The category of diagrammatic logics

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- Motivation
- The category of propagators
- Specifications, theories, models
- Deduction system
- Conclusion

A category of logics?

From the analysis of some computational effects, it follows that the logic of the language (with the effects: state, exception,...) is different from the logic of the user (where the effects are made explicit).

The logic of the language provides the syntax and the deduction system, the logic of the user provides the intended model(s), and the soundness of the language with respect to its denotation relies on some link between both logics.

Hence, there is a need for some kind of

category of logics

An example

```
class Account {
  int balance () const { }
  void deposit (int) { }
}
```

For the language ("decorated" logic):

$$\mathtt{balance}^{\mathtt{const}} : \mathtt{void} \to \mathtt{int} \ , \ \mathtt{deposit} : \mathtt{int} \to \mathtt{void}$$

For the user (logic with a distinguished sort "**state**"):

$$\mathtt{balance} : \mathtt{state} \to \mathtt{int} \ , \ \mathtt{deposit} : \mathtt{int} \times \mathtt{state} \to \mathtt{state}$$

The sort "`state`" corresponds to the set of states of an object in the class "`Account`".

Several solutions!

The institutions [Goguen, Burstall 1992]

The diagrammatic logics [Duval, Lair 2002]

The aim of this talk is to present the framework of diagrammatic logics. It should be clear from this presentation that it is fairly different from the framework of institutions.

Abstract

- A propagator is a morphism of limit sketches such that the corresponding underlying functor is full and faithful.

- A propagator defines a diagrammatic logic: syntax, models, and a sound deduction system.

- Propagators with their morphisms give rise to the category of diagrammatic logics.

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Limit sketches (*or* projective sketches)

[Ehresmann, 1965] A limit sketch is a (directed multi-)graph with:

- some (potential) identities $X \circlearrowright \mathrm{id}_X$
- some (potential) composed arrows $X \xrightarrow{f} Y \xrightarrow{g} Z$, with $X \xrightarrow{g.f} Z$
- some (potential) limits (*or* distinguished cones),
  e.g. (potential) binary products $X_1 \leftarrow X \rightarrow X_2$

A morphism of limit sketches is a graph morphism that preserves (potential) identities, composition and limits.

LSketch = the category of limit sketches

The realizations of a limit sketch

A (set-valued) realization Σ of a limit sketch $\mathbf{S}$ is a functor

$$\Sigma : \mathbf{S} \to \mathbf{Set} \; .$$

It interprets each point as a set, each arrow as a map, and each potential ... as an actual ...

A morphism of realizations of $\mathbf{S}$ is a natural transformation.

$$\mathbf{Real}(\mathbf{S}) = \text{ the category of realizations of } \mathbf{S}$$

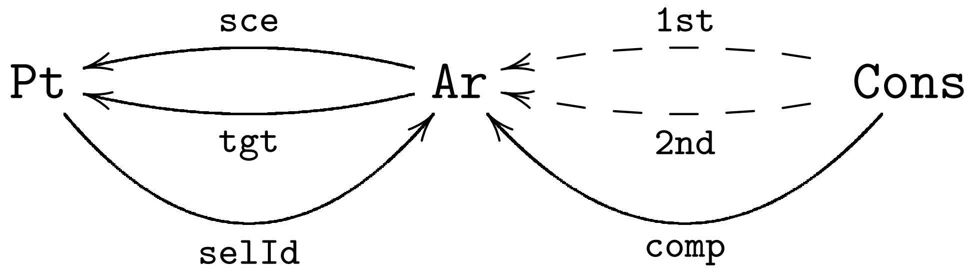
A limit sketch for graphs

$$\mathbf{S}_{\mathrm{gr}}: \quad \mathtt{Pt} \underset{\mathtt{tgt}}{\overset{\mathtt{sce}}{\leftleftarrows}} \mathtt{Ar}$$

$$\mathbf{Real}(\mathbf{S}_{\mathrm{gr}}) \cong \mathbf{Graph}$$

A limit sketch for categories

$\mathbf{S}_{\text{cat}}$:



with

$$\texttt{sce.selId} = \mathrm{id}_{\texttt{Pt}}$$

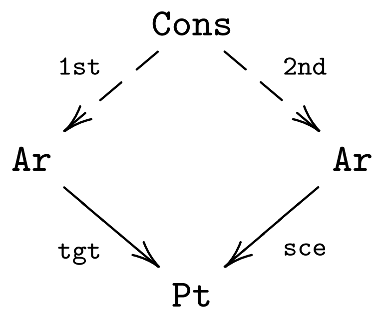
$$\texttt{tgt.selId} = \mathrm{id}_{\texttt{Pt}}$$

$$\texttt{sce.2nd} = \texttt{tgt.1st}$$

$$\texttt{sce.comp} = \texttt{sce.1st}$$

$$\texttt{tgt.comp} = \texttt{tgt.2nd}$$

and



and axioms...

$$\mathbf{Real}(\mathbf{S}_{\text{cat}}) \simeq \mathbf{Cat}$$

Propagators

A propagator is a morphism of limit sketches

$$\mathbf{S} \xrightarrow{P} \overline{\mathbf{S}}$$

such that the underlying functor U_P is full and faithful

$$\mathbf{Real}(\mathbf{S}) \xleftarrow{U_P} \mathbf{Real}(\overline{\mathbf{S}})$$

A morphism of propagators is a pair of morphisms of limit sketches such that

$$\begin{array}{ccc} \mathbf{S}_1 & \xrightarrow{P_1} & \overline{\mathbf{S}}_1 \\ {\scriptstyle\alpha}\downarrow & = & \downarrow{\scriptstyle\overline{\alpha}} \\ \mathbf{S}_2 & \xrightarrow{P_2} & \overline{\mathbf{S}}_2 \end{array}$$

$\mathbf{Propag} =$ the category of propagators

Ehresmann's theorem

Theorem. For every morphism of limit sketches

$$\mathbf{S} \xrightarrow{\quad P \quad} \overline{\mathbf{S}}$$

the underlying functor U_P has a left adjoint,
the freely generating functor F_P

$$\mathbf{Real}(\mathbf{S}) \underset{U_P}{\overset{F_P}{\rightleftarrows}} \mathbf{Real}(\overline{\mathbf{S}})$$

Corollary. P is a propagator if and only if the counit

$$\varepsilon_P : F_P.U_P \Rightarrow \mathrm{Id}$$

is a natural isomorphism

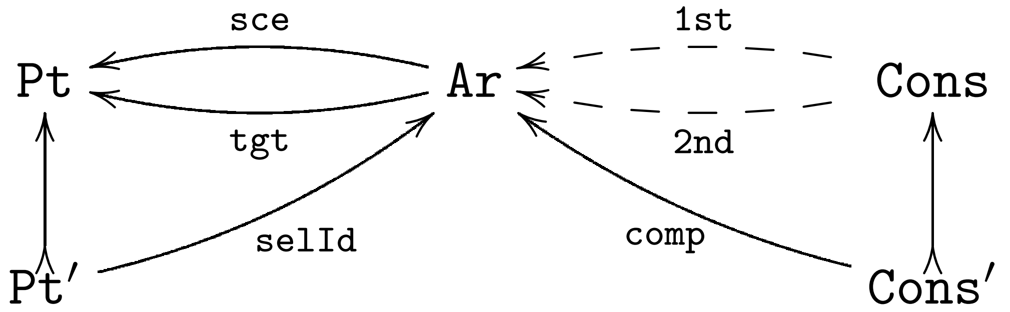
A propagator for graphs

This inclusion *is not* a propagator:

$$\mathbf{S}_{\mathrm{gr}} \xrightarrow{\subseteq} \mathbf{S}_{\mathrm{cat}}$$

A limit sketch for "graphs with partial identities and composition"

$\mathbf{S}_{\mathrm{grComp}}$:



with...

The *second* inclusion *is* a propagator:

$$\mathbf{S}_{\mathrm{gr}} \xrightarrow{\subseteq} \mathbf{S}_{\mathrm{grComp}} \xrightarrow[P]{\subseteq} \mathbf{S}_{\mathrm{cat}}$$

“Typical” propagators

“Typically”, a propagator $P : \mathbf{S} \to \overline{\mathbf{S}}$ may be such that:

- $\overline{\mathbf{S}}$ is a sketch of categories with some properties
- $\mathbf{S}$ is a sketch of graphs with some “potential” properties
- P is the inclusion

For instance :
cartesian categories (and “product” sketches)
complete categories (and limit sketches)
cartesian categories (and limit sketches with “exponentials”)
also: categories of domains, etc. . .

“Non-typical” propagators are fairly interesting:
e.g., for “decorated” features.

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A propagator is fixed:

$$\mathbf{S} \xrightarrow{\;P\;} \overline{\mathbf{S}}$$

P-specifications $=$ realizations of $\mathbf{S}$

P-theories $=$ realizations of $\overline{\mathbf{S}}$

$$\mathbf{Spec}(P) \underset{U}{\overset{F}{\rightleftarrows}} \mathbf{Theory}(P)$$

$$\mathrm{Hom}_{\mathbf{Spec}(P)}(\Sigma, U(\Theta)) \cong \mathrm{Hom}_{\mathbf{Theory}(P)}(F(\Sigma), \Theta) = \mathrm{Mod}_P(\Sigma, \Theta)$$

$=$ the set of models of Σ with values in Θ.

Models and morphisms of propagators

Given: a morphism of propagators

$$\begin{array}{ccc} \mathbf{S}_1 & \xrightarrow{P_1} & \overline{\mathbf{S}}_1 \\ {\scriptstyle \alpha}\downarrow & = & \downarrow{\scriptstyle \overline{\alpha}} \\ \mathbf{S}_2 & \xrightarrow{P_2} & \overline{\mathbf{S}}_2 \end{array}$$

a P_1-specification Σ_1 and a P_2-theory Θ_2.

Proposition.

$$\mathrm{Mod}_{P_1}(\Sigma_1, U_{\overline{\alpha}}(\Theta_2)) \cong \mathrm{Mod}_{P_2}(F_\alpha(\Sigma_1), \Theta_2)$$

Proof. By adjunction.

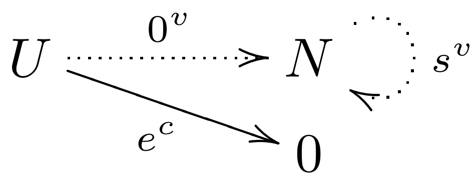
An example

P_1 is the decorated logic for exceptions

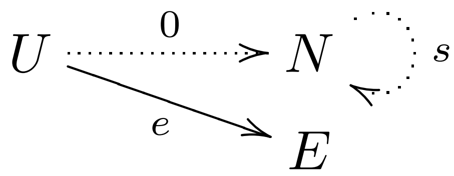
P_2 is the explicit logic for exceptions

$\Theta_2 = \mathbf{Set}$

$\Sigma_1 = \Sigma_{\text{nat.deco}}$:



$F_\alpha(\Sigma_1) = \Sigma_{\text{nat.expl}}$:



$$\text{Mod}_{\text{deco}}(\Sigma_{\text{nat.deco}}, U_{\overline{\alpha}}(\mathbf{Set})) \cong \text{Mod}_{\text{expl}}(\Sigma_{\text{nat.expl}}, \mathbf{Set})$$

Soundness

A morphism of P-specifications $\sigma : \Sigma \to \Sigma'$

is an entailment $\sigma : \Sigma \;\;\longmapsto\;\; \Sigma'$ if

$$F(\sigma) : F(\Sigma) \xrightarrow{\cong} F(\Sigma')$$

is a consequence $\sigma : \Sigma \;\;\longmapsto\!\!\!\!\!\!\mid\;\;_{\Theta}\; \Sigma'$ (with respect to Θ) if

$$\mathrm{Mod}(\sigma, \Theta) : \mathrm{Mod}(\Sigma) \xleftarrow{\cong} \mathrm{Mod}(\Sigma')$$

Theorem. Every diagrammatic logic is sound:

$$\text{if } \sigma : \Sigma \;\longmapsto\; \Sigma' \text{ then } \sigma : \Sigma \;\longmapsto\!\!\!\!\!\!\mid\;_{\Theta}\; \Sigma' \text{ for each } \Theta$$

Proof. $\mathrm{Mod}_P(\Sigma, \Theta) = \mathrm{Hom}_{\mathbf{Theory}(P)}(F(\Sigma), \Theta)$.

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Propagators are fractional

Theorem. Let P be a morphism of limit sketches. Then P is a propagator if and only if, up to equivalence, P is a fractional morphism, i.e., P consists in adding inverses to arrows.

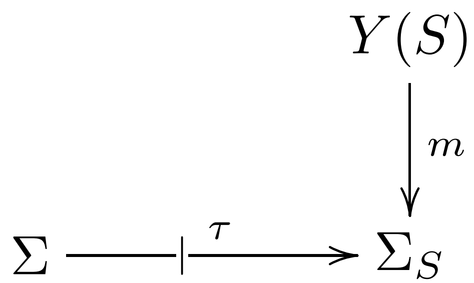
[Hebert, Adamek, Rosický 2001], [Gabriel, Zisman 1967]

Remark. A propagator describes a logic.
A fractional morphism describes a deduction system.
"Up to equivalence": for a given logic, there may be several deduction systems.

Patterns and pattern matchings

A pattern is a point in $\overline{\mathbf{S}}$.

A matching m/τ of a pattern S in a specification Σ is made of a morphism of specifications $m : Y(S) \to \Sigma_S$ and an entailment $\tau : \Sigma \nrightarrow \Sigma_S$

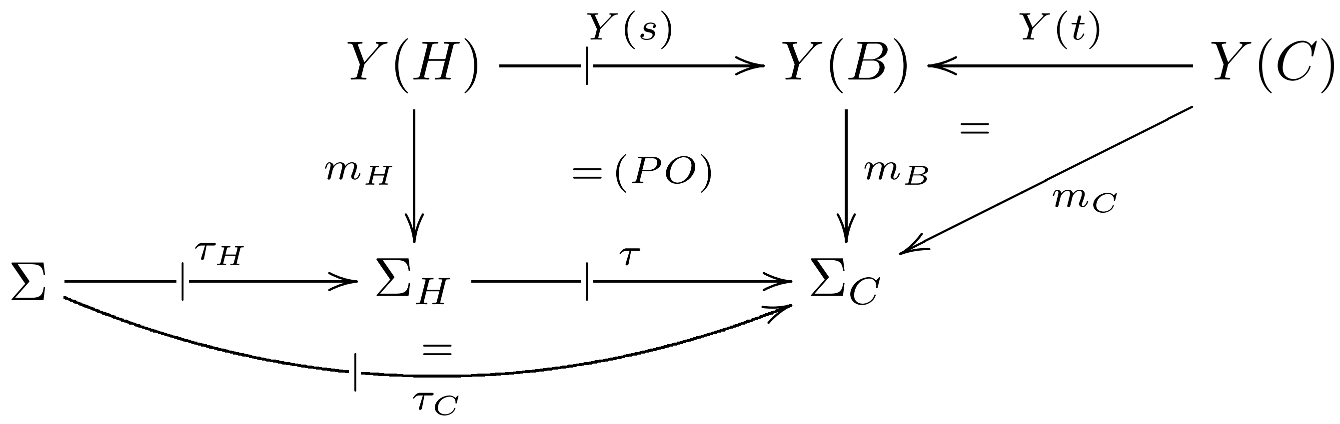


Elementary inference rules and inference steps

An elementary inference rule is an arrow $r = t \circ s^{-1}$ in $\overline{\mathbf{S}}$, with s and t in $\mathbf{S}$:

$$H \xleftarrow{\;s\;} B \xrightarrow{\;t\;} C$$

An elementary inference step consists in applying rule r to a matching of H in Σ, it builds a matching of C in Σ:



Inference rules and derivations

An inference rule is an arrow in $\overline{\mathbf{S}}$.

The derivation with respect to an inference rule is generated from the inference steps.

The inference functor is the functor $I : \overline{\mathbf{S}} \to \mathbf{Cat}$:

$$I\Sigma(S) = \text{ the category of matchings of } S$$

$$I\Sigma(r)(m_H/\tau_H) = m_C/\tau_C$$

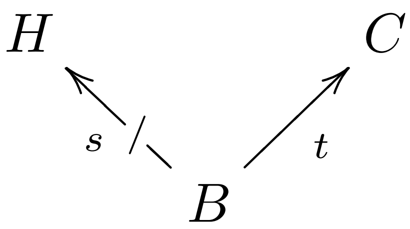
An example

The rule for substitution in equational logic:

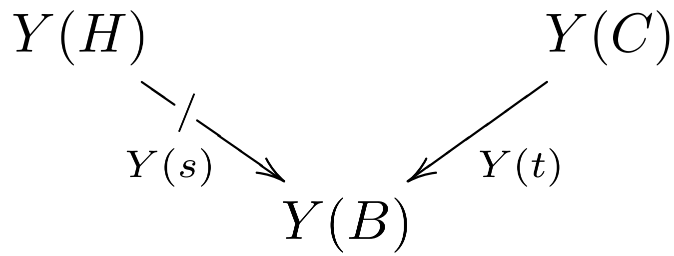
$$(R) \qquad \frac{f(x) \equiv g(x) \quad a}{f(a) \equiv g(a)}$$

or $r = t \circ s^{-1}$:

in $\mathbf{S}_{\mathrm{eq}}$:



in $\mathbf{Spec}(P_{\mathrm{eq}})$:



where:

$Y(H) = \{f(x) \equiv g(x),\ a\}$ (the hypothesis of R)

$Y(B) = \{f(x) \equiv g(x),\ a,\ f(a) \equiv g(a)\}$

$Y(C) = \{f(a) \equiv g(a)\}$ (the conclusion of R)

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We have got a

category of logics. . .