKETCH

To keep:

- a logic is a sketch $\boxed{\mathbf{E_T}}$
- the category of theories is $\mathbf{T} = \mathrm{Real}(\mathbf{E_T})$

To improve:

- a model of a theory T in a theory D is an arrow $M : T \to D$ in $\mathbf{T}$
- a rule is an arrow $H \xrightarrow{c} C$ in $\mathbf{E_T}$

Still missing:

- specifications as presentations of theories?
- rules as fractions?

III – SKETCHES and FRACTIONS

From theories to specifications

Morphisms of sketches

A morphism of sketches $\mathbf{E}_1 \xrightarrow{\sigma} \mathbf{E}_2$
induces a functor

$$\boxed{\mathrm{Real}(\mathbf{E}_1) \longleftarrow \mathrm{Real}(\mathbf{E}_2)}$$

$$\begin{tikzcd}
\mathbf{E}_1 \arrow[rr, "\sigma"] \arrow[dr, "R_1 = R_2 \circ \sigma"'] & & \mathbf{E}_2 \arrow[dl, "R_2"] \\
& \mathbf{Set} &
\end{tikzcd}$$

Theorem. This functor has a left adjoint.

$$\boxed{\begin{tikzcd} \mathrm{Real}(\mathbf{E}_1) \arrow[r, bend right] \arrow[r, phantom, "\top"] & \mathrm{Real}(\mathbf{E}_2) \arrow[l, bend right] \end{tikzcd}}$$

Thus: each realisation of $\mathbf{E}_1$ generates a realisation of $\mathbf{E}_2$

Cycles

A "cycle" in **E** is defined by considering that projections are oriented both sides

Ex. The formation rule $(IM)\ \dfrac{p,\ q : \mathrm{Form}}{p \Rightarrow q\ : \mathrm{Form}}$

$$\begin{tikzcd}
H_{IM} \arrow[rr, "c_{IM}"] \arrow[dr, bend right] \arrow[dr, bend left] & \arrow[d, phantom, "\circlearrowright"] & C_{IM} \arrow[dl] \\
 & \mathrm{Form} &
\end{tikzcd}$$

Because of cycle "$\circlearrowright$", in a theory T,
for ALL pairs of formulas (p, q) there is a formula $p \Rightarrow q$

Required: in a specification S,
for SOME pairs of formulas (p, q) there is a formula $p \Rightarrow q$

Breaking cycles

The cycles in $\mathbf{E}$ can be broken by making c partial:

$$\text{replace } \boxed{H \xrightarrow{c} C} \text{ by } \boxed{H \xleftarrow{h} H' \xrightarrow{c} C}$$

By breaking the cycles in $\mathbf{E_T}$ we get a sketch $\mathbf{E_S}$ and a morphism called a localiser

$$\boxed{\mathbf{E_S} \longrightarrow \mathbf{E_T}}$$

such that the corresponding adjunction is a reflection

$$\boxed{\begin{tikzcd} \mathrm{Real}(\mathbf{E_S}) = \mathbf{S} \arrow[rr, "\top", "F"'] & & \mathbf{T} = \mathrm{Real}(\mathbf{E_T}) \arrow[ll, bend right, "\supseteq_{\text{full}}"'] \end{tikzcd}}$$

Definitions (1/2)

A diagrammatic logic is a sketch $\boxed{\mathbf{E_T}}$

- the category of theories is $\mathbf{T} = \mathrm{Real}(\mathbf{E_T})$

Let $\sigma : \mathbf{E_S} \to \mathbf{E_T}$ be a localiser

it defines a reflector $F : \mathbf{S} \to \mathbf{T}$

- the category of specifications is $\mathbf{S} = \mathrm{Real}(\mathbf{E_S})$
- the theory generated by a specification S is $F(S)$
- a model of a specification S in a theory D is
  an arrow $M : S \to D$ in $\mathbf{S}$
  [or equivalently, an arrow $M : F(S) \to D$ in $\mathbf{T}$]

$$
\begin{tikzcd}
\mathbf{E_S}^{op} \arrow[rr, "\sigma^{op}"] \arrow[d, "\mathcal{Y}_{\mathbf{S}}"'] & & \mathbf{E_T}^{op} \arrow[d, "\mathcal{Y}_{\mathbf{T}}"] \\
\mathrm{Real}(\mathbf{E_S}) = \mathbf{S} \arrow[rr, "F"'] \arrow[rr, phantom, "\top", shift left=2] \arrow[dr, "S"'] & \arrow[d, phantom, "\xRightarrow{M}"] & \mathbf{T} = \mathrm{Real}(\mathbf{E_T}) \arrow[ll, bend right, "\supseteq_{\text{full}}"'] \arrow[dl, "T"] \\
& \mathbf{Set} &
\end{tikzcd}
$$

$\mathcal{Y}$ is the Yoneda contravariant embedding

$\boxed{\mathcal{Y} : \mathbf{E}^{op} \to \mathrm{Real}(\mathbf{E})}$ such that $\boxed{\mathcal{Y}(X) = \mathrm{Hom}_{\overline{\mathbf{E}}}(X, -)}$

Definitions (2/2)

Given a diagrammatic logic $\boxed{\mathbf{E_T}}$ with a localiser $\sigma : \mathbf{E_S} \to \mathbf{E_T}$

- a rule is a fraction in $\mathbf{E_S}$ wrt σ

Thus, using the Yoneda contravariant embedding $\mathcal{Y}$:

- a rule is a fraction in $\mathbf{S}$ wrt F (in the image of $\mathbf{E_S}$ by $\mathcal{Y}$)

The Yoneda contravariant embedding

$\mathcal{Y} : \mathbf{E}^{op} \to \mathrm{Real}(\mathbf{E})$ is "nearly as nice"
for *locally presentable* categories as for *presheaves*

- $\mathcal{Y}$ is faithful
- $\mathcal{Y}$ maps limits to colimits
- $\mathcal{Y}(\mathbf{E}^{op})$ is dense in $\mathrm{Real}(\mathbf{E})$:
  each realisation of $\mathbf{E}$ is the colimit of realisations in $\mathcal{Y}(\mathbf{E}^{op})$

The category $\mathrm{Real}(\mathbf{E})$ has all colimits (like *presheaves*)
BUT they cannot be computed sortwise (unlike *presheaves*)

Ex. Coproduct of graphs $\begin{tikzcd} v \arrow[loop left] \end{tikzcd}$ and $\begin{tikzcd} w \arrow[loop right] \end{tikzcd}$ is

$$\begin{tikzcd} v \arrow[loop left] & w \arrow[loop right] \end{tikzcd} \text{ in } \mathbf{Gr} \quad \text{BUT} \quad \begin{tikzcd} v \arrow[loop left] \arrow[r, bend left] & w \arrow[loop right] \arrow[l, bend left] \end{tikzcd} \text{ in } \mathbf{Gr}_1$$

Breaking the cycle for (IM): sketches

Adding a rule is a morphism:

$$\mathbf{E}_0 \longrightarrow \mathbf{E_T}$$

$$\boxed{\begin{tikzcd} H_{IM} \arrow[dr] \arrow[dr, bend left] & & C_{IM} \arrow[dl] \\ & \mathrm{Form} & \end{tikzcd}} \to \boxed{\begin{tikzcd} H_{IM} \arrow[rr] \arrow[dr] \arrow[dr, bend left] & & C_{IM} \arrow[dl] \\ & \mathrm{Form} & \end{tikzcd}}$$

that gets factorised by breaking cycles (theorem!):

$$\mathbf{E}_0 \longrightarrow \mathbf{E_S} \longrightarrow \mathbf{E_T}$$

$$\boxed{\begin{tikzcd} H_{IM} \arrow[dr] \arrow[dr, bend left] & & C_{IM} \arrow[dl] \\ & \mathrm{Form} & \end{tikzcd}} \to \boxed{\begin{tikzcd} & H'_{IM} \arrow[dl, tail] \arrow[dr] & \\ H_{IM} \arrow[dr] \arrow[dr, bend left] & & C_{IM} \arrow[dl] \\ & \mathrm{Form} & \end{tikzcd}} \to \boxed{\begin{tikzcd} H_{IM} \arrow[rr] \arrow[dr] \arrow[dr, bend left] & & C_{IM} \arrow[dl] \\ & \mathrm{Form} & \end{tikzcd}}$$

Breaking the cycle for (IM): realisations

$$
\begin{tikzcd}
\mathbf{E}_0^{op} \arrow[rr] \arrow[d, "\mathcal{Y}_0"'] & & {\mathbf{E_S}}^{op} \arrow[rr] \arrow[d, "\mathcal{Y}_\mathbf{S}"'] & & {\mathbf{E_T}}^{op} \arrow[d, "\mathcal{Y}_\mathbf{T}"] \\
\mathrm{Real}(\mathbf{E}_0) \arrow[rr, "\top"] & & \mathbf{S} \arrow[ll, bend right] \arrow[rr, "\top", "F"'] & & \mathbf{T} \arrow[ll, bend right, "\supseteq \mathrm{full}"']
\end{tikzcd}
$$

Thus, focusing on $\mathcal{Y}(-)(\mathrm{Form})$

$$
\boxed{\{p,q\}\ \ \{r\}} \rightarrow \boxed{\begin{tikzcd}[column sep=small] & \{p,q,r\} & \\ \{p,q\} \arrow[ur] & & \{r\} \arrow[ul] \end{tikzcd}} \rightarrow \boxed{\{p,q,p\Rightarrow q,\ldots\} \xleftarrow{r \mapsto p\Rightarrow q} \{r,\ldots\}}
$$

we get the fraction

$$
\begin{tikzcd}
& \{p,\ q,\ p\Rightarrow q\} \arrow[dl, shift left] & \\
\{p,\ q\} \arrow[ur, shift left] & & \{p\Rightarrow q\} \arrow[ul] \arrow[ll, bend left]
\end{tikzcd}
$$

"Moral": fractions as presentations

- A specification S in $\mathbf{S}$ is a presentation for the theory $T = F(S)$ in $\mathbf{T}$
- A morphism $s : S \to S'$ in $\mathbf{S}$ is a presentation for the morphism $F(s) : F(S) \to F(S')$ in $\mathbf{T}$

  One gets SOME morphisms $t : F(S) \to F(S')$ in $\mathbf{T}$

  Ex. Every ring is a monoid
- A fraction $\frac{c}{h} : S \to S'$ wrt F is a presentation for the morphism $F(h)^{-1} \circ F(c) : F(S) \to F(S')$ in $\mathbf{T}$

  One gets ALL morphisms $t : F(S) \to F(S')$ in $\mathbf{T}$

  Ex. Every boolean algebra is a ring

  "MORPHISMS OF THEORIES are presented by FRACTIONS OF SPECIFICATIONS"

To sum up (III):

A LOGIC IS A SKETCH

and

A LOGICAL RULE IS A FRACTION

IV – Application: COMPUTATIONAL EFFECTS

The state effect in object-oriented programming

```
Class BankAccount {...
    int balance (void) const ;
    void deposit (int) ;
...}
```

From this C++ syntax to an equational specification?

- apparent specification

$$\begin{array}{l} \texttt{balance} : \texttt{void} \to \texttt{int} \\ \texttt{deposit} : \texttt{int} \to \texttt{void} \end{array}$$

  ⊞ the object-oriented flavour is preserved

  ⊟ BUT the intended interpretation is not a model

- explicit specification

$$\begin{array}{l} \texttt{balance} : \texttt{state} \to \texttt{int} \\ \texttt{deposit} : \texttt{int} \times \texttt{state} \to \texttt{state} \end{array}$$

  ⊞ the intended interpretation is a model,

  ⊟ BUT the object-oriented flavour is not preserved

- decorated specification

$$\boxed{\begin{array}{l} \texttt{balance}^{\mathtt{a}} : \texttt{void} \to \texttt{int} \\ \texttt{deposit}^{\mathtt{m}} : \texttt{int} \to \texttt{void} \end{array}}$$

  where the decorations are:

  m for modifiers (methods)
  a for accessors ("const" methods)

  ⊞ the intended interpretation is a model

  ⊞ AND the object-oriented flavour is preserved

Morphisms of logics:

$$\begin{tikzcd}
 & \boxed{\begin{array}{l} \texttt{b}^{\mathtt{a}} : \texttt{void} \to \texttt{int} \\ \texttt{d}^{\mathtt{m}} : \texttt{int} \to \texttt{void} \end{array}} \arrow[dl, mapsto] \arrow[dr, mapsto] & \\
\boxed{\begin{array}{l} \texttt{b} : \texttt{void} \to \texttt{int} \\ \texttt{d} : \texttt{int} \to \texttt{void} \end{array}} & & \boxed{\begin{array}{l} \texttt{b} : \texttt{state} \to \texttt{int} \\ \texttt{d} : \texttt{int} \times \texttt{state} \to \texttt{state} \end{array}}
\end{tikzcd}$$

Conclusion: an algebraic framework for logic

- A simple framework:
  - A diagrammatic logic is a sketch $\mathbf{E_T}$
  - A diagrammatic logical rule is a fraction $\frac{c}{h}$
- A homogeneous framework:
  "the logic of logics is a logic"
- A category of logics:
  morphisms of logics are fractions of sketches

THANK YOU!

Some references

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