### Sequential products for effects

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### **Outline**

#### Introduction

What is an effect? Effect categories

What is a sequential product? Cartesian effect categories

### Motivation

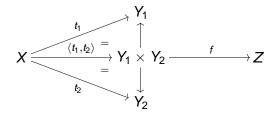
In a categorical semantics for a programming language, the construction of terms is interpreted by composition and products.

When the language has side-effects, this has to be adapted. One major issue is that the value of a term  $f(t_1, ..., t_n)$  may depend on the order of evaluation of its arguments.

The aim of this talk is to present a new framework and to compare it to existing ones.

### Categorical semantics

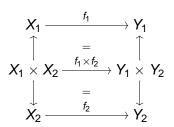
Language	Category
sort	object
operation:	morphism:
$f: X_1, \ldots, X_n \to Y$	$f: X_1 \times \cdots \times X_n \to Y$
term construction:	composition and tuple:
$f(t_1,\ldots,t_n)$	$f\circ\langle t_1,\ldots,t_n\rangle$



# The product functor

Binary products on C define a functor  $\times : \mathbb{C}^2 \to \mathbb{C}$ :

- ▶ On objects:  $X_1 \times X_2$ , with projections  $p_i : X_1 \times X_2 \rightarrow X_i$ .
- ▶ On morphims:  $f_1 \times f_2 : X_1 \times X_2 \rightarrow Y_1 \times Y_2$ , defined as  $f_1 \times f_2 = \langle f_1 \circ p_1, f_2 \circ p_2 \rangle$ , i.e., characterized by:



## Computational effects

Without effects, an operation symbol  $f: X \to Y$  stands for a (total) function  $f: X \to Y$ .

With effects, an operation symbol  $f: X \to Y$  stands for "something else", e.g.:

- Partiality: a partial function f : X → Y,
- State: a function f : S × X → S × Y
- ▶ Non-determinism: a function  $f: X \to \mathcal{L}(Y)$
- and so on...

What about term construction?
I.e., what about composition and products?

#### Frameworks for effects

#### Several frameworks, quite "similar" [Haskell]:

- Strong monads [Moggi'89]
- Premonoidal categories [Power&Robinson'97]
- Arrows [Hughes'00]

Our framework is more "restricted" and more "homogeneous":

Cartesian effect categories [Dumas&Duval&Reynaud'07,'09].

# Homogeneity

*K* is a category, *C* is a wide subcategory of *K*:

$$C \subseteq K$$

#### Freyd-category:

С	К
cartesian	
₩	
monoidal	premonoidal

### Cartesian effect category:

С	K
cartesian	"sequential cartesian"
₩	₩
monoidal	premonoidal

### Our result, in short

The universal property for the product  $f \times v$ :

$$X_{1} \xrightarrow{f} Y_{1}$$

$$\uparrow \qquad = \qquad \uparrow$$

$$X_{1} \times X_{2} \xrightarrow{f \times v} Y_{1} \times Y_{2}$$

$$\downarrow \qquad = \qquad \downarrow$$

$$X_{2} \xrightarrow{v} Y_{2}$$

has to be "decorated":

The aim of this talk is to explain what this means.



### **Example: partiality**

f is partial, v = id is total,  $\leq$  is the usual order on partial functions.

Let  $f \times id$  be such that:

then  $f \times id$  is the partial function:

$$\left\{ \begin{array}{l} \mathcal{D}(f \rtimes \mathrm{id}) = \{(x_1, x_2) \mid x_1 \in \mathcal{D}(f)\} \text{ and} \\ \forall (x_1, x_2) \in \mathcal{D}(f \rtimes \mathrm{id}), \ (f \rtimes \mathrm{id})(x_1, x_2) = (f(x_1), x_2) \end{array} \right.$$

### Two questions

- What is an effect?
  - $\rightarrow \text{effect categories}.$
- What is a sequential product?
  - $\rightarrow$  cartesian effect categories.

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# Pure vs. general morphisms

*K* is a category, *C* is a wide subcategory of *K*:

$$C \subseteq K$$

- ▶ (General) morphisms  $f: X \to Y$  in K,
- pure morphisms v : X → Y in C.

### Example. $Set \subseteq Part$

a morphism  $f: X \to Y$  is a partial function, a pure morphism  $v: X \leadsto Y$  is a total function.



### **Effects**

The effect of  $f: X \to Y$  should provide a measure of the "distance" from f to pure functions.

Let 1 be a terminal object in C:

for all X there is a unique  $\langle \ \rangle_X : X \rightsquigarrow 1$ 

The effect of  $f: X \to Y$  is  $\langle \rangle_Y \circ f: X \to 1$ .

 $f: X \to Y$  is effect-free if  $\langle \rangle_Y \circ f = \langle \rangle_X$ . Hence, every pure morphism is effect-free.

### Example. $Set \subseteq Part$

 $1 = \{*\}$  (a singleton).

The effect of f is  $\langle \rangle \circ f : X \to \{*\}$ , such that  $\mathcal{D}(\langle \rangle \circ f) = \mathcal{D}(f)$ .



### Same-effect equivalence

Let otin be the relation between morphisms such that for all  $f: X \to Y$  and  $f': X \to Y'$ ,  $f \approx f'$  if and only if f and f' have the same effect, i.e.

$$f \approx f' \iff \langle \rangle \circ f = \langle \rangle \circ f'$$

$$f \approx f' \iff \mathcal{D}(f) = \mathcal{D}(f')$$
.

# Symmetric up-to-effects consistency

Let be a relation between parallel morphisms that satisfies:

- reflexivity, symmetry,
- ▶ substitution:  $g \le g' \implies g \circ f \le g' \circ f$
- ▶ pure replacement:  $f \le f' \implies w \circ f \le w \circ f'$  when w is pure.
- ▶ complementarity wrt  $\approx$ : for all  $f, f' : X \rightarrow Y$ ,

$$f \approx f'$$
 and  $f \smile f' \implies f = f'$ 

$$f \smile f' \iff f = f' \text{ on } \mathcal{D}(f) \cap \mathcal{D}(f')$$
.

### Transitive up-to-effects consistency

Let  $\leq$  be a relation between parallel morphisms that satisfies:

- reflexivity, transitivity,
- substitution, pure replacement,
- ▶ complementarity wrt  $\approx$ : for all  $f, f', f'' : X \rightarrow Y$ ,

$$f \approx f'$$
 and  $f \leq f''$  and  $f' \leq f'' \implies f = f'$ 

Let  $f \smile f' \iff \exists f''$ ,  $f \le f''$  and  $f' \le f''$ .

Then  $\smile$  is a symmetric up-to-effects consistency.

$$f \leq f' \iff \mathcal{D}(f) \subseteq \mathcal{D}(f') \text{ and } f = f' \text{ on } \mathcal{D}(f)$$



### Hence, three relations

▶ Same-effect equivalence  $f \approx f'$ :

$$f \approx f' \iff \langle \, \rangle \circ f = \langle \, \rangle \circ f'$$

Symmetric up-to-effects consistency  $f \smile f'$ :

$$f \approx f'$$
 and  $f \smile f' \implies f = f'$ 

▶ Transitive up-to-effects consistency  $f \le f'$ :

$$f \smile f' \iff \exists f'' \ f \le f'' \ \text{and} \ f' \le f''$$

$$\begin{cases} f \approx f' & \iff \mathcal{D}(f) = \mathcal{D}(f') \\ f \smile f' & \iff f = f' \text{ on } \mathcal{D}(f) \cap \mathcal{D}(f') \\ f \le f' & \iff \mathcal{D}(f) \subseteq \mathcal{D}(f') \text{ and } f = f' \text{ on } \mathcal{D}(f) \end{cases}$$

### Effect categories

#### Definition.

An effect category is  $C \subseteq K$  with a transitive up-to-effects consistency  $\subseteq$ ,

i.e., a relation between parallel morphisms that satisfies:

- reflexivity, transitivity,
- substitution, pure replacement,
- equality on pure morphisms.

$$f \leq f' \iff \mathcal{D}(f) \subseteq \mathcal{D}(f') \text{ and } f = f' \text{ on } \mathcal{D}(f)$$



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# Semi-pure products

Let  $C \subseteq K$  with  $\leq$  be an effect category, with a binary product  $\times$  on C.

#### Definition.

The left semi-pure product  $v_1 \ltimes f_2$  and the right semi-pure product  $f_1 \rtimes v_2$  are characterized by:

$$X_{1} \xrightarrow{v_{1}} Y_{1}$$

$$\begin{cases} & \geq & \\ \\ & \downarrow \\ X_{1} \times X_{2} \xrightarrow{v_{1} \times f_{2}} Y_{1} \times Y_{2} \end{cases}$$

$$\begin{cases} & = & \\ \\ & \downarrow \\ X_{2} \xrightarrow{f_{2}} Y_{2} \end{cases}$$

$$X_{1} \xrightarrow{f_{1}} Y_{1}$$

$$\begin{cases}
 = \\
 X_{1} \times X_{2} \xrightarrow{f_{1} \times V_{2}} Y_{1} \times Y_{2}
\end{cases}$$

$$\begin{cases}
 \geq \\
 X_{2} \times X_{2} \xrightarrow{V_{2}} Y_{2}
\end{cases}$$

### Sequential products

#### Definition.

The left sequential product  $f_1 \ltimes f_2$  is defined as:

$$f_1 \ltimes f_2 = (\operatorname{id}_{Y_1} \ltimes f_2) \circ (f_1 \rtimes \operatorname{id}_{X_2})$$
 "first  $f_1$ , then  $f_2$ "

and symmetrically for the right sequential product  $f_1 \times f_2$ :

$$f_1 \times f_2 = (f_1 \times id_{Y_2}) \circ (id_{X_1} \times f_2)$$
 "first  $f_2$ , then  $f_1$ "

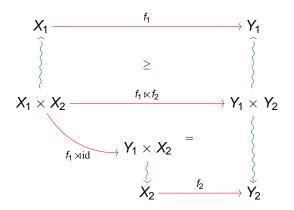
## **Example:** partiality

Then  $f_1 \ltimes f_2 = f_1 \rtimes f_2$ : every function is central.

$$\begin{cases} \mathcal{D}(f_1 \ltimes f_2) = \{(x_1, x_2) \mid x_1 \in \mathcal{D}(f_1) \land x_2 \in \mathcal{D}(f_2)\} \\ \text{and } \forall (x_1, x_2) \in \mathcal{D}(f_1 \ltimes f_2), \\ (f_1 \ltimes f_2)(x_1, x_2) = (f(x_1), f_2(x_2)) \end{cases}$$

### Sequential products, directly

Theorem. The left and right sequential products can be defined directly, in a mutually recursive way, by another "decorated" version of the product property:



and symmetrically...

## Cartesian effect categories

#### Definition.

A cartesian effect category is

- an effect category C ⊆ K with ≤
- ▶ with a binary product × on C
- ▶ and with sequential products ⋉, ⋈.

#### Theorem.

A cartesian effect category is a Freyd-category

### Example: state

S: a fixed set of states (or stores).

$$S \stackrel{\sigma}{\longleftarrow} S \times X \stackrel{\pi}{\longrightarrow} X$$

Objects of C and K: sets

Morphism  $f: X \to Y$  in K: function  $[f]: S \times X \to S \times Y$ 

Pure morphism  $v: X \rightsquigarrow Y$  in  $C: [v] = id_S \times v_0$ 

$$\begin{cases} f \approx f' & \iff \sigma \circ [f] = \sigma \circ [f'] \\ f \smile f' & \iff \pi \circ [f] = \pi \circ [f'] \\ f \le f' & \iff \pi \circ [f] = \pi \circ [f'] \end{cases}$$

$$\forall x_1, x_2, s, [f_1 \ltimes f_2](s, x_1, x_2) = (s_2, y_1, y_2)$$

where  $[f_1](s, x_1) = (s_1, y_1)$  and  $[f_2](s_1, x_2) = (s_2, y_2)$ .

# Example: non-determinism

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Cf. the monad of lists \mathcal{L}(-).
```

Objects of C and K: sets

Morphism  $f: X \to Y$  in K: function  $[f]: X \to \mathcal{L}(Y)$ 

Pure morphism  $v: X \rightsquigarrow Y$  in C: [v] of length 1.

For all  $f: X \to Y$  in K and  $k \in \mathbb{N}$ ,

let  $f^{\langle k \rangle}: X \to Y$  in K be the k-th "stutter":

$$[f^{\langle k \rangle}](x) = (y_1^k, \dots, y_n^k)$$
 where  $[f](x) = (y_1, \dots, y_n)$ 

$$\begin{cases} f \approx f' & \iff \ell(f) = \ell(f') \\ f \smile f' & \iff f = () \text{ or } f' = () \text{ or } \exists n, n' \in \mathbb{N}, \ f^{\langle n \rangle} = f'^{\langle n' \rangle} \\ f \le f' & \iff \exists k \in \mathbb{N}, \ f = f'^{\langle k \rangle} \end{cases}$$

$$\forall x_1, x_2 , [f_1 \ltimes f_2](x_1, x_2) = \\ (\langle y_{1,1}, y_{2,1} \rangle, \dots, \langle y_{1,1}, y_{2,n_2} \rangle, \dots, \langle y_{1,n_1}, y_{2,1} \rangle, \dots, \langle y_{1,n_1}, y_{2,n_2} \rangle) \\ \text{where } [f_1](x_1) = (y_{1,1}, \dots, y_{1,n_1}) \text{ and } [f_2](x_2) = (y_{2,1}, \dots, y_{2,n_2})$$

#### Conclusion

In this talk:

a new approach is provided for the major issue of dealing with multivariate operations when there are effects.

Future work:

more examples, look at the issue of combining effects.

#### Some references

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