

# Decorated proofs for computational effects: States

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# Outline

# From computer algebra to effects

About the history of the authors:

- ▶ **Computer algebra**: exact computations on large integers, matrices, polynomials, field extensions,...
- ▶ Sophisticated **programmation** in several kinds of languages: C, C++, Axiom,...
- ▶ Questions about the languages: semantics of computational **effects**? (e.g., states, exceptions,...)

# Effects and monads

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2. it should often be interpreted as a function  $[[f]] : [[X]] \rightarrow T[[Y]]$  for some **monad**  $T$

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*We agree with (1), not always with (2).  
And we get operations and equations in a different way.*



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*An effect is an apparent lack of soundness.*

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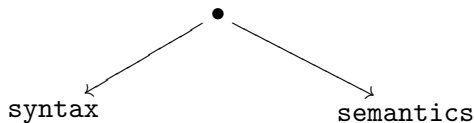
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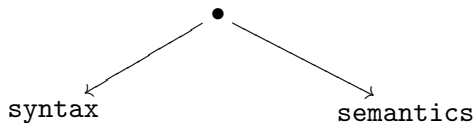
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Formally: [Domínguez&Duval MSCS'10]

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**Proof.**

Let  $n$  be the value of  $X$  in the current state.

- ▶ First “ $X$ ” (on the right) is evaluated as  $n$ .
- ▶ Then “ $X :=$ ” (on the left) puts the value of  $X$  to  $n$ , without modifying the value of other locations.

Hence the state is not modified. □

## Towards a formalization: a specification for states

**Locations** (or identifiers, or variables)  $X, Y, \dots$

The **unit** (or void, or singleton) type  $\mathbb{1}$ , with  $\langle \rangle_A : A \rightarrow \mathbb{1}$  for each  $A$ .

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For each  $X$ , a type  $V_X$  for **values**, two operations:

$$\ell_X : \mathbb{1} \rightarrow V_X \quad (\text{lookup})$$

$$u_X : V_X \rightarrow \mathbb{1} \quad (\text{update})$$

and equations:

$$\ell_X \circ u_X \equiv id$$

$$\ell_Y \circ u_X \equiv \ell_Y \circ \langle \rangle \quad \text{when } Y \neq X$$

formalizing the intended semantics:

- ▶  $\ell_X$  returns the value of  $X$  in the current state
- ▶  $u_X(n)$  modifies the current state: the value of  $X$  becomes  $n$ , and the value of  $Y$  is not modified, for every  $Y \neq X$



## A property of imperative languages: proof # 1

Let  $\Sigma$  be the specification made of  $\ell_X : \mathbb{1} \rightarrow V_X$  and  $u_X : V_X \rightarrow \mathbb{1}$  such that  $\ell_X \circ u_X \equiv id$  and  $\ell_Y \circ u_X \equiv \ell_Y \circ \langle \rangle$  when  $Y \neq X$ .

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**Proof.** By observation: prove that  $\ell_Y \circ u_X \circ \ell_X \equiv \ell_Y$  for each  $Y$ .  
When  $Y = X$ :

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Hence the state is not modified.  $\square$

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Another proof.

The (unit) rule states that  $id$  is the unique  $f : \mathbb{1} \rightarrow \mathbb{1}$ .

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**BUT** in the same way, we could prove for all  $Y$ :

$$\text{(unit)} \quad \frac{u_X \circ \ell_Y : \mathbb{1} \rightarrow \mathbb{1}}{u_X \circ \ell_Y \equiv id}$$

which obviously is **FALSE!**

# Questions

Two proofs of (ALU). Proof #1 is right, proof #2 is wrong.

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The (unit) rule should state that  $id$  is the unique  $f : \mathbb{1} \rightarrow \mathbb{1}$   
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How can we formalize this fact?

By **decorating** terms and equations.

# Decorations: terms and equations

**Terms** are classified:

- ▶  $f^{(0)}$ :  $f$  is **pure** if it cannot use nor modify the state.
- ▶  $f^{(1)}$ :  $f$  is an **accessor** if it can use the state, not modify it.
- ▶  $f^{(2)}$ :  $f$  is a **modifier** if it can use and modify the state.

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**Equations** are classified:

- ▶  $f \equiv g$ : **strong** equation:  $f$  and  $g$  return the same value and they have the same effect on the state.
- ▶  $f \sim g$ : **weak** equation:  $f$  and  $g$  return the same value but they may have different effects on the state.

Hierarchy rule:  $\frac{f \equiv g}{f \sim g}$ .

## Decorated rules

The rules of the logic are also decorated, for instance:

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Hence there are new derived rules, like:

$$\text{(1-unit)} \quad \frac{f^{(1)} : \mathbb{1} \rightarrow \mathbb{1}}{f \equiv id}$$

## Proof #2 is wrong: it cannot be properly decorated

Proof #2 of (ALU) can be decorated as follows:

$$\text{(unit)} \quad \frac{u_X \circ \ell_X : \mathbb{1} \rightarrow \mathbb{1}}{u_X \circ \ell_X \sim id}$$

which does **not** entail  $u_X \circ \ell_X \equiv id$ .



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In fact for each  $Y$  there is a proof:

$$\text{(unit)} \quad \frac{u_X \circ \ell_Y : \mathbb{1} \rightarrow \mathbb{1}}{u_X \circ \ell_Y \sim id}$$

which is right but without any interest.

# Decorated rules for substitution and replacement

Strong equations form a congruence:

$$(\equiv\text{-subs}) \frac{g_1 \equiv g_2}{g_1 \circ f \equiv g_2 \circ f} \qquad (\equiv\text{-repl}) \frac{f_1 \equiv f_2}{g \circ f_1 \equiv g \circ f_2}$$

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Weak equations do **not** form a congruence:

$$(\sim\text{-subs}) \frac{g_1 \sim g_2}{g_1 \circ f \sim g_2 \circ f} \quad (0\text{-}\sim\text{-repl}) \frac{f_1 \sim f_2 \quad g^{(0)}}{g \circ f_1 \sim g \circ f_2 : X \rightarrow Z}$$

**Indeed:**  $f_1$  and  $f_2$  may modify the state in a different way, so that  $g \circ f_1$  and  $g \circ f_2$  may return different values if  $g$  is not pure.

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For each  $X$ , a type  $V_X$  for **values**, two operations:

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and weak equations:

$$\begin{array}{l} \ell_X \circ u_X \sim id \\ \ell_Y \circ u_X \sim \ell_Y \circ \langle \rangle \quad \text{when } Y \neq X \end{array}$$

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**Proof.** By observation: prove that  $\ell_Y \circ u_X \circ \ell_X \sim \ell_Y$  for each  $Y$ .  
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The 7 properties in [Plotkin&Power 02] can be proved similarly.  
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which is formalized as:

$$u_Y \circ (u_X \times id) \equiv u_X \circ (id \times u_Y) : V_X \times V_Y \rightarrow \mathbb{1}$$

where  $\times$  is the **semi-pure** product from [Dumas&Duval&Reynaud]  
*Cartesian effect categories are Freyd-categories* JSC 2011. ACCAT'09.

$$\begin{array}{ccc} V_X & \xrightarrow{u_X^{(2)}} & \mathbb{1} \\ \uparrow & \equiv & \uparrow \\ V_X \times V_Y & \xrightarrow{(u_X \times id)^{(2)}} & \mathbb{1} \times V_Y \\ \downarrow & \sim & \downarrow \\ V_Y & \xrightarrow{id^{(0)}} & V_Y \end{array}$$

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$f^{(1)} : X \rightarrow Y$	$f : X \times S \rightarrow Y$
$f^{(2)} : X \rightarrow Y$	$f : X \times S \rightarrow Y \times S$
$f \equiv g : X \rightarrow Y$	$f \equiv g : X \times S \rightarrow Y \times S$
$f \sim g : X \rightarrow Y$	$\pi \circ f \equiv \pi \circ g : X \times S \rightarrow Y$

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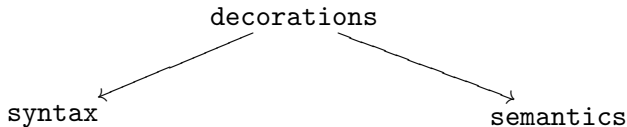
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3. **prove** in the “usual” (not decorated) logic

**But** the notion of **effect** is lost.

# A span of “logics”



- ▶  $\text{decorations} \rightarrow \text{syntax}$  :  
forget the decorations
- ▶  $\text{decorations} \rightarrow \text{semantics}$  :  
**expansion**, with an explicit  $S$  for states

# From proofs to models

The **expansion**:

- ▶ maps decorated proofs to “usual” explicit proofs



# From proofs to models

The **expansion**:

- ▶ maps decorated proofs to “usual” explicit proofs
- ▶ and provides a notion of **decorated model**

because it can be seen as a **functor**  $F$  with a right **adjoint**:

$$\text{decorations} \begin{array}{c} \xrightarrow{F} \\ \xleftarrow{G} \end{array} \text{semantics}$$

$$Mod_{\text{deco}}(\Sigma, G\Theta) \cong Mod_{\text{expl}}(F\Sigma, \Theta)$$

For instance:

$\Sigma$  is the decorated specification for states

$\Theta$  is **Set** with the distinguished set  $S = \prod_X V_X$

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[Dumas&Duval&Fousse&Reynaud] *Decorated proofs for computational effects: exceptions. Submitted for publication.*
- ▶ This is due to the **duality** between states and the core part of exceptions.  
[Dumas&Duval&Fousse&Reynaud] *A duality between exceptions and states. To appear in MSCS. ACCAT'11.*

# Conclusion and future work

We have designed a framework for **effects** which provides a **denotational semantics** and a **proof system**.

# Conclusion and future work

We have designed a framework for **effects** which provides a **denotational semantics** and a **proof system**.

Our projects include:

- ▶ Using a **proof assistant** for proving decorated properties.
- ▶ Extending our framework for **combining effects** by composing spans.

Thank you!