

LOGICAL RULES ARE FRACTIONS

Dominique Duval – LJK – University of Grenoble-Alpes

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FACT:

Logical rules **are written as** fractions

$$\frac{H}{C}$$

IN FACT:

Logical rules **ARE** fractions

$$\frac{C}{H}$$

I – FRACTIONS

P. Gabriel & M. Zisman (1967)

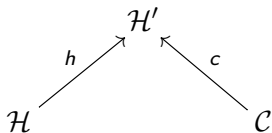
Categorical fractions

Given two categories **S**, **T** and a functor

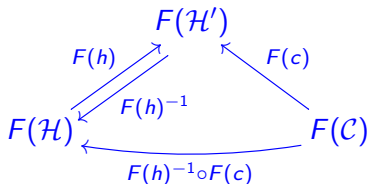
$$\mathbf{S} \xrightarrow{F} \mathbf{T}$$

a **fraction** $\frac{c}{h} : \mathcal{C} \rightarrow \mathcal{H}$ is (“essentially”)

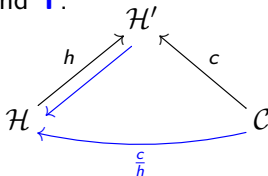
a cospan (h, c) in **S**



such that $F(h)$ is invertible in **T**



A notation for “**both**” **S** and **T**:

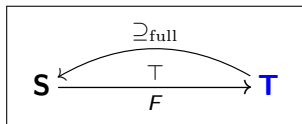


Localisation and reflection

A functor $F : \mathbf{S} \rightarrow \mathbf{T}$ is

- ▶ a **localisation** if it adds inverses for some morphisms in \mathbf{S} .
- ▶ a **reflector** if \mathbf{T} is a **full subcategory** of \mathbf{S} and F is **left adjoint** to inclusion. Such an adjunction is called a **reflection**

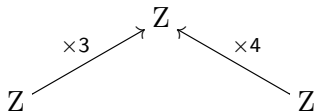
$$\mathrm{Hom}_{\mathbf{S}}(S, T) \cong \mathrm{Hom}_{\mathbf{T}}(F(S), T)$$



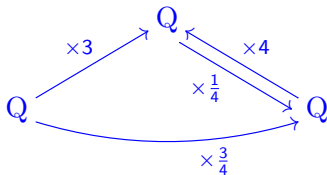
Theorem. Every reflector is a localisation.

A (usual) fraction: $\frac{3}{4}$

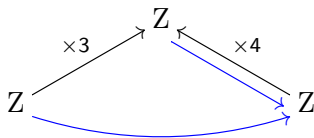
Integers:



Rationals:



“Both:”



(Usual) fractions are categorical fractions

$\mathbf{S} = \text{Module}(\mathbb{Z})$ the category of modules over \mathbb{Z}

$\mathbf{T} = \text{Vect}(\mathbb{Q})$ the category of vector spaces over \mathbb{Q}

$F : \text{Module}(\mathbb{Z}) \rightarrow \text{Vect}(\mathbb{Q})$ is the extension of scalars:

$$F(V) = \mathbb{Q} \otimes V$$

FACT. A (usual) **fraction** is a categorical **fraction** wrt F

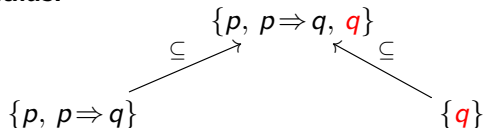
Ex. Then $F(\mathbb{Z}) = \mathbb{Q}$ and

the integer 4 **non-invertible** in \mathbb{Z} becomes

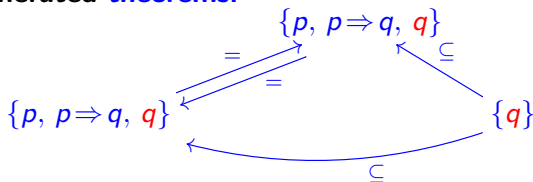
the rational 4 **invertible** in \mathbb{Q}

A logical rule: $\boxed{\frac{p \quad p \Rightarrow q}{q}}$ (*Modus Ponens*)

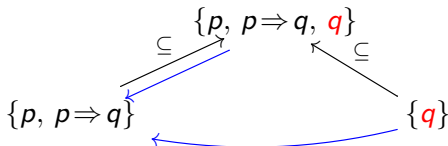
Sets of formulas:



Sets of generated **theorems**:



“Both:”



Logic, specifications, theories

INFORMALLY:

Given a *logic*, with its *formulas* and *rules*, we say that:

- ▶ a *specification* S is a family of *formulas*
- ▶ a *theory* T is a family of *formulas* which is closed under application of the *rules*

Logical rules are categorical fractions

INFORMALLY:

Let us assume the existence of:

- ▶ a category **S** of **specifications**
- ▶ a category **T** of **theories**
- ▶ and a **generating** functor $F : \mathbf{S} \rightarrow \mathbf{T}$
such that $F(S)$ is the family of formulas (or **theorems**)
deduced from the formulas (or **axioms**) in S

FACT. A logical **rule** is a categorical **fraction** wrt F

Ex. When *modus ponens* is a rule of the logic:

let $S = \{p, p \Rightarrow q\}$:

it is a specification that **does not contain** q

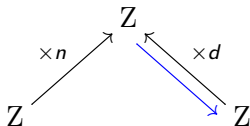
then $F(S) = \{p, p \Rightarrow q, q, \dots\}$:

it is a theory that **contains** q

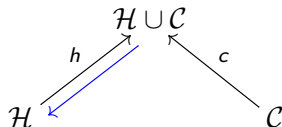
To sum up (I)

A LOGICAL RULE IS A FRACTION

fraction $\frac{n}{d}$



rule $\frac{\mathcal{H}}{\mathcal{C}}$



A logical rule $\boxed{\frac{\mathcal{H}}{\mathcal{C}}}$ IS a fraction $\boxed{\frac{c}{h}}$

"THE HYPOTHESIS BECOMES INVERTIBLE"

II – SKETCHES (“**E**squisses”)

C. Ehresmann (1968)

In this talk:

SKETCH = LIMIT SKETCH

Sketches and their realisations

A **sketch** \mathbf{E} is a *presentation* for a category with limits $\overline{\mathbf{E}}$
It is made of:

- ▶ **objects**,
- ▶ “**morphisms**” with only “some” identities and composition
- ▶ and “**limits**” with only “some” associated tuples

which become *actual* objects, morphisms and limits in $\overline{\mathbf{E}}$

A **realisation** R of \mathbf{E} is a *set-valued model* of \mathbf{E} :
it maps each object, morphism and limit in \mathbf{E}
to a set, function and limit in **Set**

Equivalently, it is a limit-preserving functor $R : \overline{\mathbf{E}} \rightarrow \mathbf{Set}$

$\boxed{\mathbf{Real}(\mathbf{E})}$ denotes the **category of realisations** of \mathbf{E}

$\mathbf{Real}(\mathbf{E})$ is a kind of generalised presheaf

- ▶ A **linear sketch** \mathbf{E} has only objects and morphisms (no limit) then $\mathbf{Real}(\mathbf{E}) = \mathbf{Func}(\overline{\mathbf{E}}, \mathbf{Set})$ is a **presheaf category**

Ex. $\mathbf{Real}(V \begin{smallmatrix} \xleftarrow{s} \\ \xrightarrow{t} \end{smallmatrix} E)$

is the category \mathbf{Gr} of directed graphs

- ▶ In general, for a **[limit] sketch** \mathbf{E} , $\mathbf{Real}(\mathbf{E})$ is a **locally presentable category**

Ex. $\mathbf{Real}(V \begin{smallmatrix} \xleftarrow{s} \\ \xrightarrow{t} \end{smallmatrix} E = V^2)$

is the category \mathbf{Gr}_1 of directed graphs with exactly one edge $n \rightarrow p$ for each pair of vertices (n, p)

“Many” properties of **presheaves**
are still valid for **locally presentable categories**

“What is a logic?”

Yet another proposal:

A LOGIC IS A SKETCH

This is a very simple and very abstract
algebraic proposal...

A logic with *modus ponens*

Syntactic entities:

formulas (Form) and theorems (Theo)

Each theorem is a formula

Formation rule:

$$(IM) \quad \frac{p, q : \text{Form}}{p \Rightarrow q : \text{Form}}$$

If p and q are formulas then $p \Rightarrow q$ is a formula

Deduction rule:

$$(MP) \quad \frac{[p, q, p \Rightarrow q : \text{Form}] \quad p, p \Rightarrow q : \text{Theo}}{q : \text{Theo}}$$

If p and $p \Rightarrow q$ are theorems then q is a theorem

A sketch for syntactic entities

Syntactic entities:

formulas (Form) and theorems (Theo)

Each theorem is a formula

Sketch:

| |
|----------------------------|
| Form \longleftarrow Theo |
|----------------------------|

A realisation R of this sketch is:

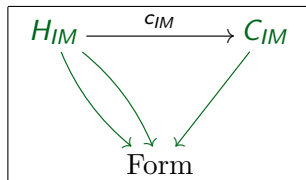
- ▶ a set of **formulas** $R(\text{Form})$
- ▶ a set of **theorems** $R(\text{Theo})$
- ▶ with $R(\text{Theo}) \subseteq R(\text{Form})$

A sketch for the formation rule

Formation rule: $(IM) \frac{p, q : \text{Form}}{p \Rightarrow q : \text{Form}}$

If p and q are formulas then $p \Rightarrow q$ is a formula

$C_{IM} = \text{Form}$, $H_{IM} = \text{Form}^2$



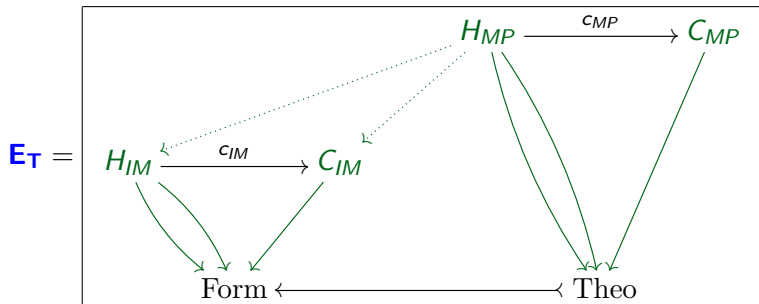
A realisation R of this sketch is:

- ▶ a set of formulas $R(\text{Form})$
- ▶ the sets $R(C_{IM}) = R(\text{Form})$ and $R(H_{IM}) = R(\text{Form})^2$
- ▶ and a function $R(c_{IM}) : R(H_{IM}) \rightarrow R(C_{IM})$
denoted $c_{IM}(p, q) = p \Rightarrow q$

A sketch for the deduction rule

Deduction rule (simplified): (MP) $\frac{p, p \Rightarrow q : \text{Theo}}{q : \text{Theo}}$
If p and $p \Rightarrow q$ are theorems then q is a theorem

$C_{MP} = \text{Theo}$, $H_{MP} \approx \text{Theo}^2$ (simplified!)



A realisation of $\mathbf{E_T}$ is a **theory**: $\boxed{\text{Real}(\mathbf{E_T}) = \mathbf{T}}$

To sum up (II)

A LOGIC IS A SKETCH

To keep:

- ▶ a **logic** is a sketch $\boxed{\mathbf{E}_T}$
- ▶ the category of **theories** is $\mathbf{T} = \mathbf{Real}(\mathbf{E}_T)$

To improve:

- ▶ a **model** of a theory T in a theory D is an arrow $M: T \rightarrow D$ in \mathbf{T}
- ▶ a **rule** is an arrow $H \xrightarrow{c} C$ in \mathbf{E}_T

Still missing:

- ▶ **specifications** as presentations of theories?
- ▶ rules as **fractions**?

III – SKETCHES and FRACTIONS

From theories to specifications

Morphisms of sketches

A morphism of sketches $\mathbf{E}_1 \xrightarrow{\sigma} \mathbf{E}_2$
induces a functor

$$\text{Real}(\mathbf{E}_1) \xleftarrow{\quad} \text{Real}(\mathbf{E}_2)$$

$$\begin{array}{ccc} \mathbf{E}_1 & \xrightarrow{\sigma} & \mathbf{E}_2 \\ & \searrow R_1 = R_2 \circ \sigma & \swarrow R_2 \\ & \mathbf{Set} & \end{array}$$

Theorem. This functor has a **left adjoint**.

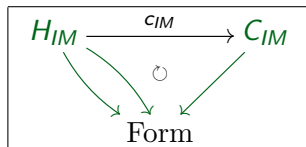
$$\text{Real}(\mathbf{E}_1) \xrightleftharpoons{\quad \top \quad} \text{Real}(\mathbf{E}_2)$$

Thus: each realisation of \mathbf{E}_1 **generates** a realisation of \mathbf{E}_2

Cycles

A “cycle” in **E** is defined by considering that projections are oriented both sides

Ex. The formation rule $(IM) \frac{p, q : \text{Form}}{p \Rightarrow q : \text{Form}}$



Because of cycle “ \circ ”, in a **theory** T ,
for **ALL** pairs of formulas (p, q) there is a formula $p \Rightarrow q$

Required: in a **specification** S ,
for **SOME** pairs of formulas (p, q) there is a formula $p \Rightarrow q$

Breaking cycles

The cycles in \mathbf{E} can be **broken** by making c **partial**:

$$\text{replace } \boxed{H \xrightarrow{c} C} \text{ by } \boxed{H \xleftarrow{h} H' \xrightarrow{c} C}$$

By breaking the cycles in $\mathbf{E_T}$ we get a sketch $\mathbf{E_S}$ and a morphism called a **localiser**

$$\boxed{\mathbf{E_S} \longrightarrow \mathbf{E_T}}$$

such that the corresponding adjunction is a **reflection**

$$\boxed{\text{Real}(\mathbf{E_S}) = \mathbf{S} \begin{array}{c} \xleftarrow{\quad \exists_{\text{full}} \quad} \\ \xrightarrow[\quad F \quad]{\quad \top \quad} \end{array} \mathbf{T} = \text{Real}(\mathbf{E_T})}$$

Definitions (1/2)

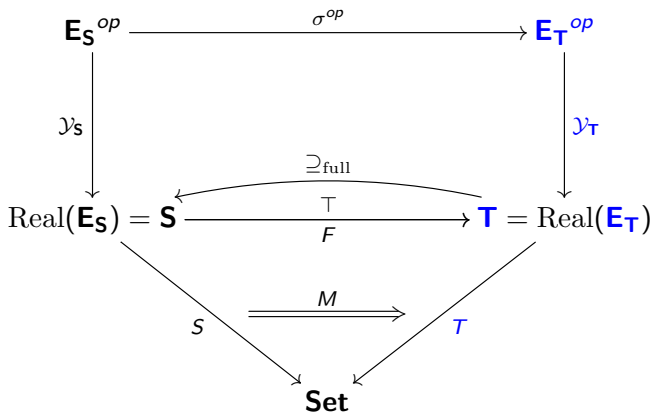
A **diagrammatic logic** is a sketch $\boxed{\mathbf{E}_T}$

- ▶ the category of **theories** is $\mathbf{T} = \text{Real}(\mathbf{E}_T)$

Let $\sigma : \mathbf{E}_S \rightarrow \mathbf{E}_T$ be a localiser

it defines a reflector $F : \mathbf{S} \rightarrow \mathbf{T}$

- ▶ the category of **specifications** is $\mathbf{S} = \text{Real}(\mathbf{E}_S)$
- ▶ the theory **generated** by a specification S is $F(S)$
- ▶ a **model** of a specification S in a theory D is
an arrow $M : S \rightarrow D$ in \mathbf{S}
[or equivalently, an arrow $M : F(S) \rightarrow D$ in \mathbf{T}]



γ is the **Yoneda contravariant embedding**

$$\boxed{\gamma : \mathbf{E}^{op} \rightarrow \mathbf{Real}(\mathbf{E})}$$

such that

$$\boxed{\gamma(X) = \mathbf{Hom}_{\mathbf{E}}(X, -)}$$

Definitions (2/2)

Given a diagrammatic logic $\boxed{\mathbf{E}_T}$ with a localiser $\sigma : \mathbf{E}_S \rightarrow \mathbf{E}_T$

- ▶ a **rule** is a fraction in \mathbf{E}_S wrt σ

Thus, using the **Yoneda contravariant embedding** \mathcal{Y} :

- ▶ a **rule** is a fraction in \mathbf{S} wrt F (in the image of \mathbf{E}_S by \mathcal{Y})

The Yoneda contravariant embedding

$\mathcal{Y} : \mathbf{E}^{op} \rightarrow \mathbf{Real}(\mathbf{E})$ is “nearly as nice”
for *locally presentable* categories as for *presheaves*

- ▶ \mathcal{Y} is **faithful**
- ▶ \mathcal{Y} maps **limits** to **colimits**
- ▶ $\mathcal{Y}(\mathbf{E}^{op})$ is **dense** in $\mathbf{Real}(\mathbf{E})$:
each realisation of \mathbf{E} is the **colimit** of realisations in $\mathcal{Y}(\mathbf{E}^{op})$

The category $\mathbf{Real}(\mathbf{E})$ has all **colimits** (like *presheaves*)
BUT they cannot be computed sortwise (unlike *presheaves*)

Ex. Coproduct of graphs $\begin{array}{c} \circlearrowright \\ v \end{array}$ and $\begin{array}{c} \circlearrowleft \\ w \end{array}$ is

$\begin{array}{c} \circlearrowright \\ v \end{array} \quad \begin{array}{c} \circlearrowleft \\ w \end{array}$ in \mathbf{Gr} BUT $\begin{array}{c} \circlearrowright \\ v \end{array} \begin{array}{c} \rightleftarrows \\ \end{array} \begin{array}{c} \circlearrowleft \\ w \end{array}$ in \mathbf{Gr}_1

Breaking the cycle for (IM): sketches

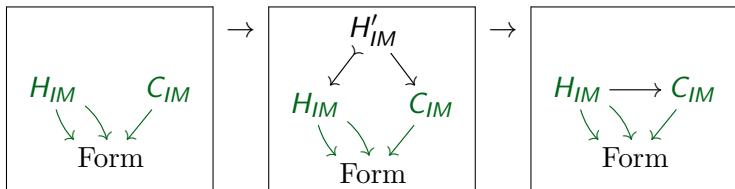
Adding a rule is a morphism:

$$\mathbf{E}_0 \longrightarrow \mathbf{E}_T$$

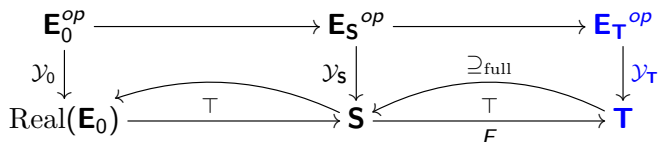


that gets factorised by breaking cycles (**theorem!**):

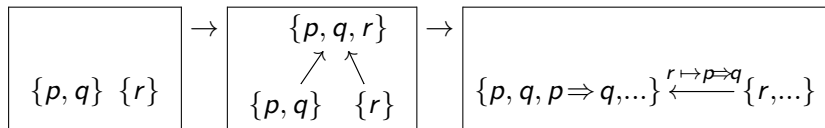
$$\mathbf{E}_0 \longrightarrow \mathbf{E}_S \longrightarrow \mathbf{E}_T$$



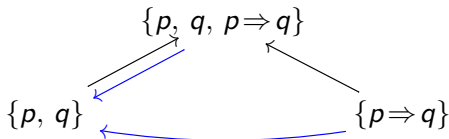
Breaking the cycle for (IM): realisations



Thus, focusing on $\mathcal{Y}(-)(\text{Form})$



we get the **fraction**



“Moral”: fractions as presentations

- ▶ A **specification** S in \mathbf{S} is a **presentation** for the theory $T = F(S)$ in \mathbf{T}
 - ▶ A **morphism** $s : S \rightarrow S'$ in \mathbf{S} is a **presentation** for the morphism $F(s) : F(S) \rightarrow F(S')$ in \mathbf{T}
- One gets **SOME** morphisms $t : F(S) \rightarrow F(S')$ in \mathbf{T}

Ex. Every ring is a monoid

- ▶ A **fraction** $\frac{c}{h} : S \rightarrow S'$ wrt F is a **presentation** for the morphism $F(h)^{-1} \circ F(c) : F(S) \rightarrow F(S')$ in \mathbf{T}

One gets **ALL** morphisms $t : F(S) \rightarrow F(S')$ in \mathbf{T}

Ex. Every boolean algebra is a ring

*“MORPHISMS OF THEORIES are presented by
FRACTIONS OF SPECIFICATIONS”*

To sum up (III):

A LOGIC IS A SKETCH

and

A LOGICAL RULE IS A FRACTION

IV – Application: COMPUTATIONAL EFFECTS

The state effect in object-oriented programming

```
Class BankAccount {...  
    int balance (void) const ;  
    void deposit (int) ;  
...}
```

From this C++ syntax to an equational specification?

► **apparent** specification

```
balance : void → int  
deposit : int → void
```

- ⊞ the object-oriented flavour is preserved
- ⊞ BUT the intended interpretation is **not** a model

► **explicit** specification

```
balance : state → int  
deposit : int × state → state
```

- ⊞ the intended interpretation is a model,
- ⊞ BUT the object-oriented flavour is **not** preserved

► decorated specification

```
balancea : void → int  
depositm : int → void
```

where the decorations are:

^m for modifiers (methods)

^a for accessors (“const” methods)

⊞ the intended interpretation is a model

⊞ AND the object-oriented flavour is preserved

Morphisms of logics:

```
ba : void → int  
dm : int → void
```



```
b : void → int  
d : int → void
```



```
b : state → int  
d : int × state → state
```

Conclusion: an algebraic framework for logic

- ⊞ A **simple** framework:
 - ▶ A diagrammatic **logic** is a **sketch** $\mathbf{E_T}$
 - ▶ A diagrammatic **logical rule** is a **fraction** $\frac{c}{h}$
- ⊞ A **homogeneous** framework:
 - “the logic of logics is a logic”
- ⊞ A **category** of logics:
 - morphisms of logics are fractions of sketches

THANK YOU!

Some references

Historical references.

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Esquisses et types de structures algébriques (1968).

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and independently, about rules as fractions:

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About computational effects:

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