Diagrammatic logic and effects: the example of exceptions

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Logic and categories

- Logic can be based on category theory:
  (*Lawvere, Ehresmann*, 1960's).
- One major result: simply typed lambda-calculus is equivalent
  to cartesian closed categories (*Lambek and Scott*).
- Many other results: "some logic is equivalent
  to some family of categories".

For dealing (also) with computational effects, such as exceptions, overloading, . . . , several logics are needed – schematically, at least:

— a logic (with effects) for the language,

— a logic (explicit) for the user.

Thus, a category of logics is needed.

This talk

1. The category of diagrammatic logics
   (also used by César Domínguez in the next talk).
2. An application to exceptions
   (involving three diagrammatic logics).

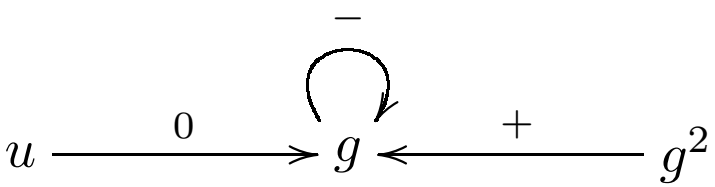
1. The category of diagrammatic logics

Graphs

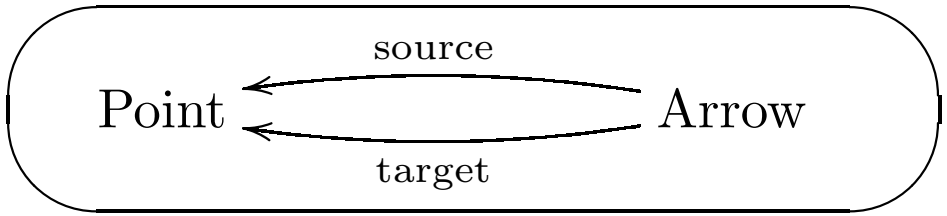
A (directed multi-)graph is made of:

- a set of *points*, a set of *arrows*,
- *source* and *target* maps, both from the arrows to the points.

Example:



The definition of graphs can be illustrated by the following graph $\mathcal{S}_{\mathrm{Gr}}$:



Categories

A category is a graph where:

- each point X has an *identity* arrow $\mathrm{id}_X : X \to X$,
- each pair of consecutive arrows $f : X \to Y$, $g : Y \to Z$ has a *composed* arrow $g \circ f : X \to Z$,
- with the usual associativity and unitarity axioms.

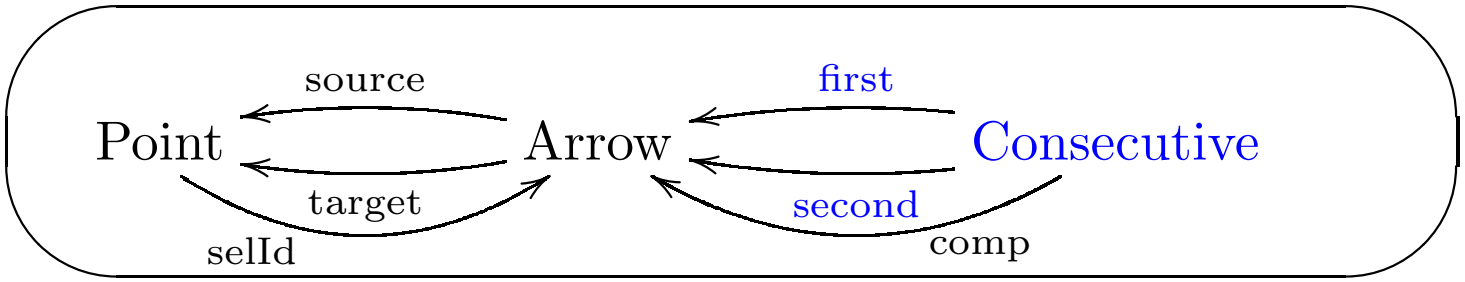
Basic examples.

Set: points are sets and arrows are maps.

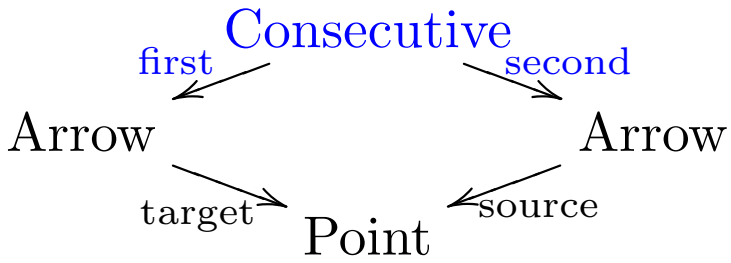
Gr: points are graphs and arrows are morphisms of graphs.

Cat: points are categories and arrows are functors.

The definition of categories can be illustrated by the following graph $\mathcal{S}_{\mathrm{Cat}}$:



with the additional information that the cone below represents a *limit*:



and with several equalities of maps:

$$\mathrm{source} \circ \mathrm{comp} = \mathrm{source} \circ \mathrm{first},\ \mathrm{target} \circ \mathrm{comp} = \mathrm{target} \circ \mathrm{second}, \ldots$$

A graph together with some distinguished cones (potential limits) and with some equalities among terms, is a projective sketch.

Graphs and categories

From the definition, every category is a graph;
there is an omitting (or forgetful) functor:

$$\mathrm{Gr} \underset{\mathrm{U}}{\longleftarrow} \mathrm{Cat}$$

which corresponds to the inclusion of projective sketches:

$$\mathcal{S}_{\mathrm{Gr}} \xrightarrow{\subseteq} \mathcal{S}_{\mathrm{Cat}}$$

On the other hand, every graph generates a category;
there is a generating functor:

$$\mathrm{Gr} \overset{\mathrm{F}}{\longrightarrow} \mathrm{Cat}$$

The pair (F, U) is an adjunction:

$$\mathrm{Hom}_{\mathrm{Gr}}(\Gamma, \mathrm{U}\Delta) \cong \mathrm{Hom}_{\mathrm{Cat}}(\mathrm{F}\Gamma, \Delta) \ .$$

Compositive graphs

A compositive graph is a graph where:

- *some* points have an identity arrow,
- *some* pairs of consecutive arrows have a composed arrow.

This definition can be illustrated by a projective sketch $\mathcal{S}_{\mathrm{Comp}}$:

$$\mathcal{S}_{\mathrm{Gr}} \longrightarrow \mathcal{S}_{\mathrm{Comp}} \longrightarrow \mathcal{S}_{\mathrm{Cat}}$$

So, there are adjunctions:

$$\mathrm{Gr} \underset{\mathrm{U}'}{\overset{\mathrm{F}'}{\rightleftarrows}} \mathrm{Comp} \underset{\mathrm{U}''}{\overset{\mathrm{F}''}{\rightleftarrows}} \mathrm{Cat}$$

Moreover, for every graph Γ and every category Δ:

$$\mathrm{U}'\mathrm{F}'\Gamma \cong \Gamma \text{ and } \mathrm{F}''\mathrm{U}''\Delta \cong \Delta \ :$$

an instance of the **decomposition theorem** (*Duval, Lair*, 2002).

Propagators

The realizations of a projective sketch $\mathcal{S}$ are the morphims from $\mathcal{S}$ to Set, they form a category $\mathrm{Real}(\mathcal{S})$. E.g., $\mathrm{Real}(\mathcal{S}_{\mathrm{Gr}}) = \mathrm{Gr}, \ldots$
Every morphism of projective sketches $M : \mathcal{S} \to \mathcal{S}'$ defines an omitting functor:

$$\mathrm{Real}(\mathcal{S}) \xleftarrow[\mathrm{U}_M]{} \mathrm{Real}(\mathcal{S}')$$

Theorem (*Ehresmann*, 1965) The functor U_M has a left-adjoint:

$$\mathrm{Real}(\mathcal{S}) \underset{\mathrm{U}_M}{\overset{\mathrm{F}_M}{\rightleftarrows}} \mathrm{Real}(\mathcal{S}')$$

A propagator is a morphism of projective sketches $P : \mathcal{S} \to \overline{\mathcal{S}}$ such that, for each realization Δ of $\overline{\mathcal{S}}$:

$$\mathrm{F}_P \mathrm{U}_P \Delta \cong \Delta \, .$$

Diagrammatic logics

Let $P : \mathcal{S} \to \overline{\mathcal{S}}$ be a propagator.

- P-specifications: $\mathrm{Spec}(P) = \mathrm{Real}(\mathcal{S})$.
- P-domains: $\mathrm{Dom}(P) = \mathrm{Real}(\overline{\mathcal{S}})$.
- P-deduction rules: arrows from $\overline{\mathcal{S}}$.
- P-deduction steps are morphisms $\sigma : \Sigma \to \Sigma'$ such that $\mathrm{F}_P(\sigma)$ is an isomorphism (see next slides).
- P-models of a specification Σ with values in a domain Δ: $\mathrm{Mod}(\Sigma, \Delta) = \mathrm{Hom}(\Gamma, \mathrm{U}_P\Delta) \cong \mathrm{Hom}(\mathrm{F}_P\Gamma, \Delta)$.

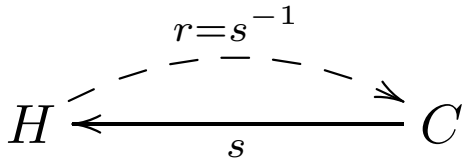
Soundness. If σ is a deduction step, then $\mathrm{Mod}(\sigma, \Delta)$ is a bijection.

Deduction rules

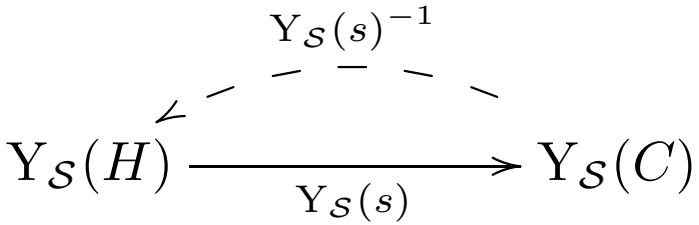
Theorem (*Hébert, Adámek and Rosický*, 2001)

A propagator consists of adding inverses to arrows.

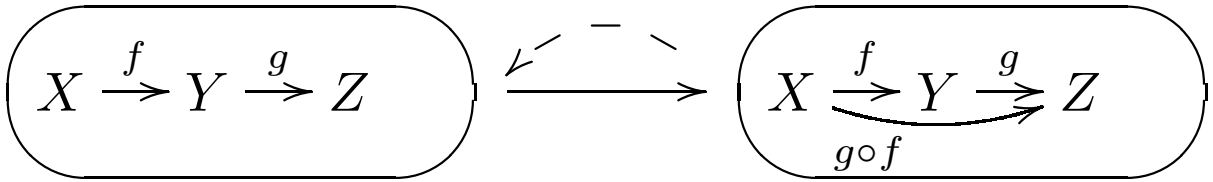
The corresponding rules $\frac{H}{C}$ are illustrated as follows ($-\!\!\succ$ only in $\overline{\mathcal{S}}$):



Or via the Yoneda contravariant morphism ($-\!\!\succ$ only in $\mathrm{Dom}(P)$):



Example:



Deduction steps

Let Σ be a P-specification. The *P-deduction step* associated to the P-deduction rule $H \xleftarrow[s]{r=s^{-1}} C$, applied to an $x \in \Sigma(H)$, is the morphism $\tau_s(x)$ in the pushout of $\mathrm{Y}(s)$ and x:

$$\begin{array}{ccc} \mathrm{Y}(H) & \xrightarrow{\mathrm{Y}(s)} & \mathrm{Y}(C) \\ \downarrow{\scriptstyle x} & & \downarrow{\scriptstyle c_s(x)} \\ \Sigma & \xrightarrow{\tau_s(x)} & \Sigma_s(x) \end{array}$$

Proposition. $\mathrm{F}_P(\tau_s(x))$ is an isomorphism.

Morphisms of diagrammatic logics

Let $P_1 : \mathcal{S}_1 \to \overline{\mathcal{S}}_1$ and $P_2 : \mathcal{S}_2 \to \overline{\mathcal{S}}_2$ be two propagators.
A morphism of propagators $P_1 \to P_2$ is a pair $(\alpha, \overline{\alpha})$ of morphisms of projective sketches such that:

$$\begin{array}{ccc} \mathcal{S}_1 & \xrightarrow{P_1} & \overline{\mathcal{S}}_1 \\ \downarrow{\scriptstyle \alpha} & = & \downarrow{\scriptstyle \overline{\alpha}} \\ \mathcal{S}_2 & \xrightarrow{P_2} & \overline{\mathcal{S}}_2 \end{array}$$

In this way, we get a category of diagrammatic logics, as required.

2. An application to exceptions

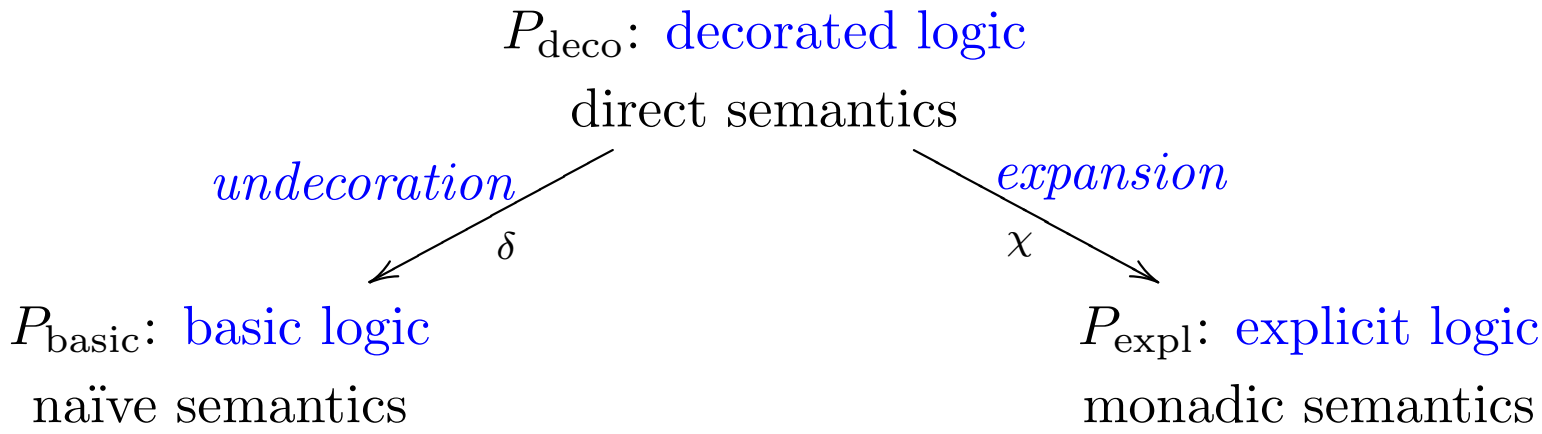
The issue

Formalizing the exception mechanism.

Previous work:
algebraic specifications, monads (*Moggi*, 1996), (`Haskell`).
Plotkin and Power, 2001: "*Evident futher work is to consider how other operations such as those for handling exceptions should be modelled. That might involve going beyond monads, as Moggi has suggested to us.*"

Our approach is influenced by the monads approach, although quite different.

Three logics
Three denotational semantics



Example, over the naturals, with z (for 0) and s (for *successor*):

Exception e

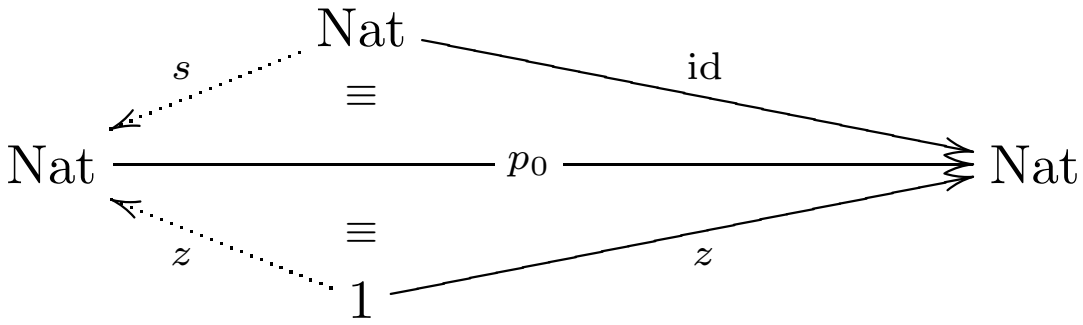
$p(x) = \text{case } x \text{ of } [\, s(y) \Rightarrow y \mid z \Rightarrow \text{raise } e \,] \text{ handle } [\, e \Rightarrow z \,]$

The basic logic

Without any exceptions.

With sum types.

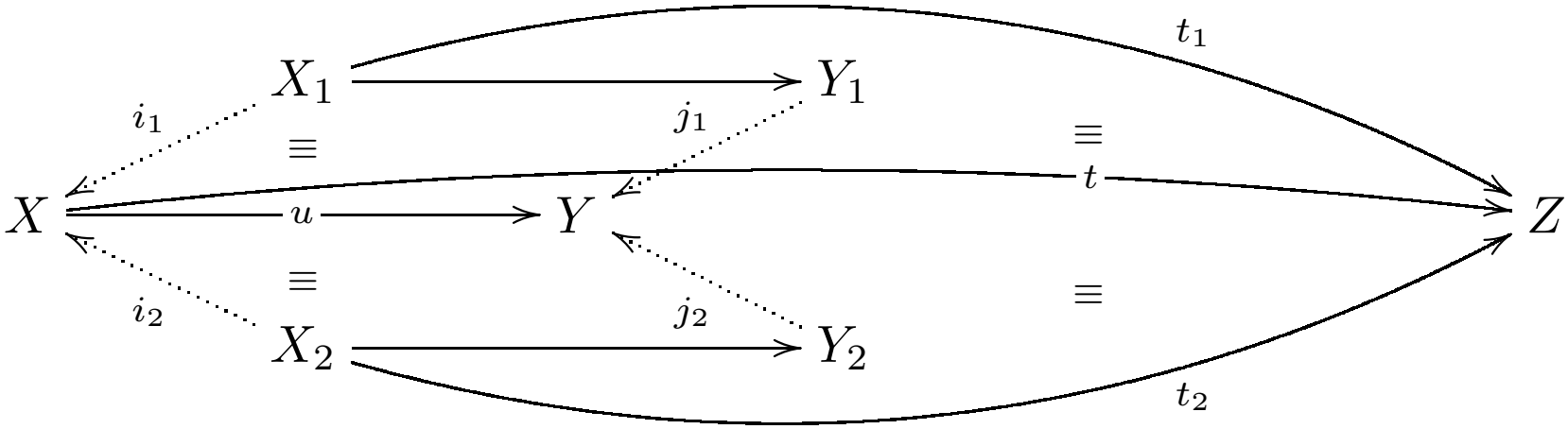
$$p_0(x) = \text{case } x \text{ of } [\, s(y) \Rightarrow y \mid z \Rightarrow z \,] : \text{Nat} \to \text{Nat}$$



The `case` construction

More generally, the `case` construction uses the extensivity property of sums (*Carboni, Lack and Walters*, 1993):

$$t(x) = \text{case } u(x) \text{ of } [\, j_1(y) \Rightarrow t_1(x,y) \mid j_2(y) \Rightarrow t_2(x,y) \,]$$
$$= [\, i_1(x) \Rightarrow t_1(x,y) \mid i_2(x) \Rightarrow t_2(x,y) \,]$$

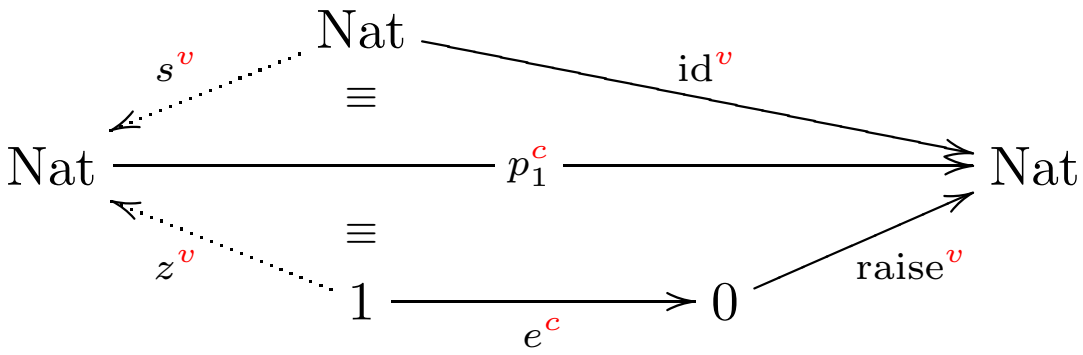


The decorated logic: raise

Exception e

$$1 \xrightarrow{\ e^c\ } 0$$

$$p_1(x) = \text{case } x \text{ of } [\, s(y) \Rightarrow y \mid z \Rightarrow \text{raise } e \,]$$



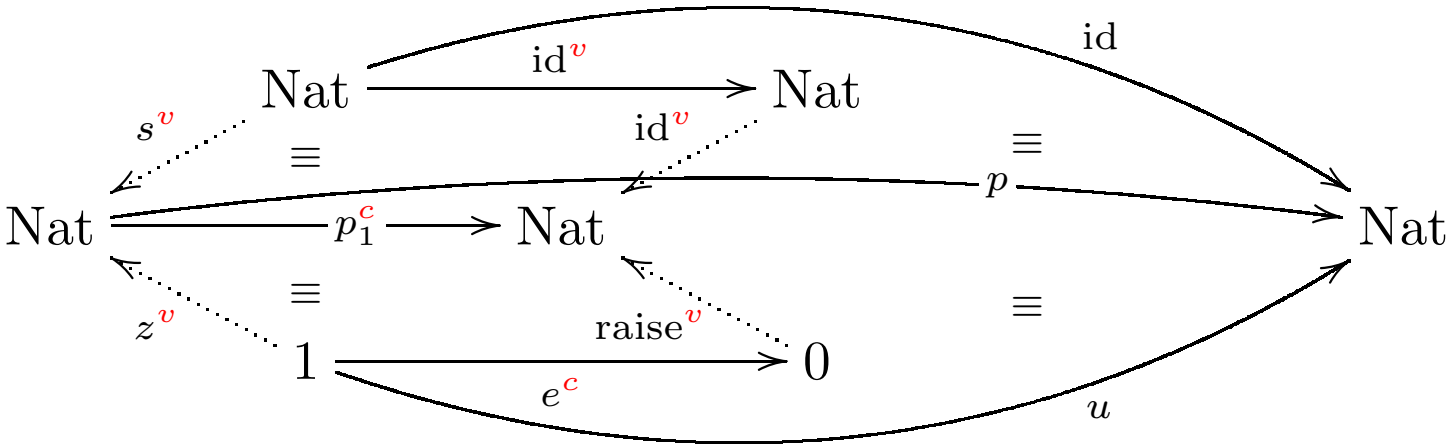
where:

$$\text{raise}_X^v = [\quad]_X^v : 0 \to X \ .$$

The decorated logic: handle

$$p(x) = p_1(x) \text{ handle } [\, e \Rightarrow z \,]$$
$$= \text{case}^c\ p_1 \text{ of } [\, \text{id} \Rightarrow \text{id} \mid \text{raise} \Rightarrow u \,]$$

where $u = \text{case}^e\ e \text{ of } [\, e \Rightarrow z \,]$



So, $p^c \equiv [\, s \Rightarrow \text{id} \mid z \Rightarrow u \,]$

and on the other hand $u^c \equiv z$.

Finally:

$$p \equiv [\, s \Rightarrow \text{id} \mid z \Rightarrow z \,] \equiv p_0.$$

The decorated logic: proofs and models

- Proofs can be made in the decorated logic.
- Models can be defined in the decorated logic, although there is no canonical interesting domain of sets.
- Decorations for arrows: v, c,
  this is rather similar to the monads approach.
- Decorations for the case rule: v, c, e,
  this is new.
- The morphism $\delta : P_{\text{deco}} \to P_{\text{basic}}$ is simply the undecoration.

The explicit logic

Similar to the basic logic, but with a distinguished type E for exceptions: "exceptions are explicit".

A (partial) description of the expansion morphism $\chi : P_{\text{deco}} \to P_{\text{expl}}$:

- A value $t^v : X \to Y$ becomes a term $t : X \to Y$.
- A computation $t^c : X \to Y$ becomes a term $t : X \to Y + E$.
  For example, an exception $e^c : 1 \to 0$ becomes $e : 1 \to E$.
- The composition of these terms is done in the Kleisli way, as in the monads approach.

Conclusion

Mathematics:

categories, adjunction, sketches,. . .

Algorithms:

can be formalized, even when they are not functional.

Proofs:

a framework for proofs of programs using effects.