Diagrammatic specifications

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1 Introduction

This paper deals with some kind of *progressive* constructions of freely generated structures. For instance, in order to generate progressively the words on the alphabet $X = \{a, b\}$, we might first generate one word ab, then add it to X, getting $X_1 = \{a, b, ab\}$, and repeat a similar process from X_1. However, for this process to result in the construction of X^*, we must be able to remember that the string ab in X_1 stands for the concatenation of a and b. This means that we have to consider X_1 not just as a set, but as a partial monoid, with a partially defined concatenation operation which maps the pair (a, b) towards the string ab. So, in this example, we have to deal with three different structures: the sets, the monoids, and the partial monoids. Clearly, the sets are the partial monoids where the operation is nowhere defined, and the monoids are the partial monoids where the operation is everywhere defined. Sets and monoids are used to define X^* from X, while partial monoids are needed for building progressively X^* from X.

Freely generated structures play a fundamental role in mathematics and computer science: for instance, words are freely generated from an alphabet, theorems from axioms, and programs from grammars. A freely generated structure may be quite "large", however it happens that only a "small" part of it is needed: in order to prove a theorem from a given set of axioms, one does not prove all the theorems first...

It is well known that the categorical notion of *adjunction* is a basic one for dealing with freely generated structures [Mac Lane, 1971]. We are interested in adjunctions which allow some kind of progressive construction of the freely generated structure. For instance, the direct definition of X^* from X is an adjunction between sets and monoids, whereas the progressive construction of X^* from X needs the adjunction between partial monoids and monoids.

In this paper, we present a general framework for such adjunctions, which includes abstract definitions of *syntactic entailment* and *semantic consequence*. Our theory of *diagrammatic specifications* is derived, in a natural and simple way, from the theory of *projective sketches*. Thanks to the use of projective sketches at the meta level, for the "specification of specifications", the theory of diagrammatic specifications is quite homogeneous and effective.

Our motivation has its roots in the study of computer languages, mainly in the links between several programming styles, including imperative and object paradigms; these applications will be considered in subsequent papers.

Adjunctions, categories and projective sketches.

An adjunction is a pair of *functors* ($F : \mathcal{A}_1 \to \mathcal{A}_2$, $U : \mathcal{A}_2 \to \mathcal{A}_1$) between two *categories* $\mathcal{A}_1$ and $\mathcal{A}_2$ such that, for all A_1 in $\mathcal{A}_1$ and A_2 in $\mathcal{A}_2$, the arrows from A_1 to $U(A_2)$ in $\mathcal{A}_1$ are naturally in one-to-one correspondence with the arrows from $F(A_1)$ to A_2 in $\mathcal{A}_2$:

$$\mathrm{Hom}_{\mathcal{A}_2}(F(A_1), A_2) \cong \mathrm{Hom}_{\mathcal{A}_1}(A_1, U(A_2)) \ .$$

For instance, $\mathcal{A}_1$ is the category of sets, $\mathcal{A}_2$ is the category of monoids, the functor U maps each monoid to its underlying set, and the functor F maps each set to its freely generated monoid.

In addition, we introduce a *meta-level*, which is based on *projective sketches*. Sketches were introduced in [Ehresmann, 1966]. We assume that $\mathcal{A}_1$ is the category of *realizations* (*i.e.* models) of a projective sketch $\mathcal{E}_1$, and that $\mathcal{A}_2$ is the category of realizations of a projective sketch $\mathcal{E}_2$. Such categories are known as *locally presentable categories* [Gabriel and Ulmer, 1971]. We also assume that U and F are the *underlying functor* and the *freely generating functor* associated to a *propagator* (*i.e.* an homomorphism) $P : \mathcal{E}_1 \to \mathcal{E}_2$:

$$\mathcal{A}_1 = \mathcal{R}eal(\mathcal{E}_1) \ , \ \mathcal{A}_2 = \mathcal{R}eal(\mathcal{E}_2) \ , \ U = U_P : \mathcal{R}eal(\mathcal{E}_2) \to \mathcal{R}eal(\mathcal{E}_1) \ , \ F = F_P : \mathcal{R}eal(\mathcal{E}_1) \to \mathcal{R}eal(\mathcal{E}_2) \ .$$

For instance, $\mathcal{E}_1$ is a projective sketch of sets, which means that the realizations of $\mathcal{E}_1$ are the sets and their morphisms are the maps, and $\mathcal{E}_2$ is a projective sketch of monoids, which means that the realizations of $\mathcal{E}_2$ are the monoids and their morphisms are the morphisms of monoids.

Decomposition.

In this context, the adjunctions which allow a progressive construction of the freely generating functor F are those with a *full and faithful* underlying functor U; then, we say that the propagator is *fractioning*. On the other hand, we say that a propagator is *filling* whenever the freely generating functor F is full and faithful. Both words "fractioning" and "filling" stem from properties of these propagators which are stated in the paper.

We prove that any propagator $P : \mathcal{E}_0 \to \overline{\mathcal{E}}$ can be decomposed as $P = K \circ J$, with $J : \mathcal{E}_0 \to \mathcal{E}$ filling and $K : \mathcal{E} \to \overline{\mathcal{E}}$ fractioning. In addition, J can be chosen in such a way that the construction of F_J is trivial. A consequence of the decomposition $P = K \circ J$ is that $F_P(A_0) = F_K(F_J(A_0))$ for all realization A_0 of $\mathcal{E}_0$. So, in order to build $F_P(A_0)$, we may replace the propagator P and the realization A_0 of $\mathcal{E}_0$ by the fractioning propagator K and the realization $F_J(A_0)$ of $\mathcal{E}$.

For instance, when P is a propagator from a sketch $\mathcal{E}_0$ of sets to a sketch $\overline{\mathcal{E}}$ of monoids, then $\mathcal{E}$ can be a sketch of partial monoids. The partial monoid $F_J(X)$ is still denoted X: it is the set X with the nowhere-defined partial concatenation.

Specifications.

Let $P : \mathcal{E}_0 \to \overline{\mathcal{E}}$ be a propagator together with a decomposition $P = K \circ J$, with $J : \mathcal{E}_0 \to \mathcal{E}$ filling and $K : \mathcal{E} \to \overline{\mathcal{E}}$ fractioning, as above. In this context, let us give some basic definitions, followed by an example.

The *specifications* are the realizations of $\mathcal{E}$, the *signatures* are the realizations of $\mathcal{E}_0$, and the *domains* are the realizations of $\overline{\mathcal{E}}$. Then, the set of *models* of a specification S with values in a domain D is:

$$\mathrm{Mod}_K(S, D) = \mathrm{Hom}_{\mathcal{R}eal(\overline{\mathcal{E}})}(F_K(S), D),$$

so that, by adjunction:

$$\mathrm{Mod}_K(S, D) \cong \mathrm{Hom}_{\mathcal{R}eal(\mathcal{E})}(S, U_K(D)).$$

For example, let us look at simple equational specifications, where all the operators are unary. A *compositive graph* is a directed graph together with a partial composition of arrows, so that a category is a compositive graph with total composition of arrows. Then, a signature is a compositive graph, a specification is a signature together with a binary relation $\equiv$ on arrows ($f \equiv g$ is called an *equation*), and a domain is a category together with a binary relation $\equiv$ on arrows which is a *congruence*, *i.e.* an equivalence relation compatible with the composition. So, $\mathcal{E}_0$ is a projective sketch of compositive graphs, $\mathcal{E}$ is a projective sketch of compositive graphs with equations, and $\overline{\mathcal{E}}$ is a projective sketch of categories with congruence. The propagators P, J and K are straightforward.

In order to specify the integers, we consider the signature $S_{int,0}$ which is made of a point I, four arrows s, p, $s \circ p$ and $p \circ s : I \to I$, and the partial composition which maps the pair (p, s) to the arrow $s \circ p$ and the pair (s, p) to the arrow $p \circ s$. The signature $S_{int,0}$ together with the equation $p \circ s \equiv s \circ p$ is a specification S_{int}. The signature $U_J(S_{int})$ which is underlying S_{int} is $S_{int,0}$. The domain $F_K(S_{int})$ which is freely generated by S_{int} is the category with one point I and all arrows composed from s and p (like $s \circ s \circ p \circ p \circ s \circ p \circ s$), and with the congruence relation $f \equiv g$ if and only if the number of s's minus the number of p's is the same one in f and in g. The arrows of $F_K(S_{int})$ are the *terms* and its equations are the *theorems* which are derived from the specification S_{int}.

On the other hand, let D_{set} be the realization of $\overline{\mathcal{E}}$ with the sets as points, the maps as arrows, and the equality as congruence. A set-valued model of S_{int} can be seen either as a morphism from $F_K(S_{int})$ to D_{set} in $\mathcal{R}eal(\overline{\mathcal{E}})$, or as a morphism from S_{int} to $U_K(D_{set})$ in $\mathcal{R}eal(\mathcal{E})$: it interprets each point of S_{int} as a set, each arrow of S_{int} as a map, and each equation of S_{int} as an equality. For instance, there is a set-valued model of the specification S_{int} which maps the point I on the set of integers, the arrows s and p on the *successor* and *predecessor* map respectively, and the arrows $p \circ s$ and $s \circ p$ on the identity.

Diagrammatic specifications are not restricted to equational ones. Actually, first-order and higher-order specifications also can be considered as diagrammatic specifications.

Entailment and consequence.

Now let us focus on a fractioning propagator $K : \mathcal{E} \to \overline{\mathcal{E}}$.

A morphism of specifications $\sigma : S \to S'$ is a *syntactic entailment*, which is denoted $S \xrightarrow{\sigma}\!\!\!\!\!\!\shortmid\, S'$, whenever the derived morphism $F_K(\sigma) : F_K(S) \to F_K(S')$ is an isomorphism:

$$S \xrightarrow{\sigma}\!\!\!\!\!\shortmid\; S' \quad \text{if and only if} \quad F_K(S) \xrightarrow[F_K(\sigma)]{\simeq} F_K(S') \ .$$

In addition, a syntactic entailment can be obtained by a succession of *deduction steps*, using the *deduction rules* which are given by the projective sketch $\overline{\mathcal{E}}$.

A morphism of specifications $\sigma : S \to S'$ is a *semantic consequence* with respect to some domain D, which is denoted $S \xrightarrow{\sigma}\!\!\!\!\!\shortmid\!\shortmid_D S'$, whenever the derived morphism $\mathrm{Mod}_K(\sigma, D) : \mathrm{Mod}_K(S', D) \to \mathrm{Mod}_K(S, D)$ is an isomorphism:

$$S \xrightarrow{\sigma}\!\!\!\!\!\shortmid\!\shortmid_D S' \quad \text{if and only if} \quad \mathrm{Mod}_K(S, D) \xleftarrow[\mathrm{Mod}_K(\sigma,D)]{\simeq} \mathrm{Mod}_K(S', D) \ .$$

In addition, a semantic consequence can also be defined from a *satisfaction relation* between models and specifications, which makes sense only when a filling propagator $J : \mathcal{E}_0 \to \mathcal{E}$ is given, besides the fractioning propagator $K : \mathcal{E} \to \overline{\mathcal{E}}$.

With these definitions, the *soundness* property is satisfied, which means that syntactic entailment implies semantic consequence:

$$\text{if} \quad S \xrightarrow{\sigma}\!\!\!\!\!\shortmid\; S' \quad \text{then} \quad S \xrightarrow{\sigma}\!\!\!\!\!\shortmid\!\shortmid_D S' \quad \text{for all } D.$$

Our diagrammatic specifications can be related to the *institutions* [Goguen and Burstall, 1992]. Then, it is possible to compare our notions of entailment and consequence with the notions which occur in logic with the same names and the symbols $\vdash$ and $\models$, respectively. The fractioning propagator K can be chosen in such a way that there is a K-domain "of sets" D_{set} and a K-specification S corresponds to a conjunction of sentences $\varphi_1, \varphi_2, \ldots, \varphi_k$. Then a morphism $\sigma : S \to S'$ can correspond to adding a sentence ψ, so that the K-specification S' corresponds to the conjunction of the sentences $\varphi_1, \varphi_2, \ldots, \varphi_k, \psi$. In such a situation:

$$S \xrightarrow{\sigma}\!\!\!\!\!\shortmid\; S' \quad \text{if and only if} \quad \varphi_1, \varphi_2, \ldots, \varphi_k \vdash \psi \ ,$$

$$S \xrightarrow{\sigma}\!\!\!\!\!\shortmid\!\shortmid_{D_{set}} S' \quad \text{if and only if} \quad \varphi_1, \varphi_2, \ldots, \varphi_k \models \psi \ .$$

Organization of the paper.

This paper begins with a review of some useful definitions and results about categories (section 2). These results are well known, they can be found in [Mac Lane, 1971] for example.

Then in section 3 are reviewed some definitions and results about projective sketches, which are not so well known, although most of them can be found in [Coppey and Lair, 1984] and [Coppey and Lair, 1988], or in [Duval and Lair, 2001].

Section 4 is devoted to the study of fractioning and filling propagators and to the decomposition theorem.

In section 5 are defined the notions of specification, domain and model, as well as syntactic entailment and semantic consequence.

Finally, in section 6, we look at equational diagrammatic specifications and we outline some links between diagrammatic specifications and institutions.

The applications of diagrammatic specifications to the study of computer languages will be the subject of forthcoming papers.

From the point of view of terminology, we have made some choices: *point* rather than *object*, *source* and *target* rather than *domain* and *codomain*, and so on. For technical issues, including the size issues, we refer to the reference manual [Duval and Lair, 2001]. So, for instance, we speak without any care about *the category of categories*.

Moreover, in order to keep distinct the specification level and the meta-specification level, we speak on one side about *morphisms* and *models* of specifications, and on the other side about *propagators* and *realizations* of projective sketches.

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2 Categories, adjoints, limits

Here are a few basic facts about *categories*, *adjunction* and *limits*. All this is very well known, it goes back to Eilenberg and Mac Lane in the 1940's, and can be found in [Mac Lane, 1971] for instance. However, the flavour of our definition of limits stems from the theory of sketches. In addition, some of our illustrations will be given a precise status in section 4.

2.1 Directed graphs

Definition 2.1.1
A *(directed) graph* $\mathcal{G}$ is made of a set of *points*, a set of *arrows*, and two maps from arrows to points, which assign to each arrow respectively its *source* and its *target*.

An arrow g with source G_1 and target G_2, *i.e.* an arrow *from* G_1 *to* G_2, is denoted $g : G_1 \to G_2$ or $G_1 \xrightarrow{g} G_2$. The set of arrows from G_1 to G_2 in $\mathcal{G}$ is denoted $\mathrm{Hom}_{\mathcal{G}}(G_1, G_2)$.

An arrow g is a *loop* if $G \xrightarrow{g} G$.

Two arrows g_1 and g_2 are *consecutive* if $G_1 \xrightarrow{g_1} G_2 \xrightarrow{g_2} G_3$.

A triple of arrows (g_1, g_2, g) is a *triangle* if $G_1 \xrightarrow{g_1} G_2 \xrightarrow{g_2} G_3$ and $G_1 \xrightarrow{g} G_3$.

The *opposite* of $\mathcal{G}$ is the directed graph $\mathcal{G}^{op}$ with the arrows in the opposite direction.

Definition 2.1.2
A *graph homomorphism* $H : \mathcal{G} \to \mathcal{G}'$ is made of two maps, both denoted H, from the points (*resp.* the arrows) of $\mathcal{G}$ towards the points (*resp.* the arrows) of $\mathcal{G}'$, such that if $g : G_1 \to G_2$ then $H(g) : H(G_1) \to H(G_2)$.

Hence, for all points G_1 and G_2 in $\mathcal{G}$, the map H on arrows restricts to a map:

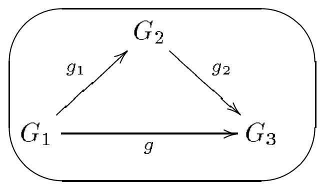
$$H_{G_1,G_2} : \mathrm{Hom}_{\mathcal{G}}(G_1, G_2) \to \mathrm{Hom}_{\mathcal{G}'}(H(G_1), H(G_2)).$$

An *inclusion* $\mathcal{G} \subseteq \mathcal{G}'$ is a graph homomorphism $H : \mathcal{G} \to \mathcal{G}'$ which is an inclusion both on the sets of points and on the sets of arrows.

A *contravariant graph homomorphism* $H : \mathcal{G} \twoheadrightarrow\!\!\!\!\leftarrow \mathcal{G}'$ is made of two maps, both denoted H, from the points (*resp.* the arrows) of $\mathcal{G}$ towards the points (*resp.* the arrows) of $\mathcal{G}'$, such that if $g : G_1 \to G_2$ then $H(g) : H(G_2) \to H(G_1)$.

A contravariant graph homomorphism $H : \mathcal{G} \twoheadrightarrow\!\!\!\!\leftarrow \mathcal{G}'$ can be identified either to a graph homomorphism $\mathcal{G}^{op} \to \mathcal{G}'$ or to a graph homomorphism $\mathcal{G} \to (\mathcal{G}')^{op}$.

A graph can be illustrated, as usual. For instance, here is an illustration of the graph made of a triangle:



2.2 Categories

Definition 2.2.1
A *category* $\mathcal{A}$ is made of a directed graph $Supp(\mathcal{A})$, called the *support* of $\mathcal{A}$, together with:
– for each point A, a loop $id_A : A \to A$ which is called the *identity at* A,
– for each consecutive pair of arrows (a_1, a_2), a triangle $(a_1, a_2, a_2 \circ a_1)$ where $a_2 \circ a_1$ is called the *composite of* a_1 *and* a_2,
which satisfies the following *unitarity* and *associativity* properties:
– $a \circ id_{A_1} = a$ and $id_{A_2} \circ a = a$ for all arrow $A_1 \xrightarrow{a} A_2$,
– $(a_3 \circ a_2) \circ a_1 = a_3 \circ (a_2 \circ a_1)$ (which is denoted $a_3 \circ a_2 \circ a_1$) for all triple of consecutive arrows $A_1 \xrightarrow{a_1} A_2 \xrightarrow{a_2} A_3 \xrightarrow{a_3} A_4$.

The following definitions hold in any category $\mathcal{A}$.
An *isomorphism* $a : A_1 \xrightarrow{\simeq} A_2$ is an arrow which has an inverse: there is an arrow $a' : A_2 \to A_1$ such that $a' \circ a = id_{A_1}$ and $a \circ a' = id_{A_2}$).
A *monomorphism* is an arrow $a : A_1 \to A_2$ such that, for all A and all $a', a'' : A \to A_1$, if $a \circ a' = a \circ a''$ then $a' = a''$.
A *split monomorphism* is an arrow $a : A_1 \to A_2$ together with a left inverse, *i.e.* with an arrow $a' : A_2 \to A_1$ such that $a' \circ a = id_{A_1}$; then a is a monomorphism.
An *epimorphism* is an arrow $a : A_1 \to A_2$ such that, for all A and all $a', a'' : A_2 \to A$, if $a' \circ a = a'' \circ a$ then $a' = a''$.
A *split epimorphism* is an arrow $a : A_1 \to A_2$ together with a right inverse, *i.e.* with an arrow $a' : A_2 \to A_1$ such that $a \circ a' = id_{A_2}$; then a is an epimorphism.

Definition 2.2.2
Let $\mathcal{A}$ and $\mathcal{A}'$ be two categories. A *functor* $H : \mathcal{A} \to \mathcal{A}'$ is a graph homomorphism $Supp(H) : Supp(\mathcal{A}) \to Supp(\mathcal{A}')$ which preserves identities and composites.

An *inclusion* $\mathcal{A} \subseteq \mathcal{A}'$ is a functor such that its support is an inclusion of graphs; then, $\mathcal{A}$ is a *subcategory* of $\mathcal{A}'$.

A *contravariant functor* $H : \mathcal{A} \twoheadrightarrow\!\!\!\!\leftarrow \mathcal{A}'$ is a contravariant graph homomorphism $Supp(H) : Supp(\mathcal{A}) \twoheadrightarrow\!\!\!\!\leftarrow Supp(\mathcal{A}')$ which preserves identities and composites.

Let $\mathcal{A}$ be a category.
For all point A of $\mathcal{A}$, the functor $\mathrm{Hom}_{\mathcal{A}}(A,-) : \mathcal{A} \to \mathcal{S}et$ maps each point B of $\mathcal{A}$ to the set $\mathrm{Hom}_{\mathcal{A}}(A,B)$ and each arrow $b : B_1 \to B_2$ of $\mathcal{A}$ to the map $\mathrm{Hom}_{\mathcal{A}}(A,b) : \mathrm{Hom}_{\mathcal{A}}(A,B_1) \to \mathrm{Hom}_{\mathcal{A}}(A,B_2)$ such that $\mathrm{Hom}_{\mathcal{A}}(A,b)(c_1) = b \circ c_1$.
For all point B of $\mathcal{A}$, the contravariant functor $\mathrm{Hom}_{\mathcal{A}}(-,B) : \mathcal{A} \twoheadrightarrow \mathcal{S}et$ maps each point A of $\mathcal{A}$ to the set $\mathrm{Hom}_{\mathcal{A}}(A,B)$ and each arrow $a : A_1 \to A_2$ of $\mathcal{A}$ to the map $\mathrm{Hom}_{\mathcal{A}}(a,B) : \mathrm{Hom}_{\mathcal{A}}(A_2,B) \to \mathrm{Hom}_{\mathcal{A}}(A_1,B)$ such that $\mathrm{Hom}_{\mathcal{A}}(a,B)(c_2) = c_2 \circ a$.

$$\begin{array}{ccc} & \overset{c_1}{\nearrow} & B_1 \\ A & & \downarrow b \\ & \underset{b \circ c_1}{\searrow} & B_2 \end{array} \qquad \begin{array}{ccc} A_1 & \overset{c_2 \circ a}{\searrow} & \\ a \downarrow & & B \\ A_2 & \underset{c_2}{\nearrow} & \end{array}$$

Definition 2.2.3
Let $H : \mathcal{A} \to \mathcal{A}'$ be a functor between two categories. For all points A_1 and A_2 in $\mathcal{A}$, there is a map $H_{A_1,A_2} : \mathrm{Hom}_{\mathcal{A}}(A_1,A_2) \to \mathrm{Hom}_{\mathcal{A}'}(H(A_1),H(A_2))$.
– The functor H is *faithful* if for all points A_1 and A_2 in $\mathcal{A}$, the map H_{A_1,A_2} is injective.
– The functor H is *full* if for all points A_1 and A_2 in $\mathcal{A}$, the map H_{A_1,A_2} is surjective.

If a functor $H : \mathcal{A} \to \mathcal{A}'$ is an inclusion and is full, then $\mathcal{A}$ is a *full subcategory* of $\mathcal{A}'$.

Definition 2.2.4
Let $\mathcal{A}$ and $\mathcal{A}'$ be two categories and H_1, $H_2 : \mathcal{A} \to \mathcal{A}'$ two functors. A *natural transformation* $\tau : H_1 \Rightarrow H_2 : \mathcal{A} \to \mathcal{A}'$ is made of an arrow $\tau_A : H_1(A) \to H_2(A)$ of $\mathcal{A}'$ for each point A of $\mathcal{A}$, such that $\tau_{A_2} \circ H_1(a) = H_2(a) \circ \tau_{A_1}$ in $\mathcal{A}'$ for each arrow $a : A_1 \to A_2$ of $\mathcal{A}$.

$$\begin{array}{ccc} H_1(A_1) & \xrightarrow{\tau_{A_1}} & H_2(A_1) \\ {\scriptstyle H_1(a)}\downarrow & \circlearrowleft & \downarrow{\scriptstyle H_2(a)} \\ H_1(A_2) & \xrightarrow[\tau_{A_2}]{} & H_2(A_2) \end{array}$$

This can be expressed as follows: for all A in $\mathcal{A}$ there is an arrow $\tau_A : H_1(A) \to H_2(A)$ of $\mathcal{A}'$ which is *natural in* A.

A *natural isomorphism* is a natural transformation $\tau : H_1 \Rightarrow H_2$ such that the arrow τ_A is an isomorphism in $\mathcal{A}'$ for all point A of $\mathcal{A}$.

For all functor $H : \mathcal{A} \to \mathcal{A}'$, the identity $id_H : H \Rightarrow H$ is the natural transformation such that $(id_H)_A = id_{H(A)}$ for all point A of $\mathcal{A}$.

For all pair of consecutive natural transformations $H_1 \overset{\tau_1}{\Longrightarrow} H_2 \overset{\tau_2}{\Longrightarrow} H_3$, where H_1, H_2, $H_3 : \mathcal{A} \to \mathcal{A}'$, the composite $H_1 \overset{\tau_2 \circ \tau_1}{\Longrightarrow} H_3$ is the natural transformation such that $(\tau_2 \circ \tau_1)_A = (\tau_2)_A \circ (\tau_1)_A$ for all point A of $\mathcal{A}$.

For all functor $H : \mathcal{A} \to \mathcal{A}'$ and all natural transformation $\tau' : H'_1 \Rightarrow H'_2 : \mathcal{A}' \to \mathcal{A}''$, the composite $\tau' \circ H : H'_1 \circ H \Rightarrow H'_2 \circ H : \mathcal{A} \to \mathcal{A}''$ is the natural transformation such that $(\tau' \circ H)_A = \tau'_{H(A)} : H'_1(H(A)) \to H'_2(H(A))$ for all point A of $\mathcal{A}$;

For all natural transformation $\tau : H_1 \Rightarrow H_2 : \mathcal{A} \to \mathcal{A}'$ and all functor $H' : \mathcal{A}' \to \mathcal{A}''$, the composite $H' \circ \tau : H' \circ H_1 \Rightarrow H' \circ H_2 : \mathcal{A} \to \mathcal{A}''$ is the natural transformation such that $(H' \circ \tau)_A = H'(\tau_A) : H'(H_1(A)) \to H'(H_2(A))$ for all point A of $\mathcal{A}$.

Example 2.2.5
Until now, up to some care about size issues, we may define the following categories:
– $\mathcal{S}et$ is the category of sets and maps. In this category, an isomorphism is a *bijection*, a monomorphism is an *injection*, and an epimorphism is a *surjection*. A split monomorphism is an injection together with a chosen *retraction* and a split epimorphism is a surjection together with a chosen *section*.

– $\mathcal{G}r$ is the category of directed graphs and graph homomorphisms.
– $\mathcal{C}at$ is the category of categories and functors.
– For all categories $\mathcal{A}$ to $\mathcal{A}'$, $\mathcal{F}unc(\mathcal{A},\mathcal{A}')$ is the category of functors from $\mathcal{A}$ to $\mathcal{A}'$ and natural transformations. It is easy to check that a natural isomorphism is an isomorphism of the category $\mathcal{F}unc(\mathcal{A},\mathcal{A}')$.

There are several functors between these categories.
– There is a functor $\mathcal{S}et \to \mathcal{G}r$, which maps a set X to the graph with X as its set of points and with no arrow; this functor identifies $\mathcal{S}et$ with a full subcategory of $\mathcal{G}r$.
– The functor $Pt : \mathcal{G}r \to \mathcal{S}et$ maps each graph to its set of points and each graph homomorphism to the underlying map on points.
– The functor $Ar : \mathcal{G}r \to \mathcal{S}et$ maps each graph to its set of arrows and each graph homomorphism to the underlying map on arrows.
– The functor $Supp : \mathcal{C}at \to \mathcal{G}r$ maps each category to its underlying graph and each functor to its underlying graph homomorphism.

2.3 Adjunction

Definition 2.3.1
Let $\mathcal{A}$ and $\mathcal{A}'$ be categories. An *adjunction from* $\mathcal{A}$ *to* $\mathcal{A}'$ is a pair of functors:

$$(\mathcal{A} \xrightarrow{F} \mathcal{A}' \ , \ \mathcal{A} \xleftarrow{U} \mathcal{A}')$$

together with, for all points A of $\mathcal{A}$ and A' of $\mathcal{A}'$, a bijection which is natural in A and A':

$$\mathrm{Hom}_{\mathcal{A}}(A, U(A')) \ \cong \ \mathrm{Hom}_{\mathcal{A}'}(F(A), A').$$

Then, F is a *left adjoint for* U, and U is a *right adjoint for* F. If a functor U has a left adjoint, it is unique up to a natural isomorphism. If a functor F has a right adjoint, it is unique up to a natural isomorphism.

Theorem 2.3.2 (adjunction)
An adjunction (F,U) *from* $\mathcal{A}$ *to* $\mathcal{A}'$ *determines two natural transformations:*

$$\eta : id_{\mathcal{A}} \Rightarrow U \circ F : \mathcal{A} \to \mathcal{A} \quad \text{and} \quad \varepsilon : F \circ U \Rightarrow id_{\mathcal{A}'} : \mathcal{A}' \to \mathcal{A}'$$

such that for all points A *of* $\mathcal{A}$ *and* A' *of* $\mathcal{A}'$, *the bijection* $\mathrm{Hom}_{\mathcal{A}}(A, U(A')) \cong \mathrm{Hom}_{\mathcal{A}'}(F(A), A')$ *maps* $a : A \to U(A')$ *towards:*

$$a^{\star} = \varepsilon_{A'} \circ F(a) : F(A) \to A',$$

and maps $a' : F(A) \to A'$ *towards:*

$$a'_{\star} = U(a') \circ \eta_A : A \to U(A').$$

In addition, both composed natural transformations below are identities:

$$U \overset{\eta \circ U}{\Longrightarrow} U \circ F \circ U \overset{U \circ \varepsilon}{\Longrightarrow} U \quad \text{and} \quad F \overset{F \circ \eta}{\Longrightarrow} F \circ U \circ F \overset{\varepsilon \circ F}{\Longrightarrow} F.$$

This result is proven in [Mac Lane, 1971, p. 80]. The last assertion means that for all points A of $\mathcal{A}$ and A' of $\mathcal{A}'$, $U(\varepsilon_{A'}) \circ \eta_{U(A')} = id_{U(A')}$ and $\varepsilon_{F(A)} \circ F(\eta_A) = id_{F(A)}$.

It follows from this theorem that $\varepsilon_{A'} = (id_{U(A')})^{\star}$ and $\eta_A = (id_{F(A)})_{\star}$.

Definition 2.3.3
Let (F,U) be an adjunction from $\mathcal{A}$ to $\mathcal{A}'$.
– The natural transformation $\eta : id_{\mathcal{A}} \Rightarrow U \circ F : \mathcal{A} \to \mathcal{A}$ is the *unit* of the adjunction.
– The natural transformation $\varepsilon : F \circ U \Rightarrow id_{\mathcal{A}'} : \mathcal{A}' \to \mathcal{A}'$ is the *counit* of the adjunction.

– The functor $M = U \circ F : \mathcal{A} \to \mathcal{A}$ together with the natural transformations $\eta : id_{\mathcal{A}} \Rightarrow M : \mathcal{A} \to \mathcal{A}$ and $\mu = U \circ \varepsilon \circ F : M^2 \Rightarrow M : \mathcal{A} \to \mathcal{A}$ is the *monad* associated to the adjunction. The functor M is the *endofunctor* of the monad, while η is its *unit* and μ is its *multiplication*; it is made of the maps $\mu_A = U(\varepsilon_{F(A)}) : U(F(U(F(A)))) \to U(F(A))$.
– A monad (M, η, μ) is *idempotent* when μ is a natural isomorphism.

A monad (M, η, μ) gives rise to natural transformations $M \circ \eta : M \Rightarrow M^2$ and $\eta \circ M : M \Rightarrow M^2$. Generally, these natural transformations are distinct, however each of them is a right inverse for the multiplication: this is known as the *unitarity* property of the monad M:

$$\mu \circ (M \circ \eta) = id_M \text{ and } \mu \circ (\eta \circ M) = id_M.$$

This means that $\mu_A \circ M(\eta_A) = id_A$ and $\mu_A \circ \eta_{M(A)} = id_A$ for all point A of $\mathcal{A}$.

There is also an *associativity* property of the monad M, which we will not use.

Now, we focus on adjunctions (F, U) where either U or F, or both, is full and faithful. Let (F, U) be an adjunction from $\mathcal{A}$ to $\mathcal{A}'$, with unit $\eta : id_{\mathcal{A}} \Rightarrow U \circ F$ and counit $\varepsilon : F \circ U \Rightarrow id_{\mathcal{A}'}$.

Theorem 2.3.4 (full and faithful functors in adjunctions)
– The functor U is full and faithful if and only if ε is a natural isomorphism.
– The functor F is full and faithful if and only if η is a natural isomorphism.

The first part of this theorem is proven in [Mac Lane, 1971, p. 88], the second part can be proven in a dual way.

Let A be a point of $\mathcal{A}$ and A' a point of $\mathcal{A}'$. Theorem 2.3.4 proves that:
– if U is full and faithful and if it is an inclusion $\mathcal{A}' \subseteq \mathcal{A}$, then $F(A') \cong A'$ as soon as A' is in $\mathcal{A}'$.
– if F is full and faithful and if it is an inclusion $\mathcal{A} \subseteq \mathcal{A}'$, then $U(A) \cong A$ as soon as A is in $\mathcal{A}$.

Corollary 2.3.5 (full and faithful U or F)
If either U or F is full and faithful, then the following natural transformations are natural isomorphisms:
– $\eta \circ U : U \stackrel{\cong}{\Longrightarrow} U \circ F \circ U$, with inverse $U \circ \varepsilon$,
– $\varepsilon \circ F : F \circ U \circ F \stackrel{\cong}{\Longrightarrow} F$, with inverse $F \circ \eta$,
– $\mu : M^2 \stackrel{\cong}{\Longrightarrow} M$, with inverse $\eta \circ M = M \circ \eta$: the monad (M, η, μ) is idempotent.

Definition 2.3.6
An *equivalence* $\mathcal{A} \simeq \mathcal{A}'$ between two categories $\mathcal{A}$ and $\mathcal{A}'$ is a pair of functors $(H : \mathcal{A} \to \mathcal{A}', H' : \mathcal{A}' \to \mathcal{A})$ and a pair of natural isomorphisms $H' \circ H \cong id_{\mathcal{A}}$ and $H \circ H' \cong id_{\mathcal{A}'}$.

This definition can be considered as a weakened notion of isomorphism. Indeed, according to the general definition of an isomorphism in a category, applied to the category $\mathcal{C}at$, an *isomorphism* $\mathcal{A} \cong \mathcal{A}'$ between two categories $\mathcal{A}$ and $\mathcal{A}'$ is a pair of functors $(H : \mathcal{A} \to \mathcal{A}', H' : \mathcal{A}' \to \mathcal{A})$ such that $H' \circ H = id_{\mathcal{A}}$ and $H \circ H' = id_{\mathcal{A}'}$.

So, theorem 2.3.4 states that when both functors U and F in an adjunction (F, U) are full and faithful, they determine an equivalence between the categories $\mathcal{A}$ and $\mathcal{A}'$. It can be proven that in this way we get all the equivalences of categories [Mac Lane, 1971, p.91].

2.4 Yoneda lemma

Let $\mathcal{A}$ be a category. Then $\mathcal{F}unc(\mathcal{A}, \mathcal{S}et)$ is the category of functors from $\mathcal{A}$ to $\mathcal{S}et$, with the natural transformations as arrows. For all points A and B of $\mathcal{A}$, the functors $\mathrm{Hom}_{\mathcal{A}}(A, -) : \mathcal{A} \to \mathcal{S}et$ and $\mathrm{Hom}_{\mathcal{A}}(-, B) : \mathcal{A} \twoheadrightarrow \mathcal{S}et$ are defined in section 2.2.

Definition 2.4.1
The *Yoneda contravariant functor* associated to $\mathcal{A}$:

$$Y_{\mathcal{A}} : \mathcal{A} \twoheadrightarrow \mathcal{F}unc(\mathcal{A}, \mathcal{S}et)$$

is such that:
– $Y_{\mathcal{A}}(A) = \mathrm{Hom}_{\mathcal{A}}(A, -) : \mathcal{A} \to \mathcal{S}et$ for all point A of $\mathcal{A}$,
– $Y_{\mathcal{A}}(a) = \mathrm{Hom}_{\mathcal{A}}(a, -) : Y_{\mathcal{A}}(A_2) \Rightarrow Y_{\mathcal{A}}(A_1) : \mathcal{A} \to \mathcal{S}et$ for all arrow $a : A_1 \to A_2$ of $\mathcal{A}$.

Then, for all point A of $\mathcal{A}$, the set $(Y_{\mathcal{A}}(A))(A) = \mathrm{Hom}_{\mathcal{A}}(A, A)$ contains id_A. So, for all functor $H : \mathcal{A} \to \mathcal{S}et$, there is a map $\mathrm{Hom}_{\mathcal{F}unc(\mathcal{A},\mathcal{S}et)}(Y_{\mathcal{A}}(A), H) \to H(A)$ which maps each natural transformation $\tau : Y_{\mathcal{A}}(A) \Rightarrow H$ to the element $\tau_A(id_A) \in H(A)$.

Theorem 2.4.2 (Yoneda lemma) *The Yoneda contravariant functor* $Y_{\mathcal{A}} : \mathcal{A} \twoheadrightarrow \mathcal{F}unc(\mathcal{A}, \mathcal{S}et)$ *is full and faithful. In addition, for each point* A *of* $\mathcal{A}$ *and each functor* H *from* $\mathcal{A}$ *to* $\mathcal{S}et$*, naturally in* A *and in* H*, the map* $\tau \mapsto \tau_A(id_A)$ *is a bijection:*

$$\mathrm{Hom}_{\mathcal{F}unc(\mathcal{A},\mathcal{S}et)}(Y_{\mathcal{A}}(A), H) \xrightarrow{\simeq} H(A).$$

Let $\mathbb{I}$ denote a one-element set. Then $X = \mathrm{Hom}_{\mathcal{S}et}(\mathbb{I}, X)$ for each set X, so that the bijection in the Yoneda lemma can be stated as a the property of a freely generated structure:

$$\mathrm{Hom}_{\mathcal{S}et}(\mathbb{I}, ev_A(H)) \cong \mathrm{Hom}_{\mathcal{F}unc(\mathcal{A},\mathcal{S}et)}(Y_{\mathcal{A}}(A), H),$$

naturally in H. So, $Y_{\mathcal{A}}(A)$ is free over $\mathbb{I}$, with respect to the functor ev_A [Ehresmann, 1965].

Example 2.4.3
Let us look at some functors from the examples in section 2.1.
– The functor $Pt : \mathcal{G}r \to \mathcal{S}et$ has a left adjoint, which is the inclusion functor $\mathcal{S}et \subseteq \mathcal{G}r$.
– The functor $U_{\mathcal{G}r,\mathcal{C}at} = Supp : \mathcal{C}at \to \mathcal{G}r$ has a left adjoint $F_{\mathcal{G}r,\mathcal{C}at} : \mathcal{G}r \to \mathcal{C}at$, which maps a graph $\mathcal{G}$ to the category $F_{\mathcal{G}r,\mathcal{C}at}(\mathcal{G})$ with the same points as $\mathcal{G}$, and with arrows the *paths* of $\mathcal{G}$, which are obtained by composing any number of consecutive arrows of $\mathcal{G}$ (considering that the identity arrows of $F_{\mathcal{G}r,\mathcal{C}at}(\mathcal{G})$ are composed of no arrow of $\mathcal{G}$).
– The functor $Pt : \mathcal{G}r \to \mathcal{S}et$ is neither full nor faithful; its left adjoint $\mathcal{S}et \subseteq \mathcal{G}r$ is full and faithful.
– The functor $U_{\mathcal{G}r,\mathcal{C}at}$ and its left adjoint $F_{\mathcal{G}r,\mathcal{C}at}$ are both faithful, but none is full.

2.5 Compositive graphs

In order to define limits in a category, in section 2.6, we will use graphs with "some" identities and composites.

Definition 2.5.1
A *compositive graph* $\mathcal{G}$ is made of a directed graph $Supp(\mathcal{G})$, called the *support* of $\mathcal{G}$, together with:
– for some points A, a loop $A \xrightarrow{id_A} A$ which is called the *identity at* A,
– for some consecutive pairs of arrows (a_1, a_2), a triangle $(a_1, a_2, a_2 \circ a_1)$ where $a_2 \circ a_1$ is called the *composite of* a_1 *and* a_2.

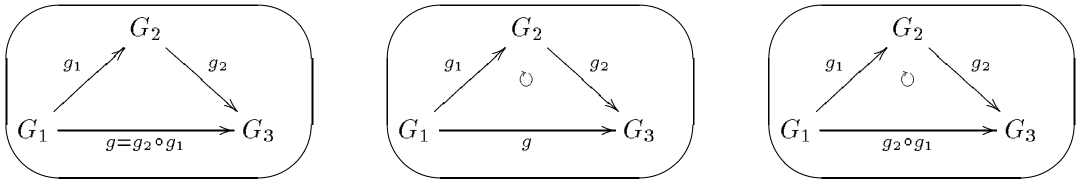
The unitarity and associativity properties do not hold in a compositive graph.

Let $\mathcal{G}$ and $\mathcal{G}'$ be two compositive graphs. A *functor* $H : \mathcal{G} \to \mathcal{G}'$ is a graph homomorphism $Supp(H) : Supp(\mathcal{G}) \to Supp(\mathcal{G}')$ which preserves identities and composites.

An *inclusion* $\mathcal{G} \subseteq \mathcal{G}'$ is a functor such that its support is an inclusion of graphs.

A *contravariant functor* $H : \mathcal{G} \twoheadrightarrow \mathcal{G}'$ is a contravariant graph homomorphism $Supp(H) : Supp(\mathcal{G}) \twoheadrightarrow Supp(\mathcal{G}')$ which preserves identities and composites.

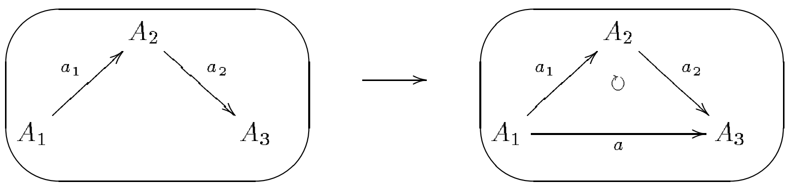
A compositive graph can be illustrated as its support together with the notations id_G for identities, and $g_2 \circ g_1$ or $\circlearrowleft$ for composites. Some identities and composites may be omitted. For instance, here are three illustrations of a compositive graph made of a commutative triangle:



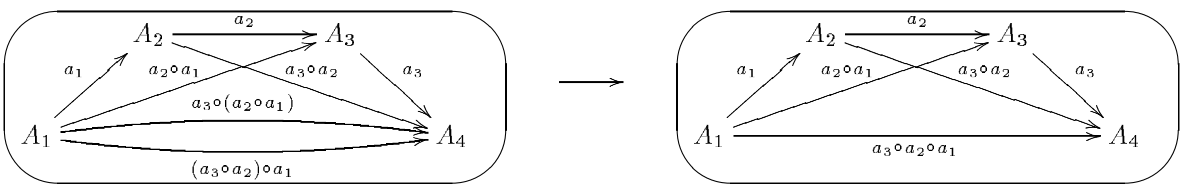
A *category* $\mathcal{A}$ can be identified to a compositive graph where there is an identity at each point, a composite for each consecutive pair of arrows, and which satisfies the unitarity and associativity properties. Then, a *functor* between categories is a functor of compositive graphs.

The definition of *natural transformations* between functors from a compositive graph to a category is an easy generalization of the definition of natural transformations in section 2.2.

This point of view upon categories can be illustrated by several functors of compositive graphs. Such illustrations will get a precise meaning in section 4.
For instance, the property "each consecutive pair of arrows has a composite" can be illustrated by the inclusion functor:



The associativity property can be illustrated by the following functor, which maps both $(a_3 \circ a_2) \circ a_1$ and $a_3 \circ (a_2 \circ a_1)$ to $a_3 \circ a_2 \circ a_1$:



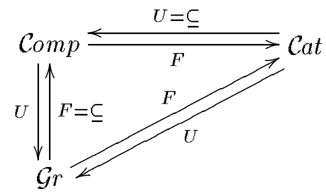
In these illustrations, the first compositive graph represents the hypothesis H of the property, while the second compositive graph represents its conclusion C. The functor represents the deduction rule $\frac{H}{C}$, *i.e.* "if H then C".

Example 2.5.2
We have just defined the category $\mathcal{C}omp$ of compositive graphs and functors between compositive graphs.

The functor $U_{\mathcal{G}r,\mathcal{C}omp} = Supp : \mathcal{C}omp \to \mathcal{G}r$ maps each compositive graph to its underlying graph and each functor to its underlying graph homomorphism. It has a left adjoint, which is the inclusion $F_{\mathcal{G}r,\mathcal{C}omp} : \mathcal{G}r \subseteq \mathcal{C}omp$: it maps a graph $\mathcal{G}$ to the compositive graph with $\mathcal{G}$ as its underlying graph and with no identity and no composite; this functor identifies $\mathcal{G}r$ with a full subcategory of $\mathcal{C}omp$. The functor $U_{\mathcal{G}r,\mathcal{C}omp}$ is faithful, but it is not full; its left adjoint $F_{\mathcal{G}r,\mathcal{C}omp}$ is full and faithful.

There is an inclusion functor $U_{\mathcal{C}omp,\mathcal{C}at} : \mathcal{C}at \subseteq \mathcal{C}omp$, since the category $\mathcal{C}at$ has just been identified to a full subcategory of $\mathcal{C}omp$. It has a left adjoint $F_{\mathcal{C}omp,\mathcal{C}at} : \mathcal{C}omp \to \mathcal{C}at$: for all compositive graph $\mathcal{G}$, in order to get the category $F_{\mathcal{C}omp,\mathcal{C}at}(\mathcal{G})$, we have to add the missing identities and composites, and to perform identifications, so that unitarity and associativity are satisfied. The functor $U_{\mathcal{C}omp,\mathcal{C}at}$ is full and faithful; its left adjoint $F_{\mathcal{C}omp,\mathcal{C}at}$ is neither full nor faithful.

Obviously $U_{\mathcal{G}r,\mathcal{C}at} = U_{\mathcal{G}r,\mathcal{C}omp} \circ U_{\mathcal{C}omp,\mathcal{C}at}$ and $F_{\mathcal{G}r,\mathcal{C}at} = F_{\mathcal{C}omp,\mathcal{C}at} \circ F_{\mathcal{G}r,\mathcal{C}omp}$.

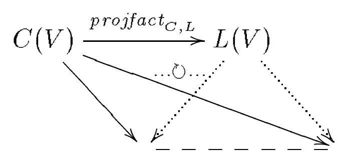


2.6 Limits

Let $\mathcal{I}$ be a compositive graph. The *typical $\mathcal{I}$-projective cone* is the compositive graph $\mathcal{C}_{pr}(\mathcal{I})$ made of $\mathcal{I}$, a point V, an arrow $pr_I : V \to I$ for all point I of $\mathcal{I}$, such that $i \circ pr_I = pr_{I'}$ for all arrow $i : I \to I'$ of $\mathcal{I}$. The inclusion functor is denoted $B_{\mathcal{I}} : \mathcal{I} \subseteq \mathcal{C}_{pr}(\mathcal{I})$.

A *$\mathcal{I}$-projective cone* in a compositive graph $\mathcal{G}$ is a functor $C : \mathcal{C}_{pr}(\mathcal{I}) \to \mathcal{G}$. Then the functor $B = C \circ B_{\mathcal{I}} : \mathcal{I} \to \mathcal{G}$ is the *base* of the projective cone C, the point $C(V)$ is its *vertex*, and the arrows $C(pr_I)$ are its *projections*.

Definition 2.6.1
A $\mathcal{I}$-projective cone L in a category $\mathcal{A}$ is a *limit projective cone* if for all $C : \mathcal{C}_{pr}(\mathcal{I}) \to \mathcal{A}$ there is a unique *projective factorisation* arrow $projfact_{C,L} : C(V) \to L(V)$ in $\mathcal{A}$ such that $L(pr_I) \circ projfact_{C,L} = C(pr_I)$ for all point I of $\mathcal{I}$.



All the limit projective cones in $\mathcal{A}$ with the same base B are isomorphic. When one of them is chosen, it is denoted $projlim(B)$ or $projlim_{I \in \mathcal{I}}(A_I)$ where $A_I = B(I)$.

A category $\mathcal{A}$ is *$\mathcal{I}$-complete* if each base $B : \mathcal{I} \to \mathcal{A}$ has a projective limit in $\mathcal{A}$. A category $\mathcal{A}$ is *with chosen $\mathcal{I}$-projective limits* if each base $B : \mathcal{I} \to \mathcal{A}$ has a chosen projective limit in $\mathcal{A}$.

When $\mathcal{I}$ is empty, then the vertex $L(V) = \mathbb{1}$ of L is a *terminal point* of $\mathcal{A}$.

When $\mathcal{I}$ is discrete (*i.e.* without any arrow), then L is the *product* of B in $\mathcal{A}$, with vertex $L(V) = \prod_{I \in \mathcal{I}} B(I)$, or $L(V) = B(I_1) \times \cdots \times B(I_n)$ when $\mathcal{I} = \{I_1, \ldots, I_n\}$.

When $\mathcal{I}$ is $I_1 \xrightarrow{i_1} I \xleftarrow{i_2} I_2$, then L is the *pullback* of B in $\mathcal{A}$, with vertex $L(V)$, sometimes written $L(V) = B(I_1) \times_{B(I)} B(I_2)$.

Among pullbacks, let $a : A_1 \to A$ be an arrow in $\mathcal{A}$, and let $B(I_1) = B(I_2) = A_1$, $B(I) = A$ and $B(i_1) = B(i_2) = a$. Then, a is a monomorphism if and only if one of the projections $L(V) \to A_1$ is an isomorphism.

When $\mathcal{I}$ is $I_1 \xrightarrow{i_1} I$, $I_1 \xrightarrow{i_2} I$, then L is the *equalizer* of i_1 and i_2 in $\mathcal{A}$.

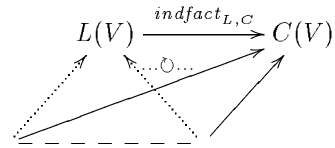
By reversing the direction of the arrows in the cones, we get the *dual* notions.

Let $\mathcal{I}$ be a compositive graph. The *typical $\mathcal{I}$-inductive cone* is the compositive graph $\mathcal{C}_{in}(\mathcal{I})$ made of $\mathcal{I}$, a point V, an arrow $in_I : I \to V$ for all point I of $\mathcal{I}$, such that $in_I = in_{I'} \circ i$ for all arrow $i : I \to I'$ of $\mathcal{I}$. The inclusion functor is denoted $B_{\mathcal{I}} : \mathcal{I} \to \mathcal{C}_{in}(\mathcal{I})$.

A *$\mathcal{I}$-inductive cone* in $\mathcal{G}$ is a functor $C : \mathcal{C}_{in}(\mathcal{I}) \to \mathcal{G}$, with its *base* $B = C \circ B_{\mathcal{I}} : \mathcal{I} \to \mathcal{G}$, its *vertex* $C(V)$ and its *inductions* $C(in_I)$:

Definition 2.6.2
A $\mathcal{I}$-inductive cone L in $\mathcal{A}$ is a *limit inductive cone* if for all $C : \mathcal{C}_{in}(\mathcal{I}) \to \mathcal{A}$ there is a unique *inductive*

factorisation arrow $indfact_{L,C} : L(V) \to C(V)$ such that $indfact_{L,C} \circ L(in_I) = C(in_I)$ for all point I of $\mathcal{I}$.



All the limit inductive cones in $\mathcal{A}$ with the same base B are isomorphic. When one of them is chosen, it is denoted $indlim(B)$ or $indlim_{I \in \mathcal{I}}(A_I)$ where $A_I = B(I)$.

A category $\mathcal{A}$ is *$\mathcal{I}$-cocomplete* if each base $B : \mathcal{I} \to \mathcal{A}$ has an inductive limit in $\mathcal{A}$. A category $\mathcal{A}$ is *with chosen $\mathcal{I}$-inductive limits* if each base $B : \mathcal{I} \to \mathcal{A}$ has a chosen inductive limit in $\mathcal{A}$.

The dual of a terminal point is an *initial point*, the dual of a product is a *sum*, the dual of a pullback is *pushout*, and the dual of an equalizer is a *coequalizer*.

The first following result is obvious, the second one is proven in [Mac Lane, 1971, p. 114].

Proposition 2.6.3 (arrows on limits)
Let $\mathcal{A}$ be a category and A a point of $\mathcal{A}$.
– The functor $\mathrm{Hom}_{\mathcal{A}}(A, -) : \mathcal{A} \to \mathcal{S}et$ *maps projective limits to projective limits.*
– The contravariant functor $\mathrm{Hom}_{\mathcal{A}}(-, A) : \mathcal{A} \twoheadrightarrow \mathcal{S}et$ *maps inductive limits to projective limits.*

Proposition 2.6.4 (adjoints on limits)
Let (F, U) be an adjunction. Then the functor U preserves the projective limits, and the functor F preserves the inductive limits.

Example 2.6.5
The category $\mathcal{S}et$ is $\mathcal{I}$-complete and $\mathcal{I}$-cocomplete for all (sufficiently small) $\mathcal{I}$. The usual way to build projective and inductive limits via cartesian products, disjoint unions and quotients, yields a choice of limits and colimits.

In the category $\mathcal{S}et$, a terminal point is a one-element set, a product is a cartesian product, and a monomorphism is an injection. The initial point is the empty set, a sum is a disjoint union, and an epimorphism is a surjection.

3 Projective sketches, propagators, realizations

Basic notions about projective sketches are presented here. We define *projective sketches* and their homomorphisms, which we call *propagators*, as well as the category of *realizations* of a projective sketch. We state the fundamental theorem about the *freely generated realization*, which associates an adjunction (F_P, U_P) to each propagator P. These notions are rather well known, from Ehresmann's pioneering work in the 1960's [Ehresmann, 1966]. Some of these notions can be found in [Coppey and Lair, 1984] and [Coppey and Lair, 1988], others in [Duval and Lair, 2001]. The fundamental theorem 3.4.1 is known as the *associated sheaf theorem*.

3.1 Projective sketches

Definition 3.1.1
A *projective sketch* $\mathcal{E}$ is made of a compositive graph $Supp(\mathcal{E})$, called the *support* of $\mathcal{E}$, where some projective cones are called *distinguished projective cones* (or *dpcs*).

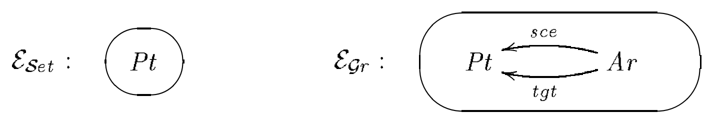
Let $\mathcal{E}$ be a projective sketch.
A *(potential) isomorphism* is an arrow $e_1 : E_1 \to E_2$ with a (potential) inverse, *i.e.* such that there are

an arrow $e_2 : E_2 \to E_1$, two identities id_{E_1} and id_{E_2}, and two composites $e_1 \circ e_2 = id_{E_2}$ and $e_2 \circ e_1 = id_{E_1}$.
A *(potential) monomorphism* is an arrow $e_1 : E_1 \to E_2$ such that there is a distinguished projective cone with base $E_1 \xrightarrow{e_1} E_2 \xleftarrow{e_1} E_1$, and one of the projections from the vertex of this cone to E_1 is a potential isomorphic arrow.
A *(potential) split monomorphism* is an arrow $e_1 : E_1 \to E_2$ with a (potential) left inverse, *i.e.* such that there are an arrow $e_2 : E_2 \to E_1$, an identity id_{E_1}, and a composite $e_2 \circ e_1 = id_{E_1}$.
A *(potential) split epimorphism* is an arrow $e_1 : E_1 \to E_2$ with a (potential) right inverse, *i.e.* such that there are an arrow $e_2 : E_2 \to E_1$, an identity id_{E_2}, and a composite $e_1 \circ e_2 = id_{E_2}$.
A *(potential) factorization arrow* is an arrow $fact_{C,L} : C \to L$ where C and L are projective cones with the same base $B : \mathcal{I} \to Supp(\mathcal{E})$, with L distinguished, together with the composites $L(pr_I) \circ fact_{L,C} = C(pr_I)$ for all point I of $\mathcal{I}$.
A *(potential) terminal point* is a point U together with a distinguished projective cone with empty base and vertex U; this is denoted $U = \mathbb{1}$. Then, for each point E of $\mathcal{E}$, there may be a potential factorization arrow $fact_{E,U} : E \to U$.

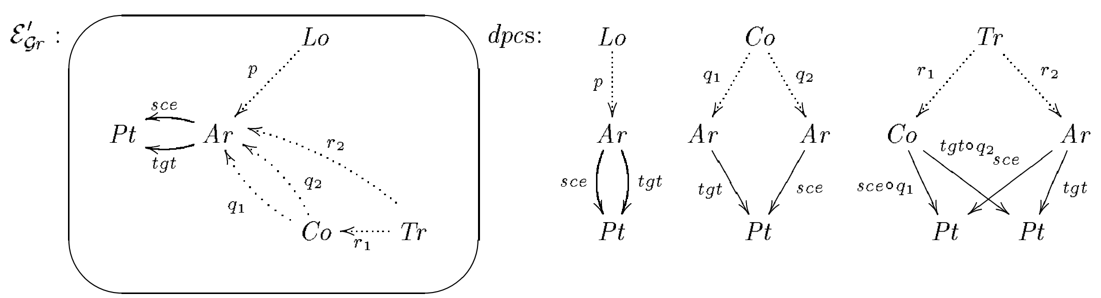
By adding *distinguished inductive cones*, in a dual way, we get the *(mixed) sketches*, which will not play any important role in this paper. In mixed sketches, we could define a (potential) epimorphism and a (potential) initial point.
The generalization of this paper to mixed sketches would be far from trivial. It should use results from [Guitart and Lair, 1980] in order to generalize the freely generated realization theorem 3.4.1.

In this paper, we illustrate a projective sketch as its underlying compositive graph, together with the symbols $\cdots\cdots\!\!>$ for the projection arrows and $\rightarrowtail$ for potential monomorphisms. There is a lot of ambiguity in such an illustration, which has to come with some additional information about the distinguished projective cones. The representation of composite projections may be omitted.

Example 3.1.2
Here are two projective sketches $\mathcal{E}_{\mathcal{S}et}$ and $\mathcal{E}_{\mathcal{G}r}$ without any distinguished projective cone. As will be seen in section 3.3, the names *Pt*, *Ar*, *sce* and *tgt* stand respectively for *points*, *arrows*, *source* and *target*.



Here is a projective sketch $\mathcal{E}'_{\mathcal{G}r}$, with three distinguished projective cones. As will be seen in section 3.3, the names *Lo*, *Co* and *Tr* stand respectively for *loops*, *consecutive arrows* and *triangles*.



3.2 Propagators

Definition 3.2.1
Let $\mathcal{E}$ and $\mathcal{E}'$ be two projective sketches. A *propagator* $P : \mathcal{E} \to \mathcal{E}'$ is a functor $Supp(P) : Supp(\mathcal{E}) \to Supp(\mathcal{E}')$ which preserves the distinguished projective cones.

Obviously, up to size issues, the projective sketches and their propagators form a category $\mathcal{S}ketch$.

An *inclusion* of projective sketches $\mathcal{E} \subseteq \mathcal{E}'$ is a propagator $P : \mathcal{E} \to \mathcal{E}'$ such that $Supp(P)$ is an inclusion of compositive graphs.

Example 3.2.2
$\mathcal{E}_{\mathcal{S}et} \subseteq \mathcal{E}_{\mathcal{G}r} \subseteq \mathcal{E}'_{\mathcal{G}r}$.

3.3 Realizations

Definition 3.3.1
Let $\mathcal{E}$ be a projective sketch and $\mathcal{A}$ a category. A *realization* $S : \mathcal{E} \to \mathcal{A}$ *of* $\mathcal{E}$ *with values in* $\mathcal{A}$ is a functor $Supp(S) : Supp(\mathcal{E}) \to \mathcal{A}$ which maps each distinguished projective cone in $\mathcal{E}$ to a limit projective cone in $\mathcal{A}$.

So, a realization of $\mathcal{E}$ maps a potential isomorphism (*resp.* monomorphism, split monomorphism, split epimorphism) of $\mathcal{E}$ to a (real) isomorphism (*resp.* monomorphism, split monomorphism, split epimorphism) of $\mathcal{A}$.

The category $\mathcal{A}$ can be considered as a projective sketch: its support is the underlying compositive graph, and its *dpc*s are all its projective limit cones (with some care about the size of the indexations of the cones). Then, a realization of $\mathcal{E}$ with values in $\mathcal{A}$ is a propagator from $\mathcal{E}$ to the projective sketch $\mathcal{A}$.

Definition 3.3.2
Let S_1 and S_2 be two realizations of $\mathcal{E}$ with values in $\mathcal{A}$. A *morphism* $\sigma : S_1 \to S_2$ is a natural transformation between the underlying functors.

Obviously, the realizations of $\mathcal{E}$ with values in $\mathcal{A}$ and their morphisms form a category $\mathcal{R}eal(\mathcal{E}, \mathcal{A})$. In addition, for each point E of $\mathcal{E}$, there is a functor $ev_E : \mathcal{R}eal(\mathcal{E}, \mathcal{A}) \to \mathcal{A}$, called the *evaluation at* E, such that $ev_E(S) = S(E)$ for all realization and $ev_E(\sigma) = \sigma(E)$ for all morphism of realizations.

In addition, for all propagator $P : \mathcal{E} \to \mathcal{E}'$ there is a functor $\mathcal{R}eal(P, \mathcal{A}) : \mathcal{R}eal(\mathcal{E}', \mathcal{A}) \to \mathcal{R}eal(\mathcal{E}, \mathcal{A})$, which maps all realization S' of $\mathcal{E}'$ to the realization $S' \circ P$ of $\mathcal{E}$, and all morphism of realizations $\sigma' : S'_1 \to S'_2$ of $\mathcal{E}'$ to the morphism of realizations $\sigma' \circ P : S'_1 \circ P \to S'_2 \circ P$ of $\mathcal{E}$. Altogether, we get a contravariant functor:

$$\mathcal{R}eal(-, \mathcal{A}) : \mathcal{S}ketch \twoheadrightarrow \mathcal{C}at.$$

Proposition 3.3.3
The functor $\mathcal{R}eal(-, \mathcal{A})$ *maps inductive limits to projective limits.*

A *contravariant realization* $Z : \mathcal{E} \twoheadrightarrow \mathcal{A}$ *of* $\mathcal{E}$ *with values in* a category $\mathcal{A}$ is a contravariant functor $Supp(Z) : Supp(\mathcal{E}) \to \mathcal{A}$ which maps each distinguished projective cone in $\mathcal{E}$ to a limit inductive cone in $\mathcal{A}$.

Example 3.3.4
A realization S of $\mathcal{E}_{\mathcal{S}et}$ is a set $S(Pt)$, and a morphism $\sigma : S_1 \to S_2$ is a map $\sigma(Pt) : S_1(Pt) \to S_2(Pt)$. So, there is an isomorphim $\mathcal{R}eal(\mathcal{E}_{\mathcal{S}et}) \cong \mathcal{S}et$.

A realization S of $\mathcal{E}_{\mathcal{G}r}$ is made of two sets $S(Pt)$ and $S(Ar)$, and two maps $S(sce), S(tgt) : S(Ar) \to S(Pt)$: it is a directed graph. And indeed, there is an isomorphim $\mathcal{R}eal(\mathcal{E}_{\mathcal{G}r}) \cong \mathcal{G}r$.

There is an equivalence $\mathcal{R}eal(\mathcal{E}'_{\mathcal{G}r}) \simeq \mathcal{G}r$. Indeed, a realization S of $\mathcal{E}'_{\mathcal{G}r}$ is a directed graph, together with sets $S(Lo)$, $S(Co)$ and $S(Tr)$ which are, because of the distinguished projective cones, isomorphic to, respectively, the set of loops, the set of consecutive arrows, and the set of triangles, of this directed graph.

3.4 Adjunction

The category of *set-valued realizations* of $\mathcal{E}$ is $\mathcal{R}eal(\mathcal{E}) = \mathcal{R}eal(\mathcal{E}, \mathcal{S}et)$. Up to some care about size issues, the category $\mathcal{R}eal(\mathcal{E})$ is both complete and cocomplete.

To each propagator $P : \mathcal{E} \to \mathcal{E}'$ is associated the *underlying functor*:

$$U_P = \mathcal{R}eal(P) : \mathcal{R}eal(\mathcal{E}') \to \mathcal{R}eal(\mathcal{E}).$$

The following result is a fundamental one, it is known as the *associated sheaf theorem*. A proof can be found in [Duval and Lair, 2001]. The generalization of this result to mixed sketches, which is far from trivial, is done in [Guitart and Lair, 1980].

Theorem 3.4.1 (freely generated realization)
Let $P : \mathcal{E} \to \mathcal{E}'$ be a propagator. The functor $U_P : \mathcal{R}eal(\mathcal{E}') \to \mathcal{R}eal(\mathcal{E})$ has a left adjoint:

$$F_P : \mathcal{R}eal(\mathcal{E}) \to \mathcal{R}eal(\mathcal{E}').$$

The functor F_P is the *freely generating functor* associated to P.

From the definition of an adjunction, it follows that, for all realizations S of $\mathcal{E}$ and S' of $\mathcal{E}'$, there is a bijection, which is natural in S and in S':

$$\mathrm{Hom}_{\mathcal{R}eal(\mathcal{E})}(S, U_P(S')) \;\cong\; \mathrm{Hom}_{\mathcal{R}eal(\mathcal{E}')}(F_P(S), S').$$

The corresponding monad and counit are respectively denoted (the subscript P may be omitted):
$(M_P : \mathcal{R}eal(\mathcal{E}) \to \mathcal{R}eal(\mathcal{E})\,,\ \eta_P : id_{\mathcal{R}eal(\mathcal{E})} \Rightarrow M_P\,,\ \mu_P : M_P^2 \Rightarrow M_P)$ and $\varepsilon_P : F_P \circ U_P \Rightarrow id_{\mathcal{R}eal(\mathcal{E}')}$.

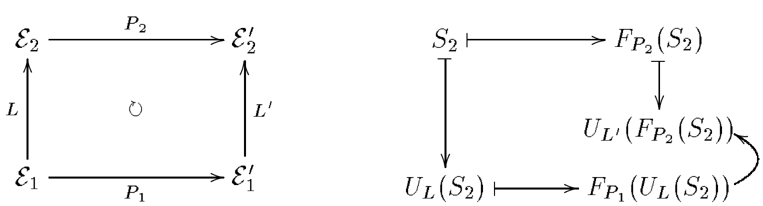
Proposition 3.4.2
Let $P_1 : \mathcal{E}_1 \to \mathcal{E}'_1$, $P_2 : \mathcal{E}_2 \to \mathcal{E}'_2$, $Ty : \mathcal{E}_1 \to \mathcal{E}_2$ and $Ty' : \mathcal{E}'_1 \to \mathcal{E}'_2$ be a commutative square in the category of projective sketches. Then, there is a natural transformation:

$$(F_{P_2} \circ \varepsilon_{Ty})_\star : F_{P_1} \circ U_{Ty} \Rightarrow U_{Ty'} \circ F_{P_2} : \mathcal{R}eal(\mathcal{E}_2) \to \mathcal{R}eal(\mathcal{E}'_1).$$

This natural transformations is not, in general, a natural isomorphism.

Proof. From the counit $\varepsilon_L : F_L \circ U_L \Rightarrow id_{\mathcal{R}eal(\mathcal{E}_2)}$, we get the natural transformation $F_{P_2} \circ \varepsilon_L : F_{P_2} \circ F_L \circ U_L \Rightarrow F_{P_2} : \mathcal{R}eal(\mathcal{E}_2) \to \mathcal{R}eal(\mathcal{E}'_2)$. Since $P_2 \circ L = L' \circ P_1$, the previous result can be written as $F_{P_2} \circ \varepsilon_L : F_{L'} \circ F_{P_1} \circ U_L \Rightarrow F_{P_2}$. So, by adjunction, we get the proposition. □

It follows that, for all realization S_2 of $\mathcal{E}_2$, there is a morphism $F_{P_1}(U_L(S_2)) \to U_{L'}(F_{P_2}(S_2))$ in $\mathcal{R}eal(\mathcal{E}'_1)$.



Example 3.4.3
Let P denote the inclusion $P : \mathcal{E}_{\mathcal{S}et} \subseteq \mathcal{E}_{\mathcal{G}r}$. The underlying functor $U_P : \mathcal{R}eal(\mathcal{E}_{\mathcal{G}r}) \to \mathcal{R}eal(\mathcal{E}_{\mathcal{S}et})$ is the functor $Pt : \mathcal{G}r \to \mathcal{S}et$, from section 2.2, which forgets the arrows. The freely generating functor $F_P : \mathcal{R}eal(\mathcal{E}_{\mathcal{S}et}) \to \mathcal{R}eal(\mathcal{E}_{\mathcal{G}r})$ is the inclusion functor $\mathcal{S}et \subseteq \mathcal{G}r$ from section 2.2.

3.5 Equivalence of sketches.

The following definition of conservative propagators is semantic: it is relative to the set-valued realizations of the sketches involved.

Definition 3.5.1
A propagator $Q : \mathcal{E} \to \mathcal{E}'$ is *conservative* if both functors F_Q and U_Q are full and faithful.

From theorem 2.3.4, Q is conservative if and only if the unit η_Q and the counit ε_Q are natural isomorphisms.

Definition 3.5.2
The *equivalence* of projective sketches is the equivalence relation generated by:
– $\mathcal{E} \equiv \mathcal{E}'$ as soon as there is a conservative propagator from $\mathcal{E}$ to $\mathcal{E}'$.

A *zig-zag* of propagators $(P_1, \ldots, P_n)$ from $\mathcal{E}$ to $\mathcal{E}'$ is made of projective sketches $\mathcal{E}_0, \mathcal{E}_1, \ldots, \mathcal{E}_n$ such that $\mathcal{E}_0 = \mathcal{E}$ and $\mathcal{E}_n = \mathcal{E}'$, and of propagators $P_1, \ldots, P_n$ with, for each k from 1 to n, either $P_k : \mathcal{E}_{k-1} \to \mathcal{E}_k$ or $P_k : \mathcal{E}_k \to \mathcal{E}_{k-1}$. Then, clearly, two projective sketches $\mathcal{E}$ and $\mathcal{E}'$ are equivalent if there is a zig-zag of conservative propagators from $\mathcal{E}$ to $\mathcal{E}'$.

From theorem 2.3.4, if two projective sketches $\mathcal{E}$ and $\mathcal{E}'$ are equivalent then the categories $\mathcal{R}eal(\mathcal{E})$ and $\mathcal{R}eal(\mathcal{E}')$ are equivalent: if $\mathcal{E} \equiv \mathcal{E}'$ then $\mathcal{R}eal(\mathcal{E}) \simeq \mathcal{R}eal(\mathcal{E}')$.

In the following result are listed some families of conservative propagators, which can be composed or used in zig-zag in order to get equivalences of projective sketches. There are many other ways to get conservative propagators and equivalences of projective sketches.

Proposition 3.5.3 (construction of conservative propagators)
Let $Q : \mathcal{E} \to \mathcal{E}'$ be a propagator such that, either:
– Q adds an identity loop at a point of $\mathcal{E}$,
– Q adds a composite for a pair of consecutive arrows of $\mathcal{E}$,
– Q adds a distinguished projective cone for a base in $\mathcal{E}$,
– Q adds a potential factorization arrow, or identifies two potential factorisation arrows, between a projective cone and a distinguished projective cone with the same base, both in $\mathcal{E}$,
– Q states that an invertible arrow or an identity arrow is a monomorphic arrow.
– Q adds a new point E', the identities id_E (if it is not yet in $\mathcal{E}$) and $id_{E'}$, two arrows $e'_1 : E \to E'$ and $e'_2 : E' \to E$ with the composites $e'_2 \circ e'_1 = id_E$ and $e'_1 \circ e'_2 = id_{E'}$.
– Q maps an invertible arrow $e : E_1 \to E_2$, with $E_1 \neq E_2$, towards an identity arrow.
Then Q is a conservative propagator.

Proof. This result is easily derived from the properties of the complete category $\mathcal{S}et$: for instance the image of a point of $\mathcal{E}$ is a point in $\mathcal{S}et$, so that it has one identity arrow, and so on. □

On the contrary, a propagator which maps an invertible arrow $e : E \to E$ towards an identity arrow is not conservative, in general. Indeed, let $\mathcal{E}$ be made of one point E, the identity id_E, and two arrows $e_1, e_2 : E \to E$ with the composites $e_2 \circ e_1 = id_E$ and $e_1 \circ e_2 = id_E$. Let $\mathcal{E}'$ be made of one point E' and the identity $id_{E'}$, and let $P : \mathcal{E} \to \mathcal{E}'$ be the unique propagator from $\mathcal{E}$ to $\mathcal{E}'$. Now, let S be a realization of $\mathcal{E}$ such that $S(E)$ has two elements x and y, and $S(e_1) = S(e_2)$ permutes x and y. Then $F_P(S)$ identifies x and y, so that $M_P(S)(E)$ is made of only one element, and $\eta_{P,S}$ cannot be an isomorphism.

Definition 3.5.4
The *equivalence* of propagators is the equivalence relation $P \equiv P'$ (where $P : \mathcal{E}_1 \to \mathcal{E}_2$ and $P' : \mathcal{E}'_1 \to \mathcal{E}'_2$) generated by:
– $P \equiv P'$ as soon as $\mathcal{E}_2 = \mathcal{E}'_2$ and there is a conservative propagator $Q_1 : \mathcal{E}_1 \to \mathcal{E}'_1$ such that $P' \circ Q_1 = P$,
– $P \equiv P'$ as soon as $\mathcal{E}_1 = \mathcal{E}'_1$ and there is a conservative propagator $Q_2 : \mathcal{E}_2 \to \mathcal{E}'_2$ such that $Q_2 \circ P = P'$.

If $P \equiv P'$, then clearly $\mathcal{E}_1 \equiv \mathcal{E}'_1$ and $\mathcal{E}_2 \equiv \mathcal{E}'_2$.

Example 3.5.5
The inclusion of $\mathcal{E}_{\mathcal{G}r}$ in $\mathcal{E}'_{\mathcal{G}r}$ (from section 3.1) is a conservative propagator: indeed, it may easily be decomposed in several steps, which are either the addition of a distinguished projective cone for a given base, or the addition of a composite for a pair of consecutive arrows. In this way, from the isomorphism $\mathcal{R}eal(\mathcal{E}_{\mathcal{G}r}) \cong \mathcal{G}r$, we get another proof of the equivalence $\mathcal{R}eal(\mathcal{E}'_{\mathcal{G}r}) \simeq \mathcal{G}r$.

3.6 Prototypes and types

Definition 3.6.1
A *projective prototype* is a projective sketch such that its support is a category and its distinguished projective cones are limit cones.

It can be proven that each projective sketch $\mathcal{E}$ freely generates a projective prototype $Pr(\mathcal{E})$. The unit propagator $\mathcal{E} \to Pr(\mathcal{E})$ maps each distinguished projective cone of $\mathcal{E}$ to a distinguished limit projective cone of $Pr(\mathcal{E})$. It follows that $\mathcal{R}eal(Pr(\mathcal{E})) \cong \mathcal{R}eal(\mathcal{E})$.

Definition 3.6.2
With respect to some family of compositive graphs for indexations, a *projective type* is a category with with chosen projective limit cones (as defined in section 2.6).

A projective type can be considered as a projective prototype, by distinguishing all its chosen projective cones.

It can be proven that each projective sketch $\mathcal{E}$ freely generates a projective type $Ty(\mathcal{E})$. The unit propagator $\mathcal{E} \to Ty(\mathcal{E})$ maps each distinguished projective cone of $\mathcal{E}$ to a chosen (hence distinguished) limit projective cone of $Ty(\mathcal{E})$. It follows that $\mathcal{R}eal(Ty(\mathcal{E})) \cong \mathcal{R}eal(\mathcal{E})$.

Usually, the same notation is used for the points and arrows of $\mathcal{E}$ and their images in $Pr(\mathcal{E})$ and in $Ty(\mathcal{E})$, although the unit propagators $\mathcal{E} \to Pr(\mathcal{E})$ and $\mathcal{E} \to Ty(\mathcal{E})$ need not be inclusions.

3.7 Yoneda lemma for projective sketches

Let $\mathcal{E}$ be a projective sketch, then from section 2.4, there is a Yoneda contravariant functor:

$$Y_{Pr(\mathcal{E})} : Pr(\mathcal{E}) \twoheadrightarrow \mathcal{F}unc(Pr(\mathcal{E}), \mathcal{S}et).$$

From proposition 2.6.3, the functor $\mathrm{Hom}_{\mathrm{Pr}(\mathcal{E})}(E, -) : \mathrm{Pr}(\mathcal{E}) \to \mathcal{S}et$ maps projective limits to projective limits. So, the functor $Y_{Pr(\mathcal{E})}(E) : Pr(\mathcal{E}) \to \mathcal{S}et$ preserves the projective limit cones, which means that the image of $Y_{Pr(\mathcal{E})}$ is contained in $\mathcal{R}eal(Pr(\mathcal{E}))$:

$$Y_{Pr(\mathcal{E})} : Pr(\mathcal{E}) \twoheadrightarrow \mathcal{R}eal(Pr(\mathcal{E})).$$

In addition, since $\mathcal{R}eal(Pr(\mathcal{E}))$ is isomorphic to $\mathcal{R}eal(\mathcal{E})$, by composition of $Y_{Pr(\mathcal{E})}$ with the unit propagator $\mathcal{E} \to Pr(\mathcal{E})$, we get a contravariant functor:

$$Y_{\mathcal{E}} : \mathcal{E} \twoheadrightarrow \mathcal{R}eal(\mathcal{E}).$$

From proposition 2.6.3, the contravariant functor $\mathrm{Hom}_{\mathcal{R}eal(\mathcal{E})}(-, S) : \mathcal{R}eal(\mathcal{E}) \twoheadrightarrow \mathcal{S}et$ maps inductive limits to projective limits. So, the functor $Y_{\mathcal{E}}$ maps distinguished projective cones to limit inductive cones, which means that it is a contravariant realization of $\mathcal{E}$.

Theorem 3.7.1 (Yoneda lemma for projective sketches)
The Yoneda contravariant realization $Y_{\mathcal{E}} : \mathcal{E} \twoheadrightarrow \mathcal{R}eal(\mathcal{E})$ is such that, for each point E of $\mathcal{E}$ and each realization S of $\mathcal{E}$, naturally in E and in S, the map $\sigma \mapsto \sigma_E(id_E)$ is a bijection:

$$\mathrm{Hom}_{\mathcal{R}eal(\mathcal{E})}(Y_{\mathcal{E}}(E), S) \xrightarrow{\simeq} S(E).$$

A consequence of theorem 3.7.1 is the *density* result of corollary 3.7.3 below: any set-valued realization of $\mathcal{E}$ is the vertex of an inductive limit cone which has its base in $Y_{\mathcal{E}}(\mathcal{E})$. The description of this cone makes use of a *blow-up* of $Supp(\mathcal{E})$.

Definition 3.7.2
Let $\mathcal{G}$ be a directed graph and $H : \mathcal{G} \to \mathcal{S}et$ a functor. The *blow-up* $\mathcal{G}\backslash H$ of $\mathcal{G}$ by H is the directed graph with:
– a point $[G, x]$ for all point G of $\mathcal{G}$ and all $x \in H(G)$,
– an arrow $[g, x] : [G, x] \to [G', x']$ for all arrow $g : G \to G'$ of $\mathcal{G}$ and all $x \in H(G)$, where $x' = H(g)(x)$,
– an identity $id_{[G,x]} = [id_G, x]$ for all identity id_G of $\mathcal{G}$ and all $x \in H(G)$,
– a composite $[g_2 \circ g_1, x_1] = [g_2, x_2] \circ [g_1, x_1]$ for all composite $g_2 \circ g_1$ of $\mathcal{G}$ and all $x_1 \in sce(g_1)$, where $x_2 = H(g_1)(x_1)$.

Let us write Y for $Y_{\mathcal{E}}$. Let S be a set-valued realization of $\mathcal{E}$, and $\mathcal{I} = (Supp(\mathcal{E})\backslash Supp(S))^{op}$. Let C_S denote the $\mathcal{I}$-inductive cone in $\mathcal{R}eal(\mathcal{E})$ with:
– vertex S,
– base $B : \mathcal{I} \to \mathcal{R}eal(\mathcal{E})$ such that $B([E, x]) = Y(E)$ for all point $[E, x]$ of $\mathcal{I}$ and $B([e, x]) = Y(e)$ for all arrow $[e, x]$ of $\mathcal{I}$.
– inductions $in_{[E,x]} : Y(E) \to S$ such that for each point E' in $\mathcal{E}$ the map $in_{[E,x]}(E') : \mathrm{Hom}_{\mathrm{Pr}(\mathcal{E})}(E, E') \to S(E')$ maps e towards $S(e)(x)$.
It is easy to check that this is indeed an inductive cone. The density of Yoneda realization states that it is an inductive limit cone.

Corollary 3.7.3 (density of Yoneda realization)
Let S be a realization of $\mathcal{E}$. Then the inductive cone C_S in $\mathcal{R}eal(\mathcal{E})$ is a limit cone, which is written: $S \cong indlim_{\mathcal{E}\backslash S}(Y_{\mathcal{E}}(E))$.

Proposition 3.7.4
Let $P : \mathcal{E} \to \mathcal{E}'$ be a propagator. Then there is an isomorphism of contravariant models of $\mathcal{E}$ with values in $\mathcal{R}eal(\mathcal{E}')$: $F_P \circ Y_{\mathcal{E}} \cong Y_{\mathcal{E}'} \circ P$.

Proof. Let E be a point of $\mathcal{E}$, and S' a realization of $\mathcal{E}'$. Then, from Yoneda lemma applied to $\mathcal{E}$, $\mathrm{Hom}_{\mathcal{R}eal(\mathcal{E})}(Y_{\mathcal{E}}(E), U_P(S')) \cong U_P(S')(E) = S'(P(E))$. On the other hand, from Yoneda lemma applied to $\mathcal{E}'$, $\mathrm{Hom}_{\mathcal{R}eal(\mathcal{E}')}(Y_{\mathcal{E}'}(P(E)), S') \cong S'(P(E))$. So that:
$\mathrm{Hom}_{\mathcal{R}eal(\mathcal{E})}(Y_{\mathcal{E}}(E), U_P(S')) \cong \mathrm{Hom}_{\mathcal{R}eal(\mathcal{E}')}(Y_{\mathcal{E}'} \circ P(E), S')$, naturally in E and S',
which means that $Y_{\mathcal{E}'} \circ P$ is isomorphic to $F_P \circ Y_{\mathcal{E}}$. □

4 Fractioning and filling propagators

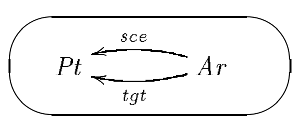
In this section, we focus on two families of propagators. A *fractioning propagator* K is such that U_K is full and faithful, while a *filling propagator* J is such that F_J is full and faithful. We prove that any propagator P can be decomposed as $P = K \circ J$ with K fractioning and J filling. The words “fractioning” and “filling” stem from theorems 4.2.2 and 4.3.2, respectively.

4.1 A basic example

Directed graphs.

As in section 3.1, let $\mathcal{E}_{\mathcal{G}r}$ be the following projective sketch (without any distinguished projective cone), such that $\mathcal{R}eal(\mathcal{E}_{\mathcal{G}r}) \cong \mathcal{G}r$:

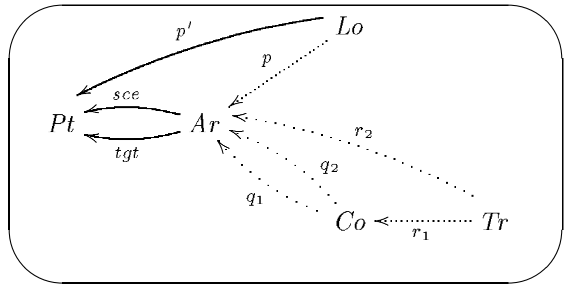
$\mathcal{E}_{\mathcal{G}r}$:



Let $\mathcal{E}'_{\mathcal{G}r}$ denote the projective sketch described in section 3.1, together with the composite $p' = sce \circ p = tgt \circ p : Lo \to Pt$. The precise description of its distinguished projective cones is given in section 3.1.

The inclusion of $\mathcal{E}_{\mathcal{G}r}$ in $\mathcal{E}'_{\mathcal{G}r}$ is conservative, so that $\mathcal{E}_{\mathcal{G}r} \equiv \mathcal{E}'_{\mathcal{G}r}$ and $\mathcal{R}eal(\mathcal{E}'_{\mathcal{G}r}) \simeq \mathcal{G}r$.

$\mathcal{E}'_{\mathcal{G}r}$:



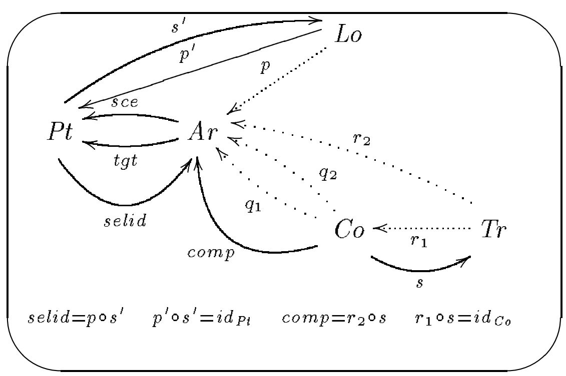
Categories.

Let us add to $\mathcal{E}'_{\mathcal{G}r}$:
– two identities id_{Co} and id_{Pt} (not represented in the picture),
– two arrows $s : Co \to Tr$ and $s' : Pt \to Lo$ such that $r_1 \circ s = id_{Co}$ and $p' \circ s' = id_{Pt}$,
– and whatever is needed to express the unitarity and associativity of categories.
It is easy to check that the resulting projective sketch $\mathcal{E}_{\mathcal{C}at}$ is such that $\mathcal{R}eal(\mathcal{E}_{\mathcal{C}at}) \simeq \mathcal{C}at$.

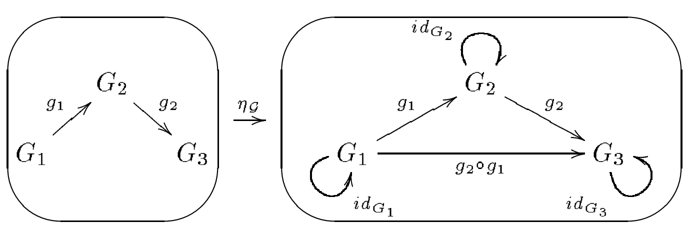
Let us add to $\mathcal{E}_{\mathcal{C}at}$:
– two *dpc*s such that the identity arrows id_{Co} and id_{Pt} are potential monomorphisms,
– two composite arrows $comp = r_2 \circ s : Co \to Ar$, for the composition, and $selid = p \circ s' : Pt \to Ar$, for the selection of identities.
Then the inclusion of $\mathcal{E}_{\mathcal{C}at}$ in $\mathcal{E}'_{\mathcal{C}at}$ is conservative, so that $\mathcal{E}_{\mathcal{C}at} \equiv \mathcal{E}'_{\mathcal{C}at}$ and $\mathcal{R}eal(\mathcal{E}'_{\mathcal{C}at}) \simeq \mathcal{C}at$.

The following illustration does not represent the identities, nor the unitarity and associativity properties.

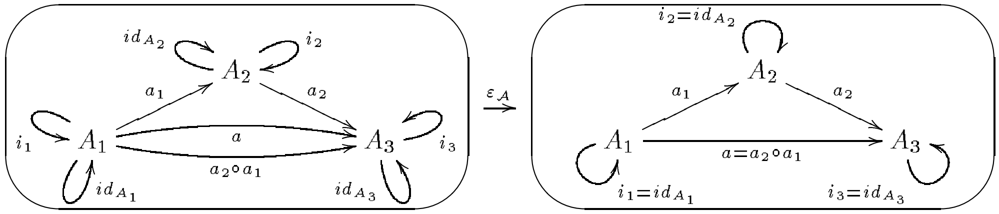
$\mathcal{E}'_{\mathcal{C}at}$:



Let $P : \mathcal{E}_{\mathcal{G}r} \to \mathcal{E}_{\mathcal{C}at}$ be the inclusion. Let $\mathcal{G}$ be a graph, then the graph $U_P(F_P(\mathcal{G}))$ is not isomorphic to $\mathcal{G}$. Indeed, the freely generating functor F_P adds the required identities and composites, which are not removed by the underlying functor U_P. So, the unit $\eta_{\mathcal{G}} : \mathcal{G} \to U_P(F_P(\mathcal{G}))$ is far from an isomorphism. For instance:



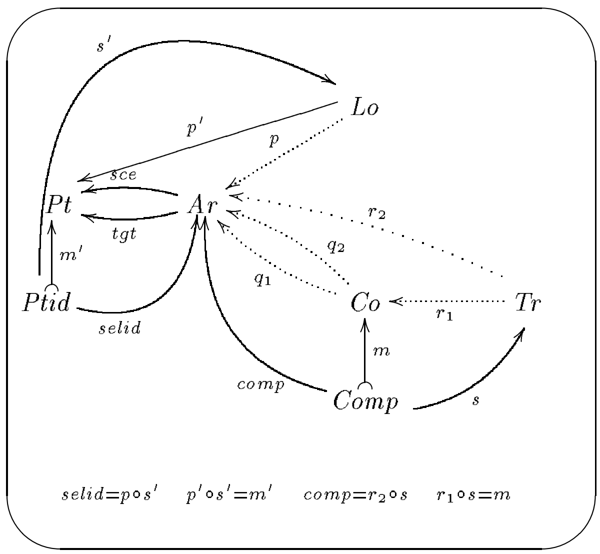
Let $\mathcal{A}$ be a category, then the category $F_P(U_P(\mathcal{A}))$ is not isomorphic to $\mathcal{A}$. Indeed, the underlying functor U_P forgets that some arrows are identities or composites. Then, the freely generating functor F_P adds to the graph $U_P(\mathcal{A})$ a new copy of these identities or composites. So, the counit $\varepsilon_{\mathcal{A}} : F(U(\mathcal{A})) \to \mathcal{A}$ is far from an isomorphism. For instance:



Compositive graphs.

Let us add to $\mathcal{E}'_{\mathcal{G}r}$:
– two points, *Comp* for the consecutive arrows with a composite, and *Ptid* for the points with an identity, together with arrows $m' : Ptid \to Pt$ and $m : Comp \to Co$ which are potential monomorphisms,
– two arrows $s : Comp \to Tr$ and $s' : Ptid \to Lo$ such that $r_1 \circ s = m$ and $p' \circ s' = m'$,
– two composite arrows $comp = r_2 \circ s : Comp \to Ar$, for the composition, and $selid = p \circ s' : Ptid \to Ar$, for the selection of identities.
It is easy to check that the resulting projective sketch $\mathcal{E}_{\mathcal{C}omp}$ is such that $\mathcal{R}eal(\mathcal{E}_{\mathcal{C}omp}) \simeq \mathcal{C}omp$.

$\mathcal{E}_{\mathcal{C}omp}$:



Decomposition of P.

The propagator $P : \mathcal{E}_{\mathcal{G}r} \to \mathcal{E}_{\mathcal{C}at}$ is equivalent to $P' : \mathcal{E}_{\mathcal{G}r} \to \mathcal{E}'_{\mathcal{C}at}$, which can be decomposed as $P' = K' \circ J$ where $J : \mathcal{E}_{\mathcal{G}r} \to \mathcal{E}_{\mathcal{C}omp}$ is the inclusion and $K' : \mathcal{E}_{\mathcal{C}omp} \to \mathcal{E}'_{\mathcal{C}at}$ is such that m and m' are mapped to id_{Co} and id_{Pt}, respectively.

Let $\mathcal{G}$ be a graph, then the graph $U_J(F_J(\mathcal{G}))$ is isomorphic to $\mathcal{G}$, because the compositive graph $F_J(\mathcal{G})$ has neither identities nor composites.

Let $\mathcal{A}$ be a category, then clearly the category $F_{K'}(U_{K'}(\mathcal{A}))$ is isomorphic to $\mathcal{A}$.

Now, let us come back to the propagator $P : \mathcal{E}_{\mathcal{G}r} \to \mathcal{E}_{\mathcal{C}at}$ and to the construction of the category $F_P(\mathcal{G})$ which is freely generated by some given graph $\mathcal{G}$. Up to equivalence, we can rather consider the propagator $P' : \mathcal{E}_{\mathcal{G}r} \to \mathcal{E}'_{\mathcal{C}at}$ and build the category $F_{P'}(\mathcal{G})$. The intermediate sketch $\mathcal{E}_{\mathcal{C}omp}$ can be used in order to get a progressive construction of $F_{P'}(\mathcal{G})$. First, $F_{P'}(\mathcal{G}) = F_{K'}(F_J(\mathcal{G}))$, where $F_J(\mathcal{G})$ is easily obtained: it is $\mathcal{G}$ together with no identity and no composite. So, we can assume that $\mathcal{G}$ is a compositive graph, and look for a progressive construction of $F_{K'}(\mathcal{G})$. If G is a point in $\mathcal{G}$ without an identity, we can build a compositive graph by adding $id_G : G \to G$. If $g_1 : G_1 \to G_2$ and $g_2 : G_2 \to G_3$ are successive arrows in $\mathcal{G}$ without a composite, we can build a compositive graph by adding $g_2 \circ g_1 : G_1 \to G_3$. In both cases, the resulting compositive graph $\mathcal{G}'$ is such that $F_{K'}(\mathcal{G}) = F_{K'}(\mathcal{G}')$, so that the construction may start again

from $\mathcal{G}'$.
So, the composites and identities can be built little by little, from a directed graph (where they are nowhere defined) to a category (where they are everywhere defined), thanks to intermediate compositive graphs (where they are partially defined).
In the following, we prove that this property of $P : \mathcal{E}_{\mathcal{G}r} \to \mathcal{E}_{\mathcal{C}at}$ can be generalized to any propagator.

4.2 Fractioning propagators

Definition 4.2.1
A propagator $K : \mathcal{E} \to \overline{\mathcal{E}}$ is *fractioning* if the underlying functor U_K is full and faithful.

From theorem 2.3.4, K is fractioning if and only if the counit ε_K is a natural isomorphism:

$$\varepsilon_K : F_K \circ U_K \stackrel{\simeq}{\Longrightarrow} id_{\mathcal{R}eal(\overline{\mathcal{E}})}.$$

Then, the multiplication μ_K is a natural isomorphism, *i.e.* the monad associated to K is idempotent:

$$\mu_K : M_K^2 \stackrel{\simeq}{\Longrightarrow} M_K.$$

Obviously, a conservative propagator is fractioning, the composite of fractioning propagators is fractioning, and a propagator which is equivalent to a fractioning one is also fractioning.

On the other hand, we say that a propagator $K : \mathcal{E} \to \overline{\mathcal{E}}$ adds an inverse to an arrow $e : E_1 \to E_2$ of $\mathcal{E}$ if it adds an arrow $e^{-1} : E_2 \to E_1$, two identities id_{E_1} and id_{E_2} if they are needed, and two composites $e^{-1} \circ e = id_{E_1}$ and $e \circ e^{-1} = id_{E_2}$.

Theorem 4.2.2 (fractioning propagators)
A propagator is fractioning if and only if, up to equivalence, it adds inverses to arrows.

Proof (partial). We only prove here the easy part of this result. Similar results can be found in [Gabriel and Zisman, 1967] and in [Hebert, Adamek and Rosicky, 2001].

Let us assume that K adds an inverse to an arrow $e : E_1 \to E_2$ of $\mathcal{E}$. Let D be a realization of $\overline{\mathcal{E}}$, so that the map $D(e^{-1})$ is the inverse of $D(e)$. In $U(D)$, the map $U(D)(e)$ is equal to $D(e)$, so that it is invertible. Then, $F(U(D))$ only gives a name to the inverse of $U(D)(e)$, so that $\varepsilon(D) : F \circ U(D) \to D$ is an isomorphism. It follows that K is fractioning, so that any propagator which adds inverses to arrows is fractioning. □

Theorem 4.2.3
A propagator is fractioning if and only if, up to equivalence, it consists in the distinction of projective cones.

Proof. By theorem 4.2.2, we have to prove that, up to equivalence, a propagator K adds inverses to arrows if and only if it distinguishes projective cones.

Let $e : E_1 \to E_2$ be an arrow in $\mathcal{E}$, and let us distinguish the projective cone with vertex E_1, base E_2 and projection e. Then, up to equivalence, we can add the factorization arrow $f = fact(id_{E_2}, e) : E_2 \to E_1$. The property of factorization arrows states that $e \circ f = id_{E_2}$. It follows that $e \circ (f \circ e) = (e \circ f) \circ e = e$, which means that $f \circ e = fact(e, e)$, but clearly $fact(e, e) = id_{E_1}$, so that the unicity of factorization arrows proves that $f \circ e = id_{E_1}$. So, f is an inverse of e.

Let C be a projective cone in $\mathcal{E}$ with base B and vertex E_1. Then, up to equivalence, we can add a distinguished projective cone C' with the same base B and some vertex E_2, and the factorization arrow $e = fact(C, C') : E_1 \to E_2$. Let us add an inverse e^{-1} to e. Then, up to equivalence, we can distinguish the cone C. □

Proposition 4.2.4
A propagator which maps an arrow to an identity is fractioning.

Proof. Let us assume that $K : \mathcal{E} \to \overline{\mathcal{E}}$ maps an arrow $e : E_1 \to E_2$ of $\mathcal{E}$ to an identity $id_{E'} : E' \to E'$ of $\overline{\mathcal{E}}$. Let D be a realization of $\overline{\mathcal{E}}$, so that the map $D(K(e))$ is the identity of $D(E')$. In $U(D)$, the sets $U(D)(E_1)$ and $U(D)(E_2)$ are both equal to $D(E')$, and the map $U(D)(e)$ is the identity. So, $\varepsilon(D) : F \circ U(D) \to D$ is an isomorphism. It follows that K is fractioning. □

Let $e : E_1 \to E_2$ be an arrow in a projective sketch $\mathcal{E}$. A propagator $P : \mathcal{E} \to \mathcal{E}'$ adds a *restriction* to e with respect to m_1 and m_2, where $m_1 : E'_1 \to E_1$ and $m_2 : E'_2 \rightarrowtail E_2$ are arrows of $\mathcal{E}$ and m_2 is a potential monomorphism, if it adds an arrow $e' : E'_1 \to E'_2$ with a commutative square $e \circ m_1 = m_2 \circ e'$.

Proposition 4.2.5
A propagator which adds a restriction to an arrow is fractioning.

Proof. Let us assume that $K : \mathcal{E} \to \overline{\mathcal{E}}$ adds a restriction $e' : E'_1 \to E'_2$ to an arrow $e : E_1 \to E_2$ with respect to m_1 and m_2. Let D be a realization of $\overline{\mathcal{E}}$, so that the map $D(K(e')) : D(K(E'_1) \to D(K(E'_2))$ is the restriction of $D(K(e))$. In $U(D)$, it remains true that $U(D)(e) \circ U(D)(m_1) = U(D)(m_2) \circ f$ for some map f. Since the map $U(D)(m_2)$ is injective, the map f is characterized by this equality. So, $F(U(D))$ only gives the name $F(U(D))(e')$ to the map f, hence $\varepsilon(D) : F \circ U(D) \to D$ is an isomorphism. It follows that K is fractioning. □

Definition 4.2.6
Let $K : \mathcal{E} \to \overline{\mathcal{E}}$ and $K' : \mathcal{E}' \to \overline{\mathcal{E}}'$ be two fractioning propagators. A *morphism* from K to K' is a pair $(L, \overline{L})$ of propagators $L : \mathcal{E} \to \mathcal{E}'$, $\overline{L} : \overline{\mathcal{E}} \to \overline{\mathcal{E}}'$, such that $\overline{L} \circ K = K' \circ L$.

Example 4.2.7
In section 4.1, the propagator $P : \mathcal{E}_{\mathcal{G}r} \to \mathcal{E}_{\mathcal{C}at}$ is not fractioning, whereas the propagator $K : \mathcal{E}_{\mathcal{C}omp} \to \mathcal{E}_{\mathcal{C}at}$ is fractioning.

4.3 Filling propagators

Definition 4.3.1
A propagator $J : \mathcal{E}_0 \to \mathcal{E}$ is *filling* if the freely generating functor F_J is full and faithful. Then the underlying functor U_J is the *support* functor with respect to J.

From theorem 2.3.4, J is a filling propagator if and only if the unit η_J is a natural isomorphism:

$$\eta_J : id_{\mathcal{R}eal(\mathcal{E}_0)} \stackrel{\cong}{\Longrightarrow} U_J \circ F_J \; (= M_J).$$

Obviously, a conservative propagator is filling, the composite of filling propagators is filling, and a propagator which is equivalent to a filling one is also filling.

The next result gives a characterization of filling propagators in terms of their types, as defined in section 3.6. This result will not be used in this paper, and it is not proven either.

Theorem 4.3.2 (filling propagators)
A propagator J is filling if and only if the functor which underlies the morphism of projective types $Ty(J)$ is full and faithful.

We now define a notion of *distributor*, which is a variant of the notion of distributor defined originally in [Bénabou, 1973].

Definition 4.3.3
In this paper, a *distributor* is a propagator $J : \mathcal{E}_0 \to \mathcal{E}$, which is an inclusion and which adds to $\mathcal{E}_0$:
– a copy of a projective sketch $\mathcal{E}_1$ which has no distinguished projective cone with empty base,
– some *transition* arrows from $\mathcal{E}_1$ to $\mathcal{E}_0$, *i.e.* some arrows with their source in $\mathcal{E}_1$ and their target in $\mathcal{E}_0$,
– some *transverse* commutative squares, *i.e.* some commutative squares $tr' \circ e_1 = e_0 \circ tr$, where tr and tr' are transition arrows, e_1 is in $\mathcal{E}_1$ and e_0 in $\mathcal{E}_0$,
– and some distinguished *transverse* projective cones, where a transverse projective cone has its vertex in $\mathcal{E}_1$, at least a point of its base in $\mathcal{E}_1$, and at least a point of its base in $\mathcal{E}_0$.

Proposition 4.3.4
A propagator which is equivalent to a distributor is filling.

Proof. Let $J : \mathcal{E}_0 \to \mathcal{E}$ be a filling propagator. For each realization S of $\mathcal{E}_0$, the realization $F_J(S)$ of $\mathcal{E}$ is easy to compute: it coincides with S on $\mathcal{E}_0$, and $F_J(S)(E) = \emptyset$ for all point E not in $\mathcal{E}_0$. It follows immediately that $U_J \circ F_J(S) \xrightarrow{\simeq} S$, so that F_J is full and faithful.

This proves that a distributor is a filling propagator, hence the proposition follows. □

In a distributor, the base of a transverse projective cone can be $E_1 \xrightarrow{tr} E_0 \xleftarrow{tr} E_1$ for some transition arrow tr, so that it is possible to state that some transition arrows are monomorphic.

Proposition 4.3.5
Let J be a distributor with at least one potential monomorphic transition arrow with source E_1 for each point E_1 of $\mathcal{E}_1$. Then the underlying functor $U_J : \mathcal{R}eal(\mathcal{E}) \to \mathcal{R}eal(\mathcal{E}_0)$ is faithful.

Proof. Let $\sigma_1, \sigma_2 : S \to S'$ be two morphisms of realizations of $\mathcal{E}$ such that $U(\sigma_1) = U(\sigma_2) : U(S) \to U(S')$. We have to prove that $\sigma_1(E) = \sigma_2(E)$ for all point E of $\mathcal{E}$.
If $E = E_0$ is a point of $\mathcal{E}_0$, then $\sigma_i(E_0) = U(\sigma_i)(E_0)$ for $i = 1$ and 2, so that $\sigma_1(E_0) = \sigma_2(E_0)$.
Otherwise, $E = E_1$ is a point of $\mathcal{E}_1$, and there is a monomorphic transition arrow $tr : E_1 \to E_0$. For $i = 1$ and 2, from the naturality of σ_i we get $S'(tr) \circ \sigma_i(E_1) = \sigma_i(E_0) \circ S(tr)$. Since $\sigma_1(E_0) = \sigma_2(E_0)$, we get $S'(tr) \circ \sigma_1(E_1) = S'(tr) \circ \sigma_2(E_1)$. Since $S'(tr)$ is a monomorphism, we get $\sigma_1(E_1) = \sigma_2(E_1)$. □

Example 4.3.6
In section 4.1, the propagator $P : \mathcal{E}_{\mathcal{G}r} \to \mathcal{E}_{\mathcal{C}at}$ is not filling, whereas the propagator $J : \mathcal{E}_{\mathcal{G}r} \to \mathcal{E}_{\mathcal{C}omp}$ is filling. Indeed, it is equivalent to $J' : \mathcal{E}'_{\mathcal{G}r} \to \mathcal{E}_{\mathcal{C}omp}$ (see section 4.1), and it is easily checked that J' is a distributor.

4.4 Decomposition of propagators

A propagator is, in general, neither fractioning nor filling. The following result proves that, up to equivalence, it can be decomposed as a filling propagator followed by a fractioning one. Actually, there are several ways to achieve such a decomposition. One systematic way stems from the proof which is given below.

Theorem 4.4.1 (decomposition of propagators)
Let $P : \mathcal{E}_0 \to \overline{\mathcal{E}}$ be a propagator. There are a projective sketch $\mathcal{E}$, a fractioning propagator $K : \mathcal{E} \to \overline{\mathcal{E}}$ and a filling propagator $J : \mathcal{E}_0 \to \mathcal{E}$ such that:

$$P \equiv K \circ J.$$

In addition, it can be assumed that J is a distributor.

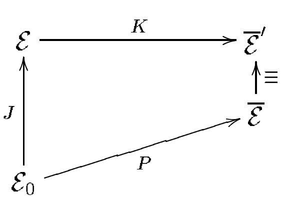
Proof. Let $J : \mathcal{E}_0 \to \mathcal{E}$ be the distributor which adds to $\mathcal{E}_0$:
– a copy of the support $\mathcal{E}_1 = Supp(\overline{\mathcal{E}})$ of $\overline{\mathcal{E}}$ (so, $\mathcal{E}_1$ is a projective sketch without any distinguished projective cone),
– the transition arrows $tr_{E_1,E_0} : E_1 \to E_0$ for all points E_1 of $\mathcal{E}_1$ and E_0 of $\mathcal{E}_0$ such that $P(E_0) = E_1$,

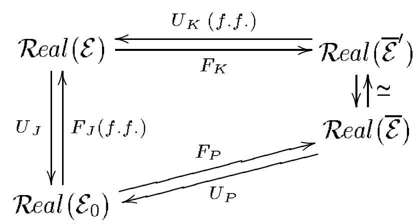
– the transverse commutative squares $tr_{E_1',E_0'} \circ e_1 = e_0 \circ tr_{E_1,E_0}$ for all arrows $e_1 : E_1 \to E_1'$ of $\mathcal{E}_1$ and $e_0 : E_0 \to E_0'$ of $\mathcal{E}_0$ such that $P(e_0) = e_1$. There is no distinguished transverse projective cone.

Now, let $\overline{\mathcal{E}}'$ be made of $\overline{\mathcal{E}}$ together with one identity for each point, so that the inclusion $\overline{\mathcal{E}} \subseteq \overline{\mathcal{E}}'$ is an equivalence. Let $K : \mathcal{E} \to \overline{\mathcal{E}}'$ be the propagator such that:
– on $\mathcal{E}_0$, it coincides with P,
– on $\mathcal{E}_1$, it coincides with the inclusion $Supp(\overline{\mathcal{E}}) \subseteq \overline{\mathcal{E}} \subseteq \overline{\mathcal{E}}'$,
– all transition arrow $tr_{E_1,E_0} : E_1 \to E_0$ is mapped to $id_{E_1} : E_1 \to E_1$: indeed $K(E_0) = P(E_0) = E_1$ and $K(E_1) = E_1$.
Then all transverse commutative square $tr_{E_1',E_0'} \circ e_1 = e_0 \circ tr_{E_1,E_0}$ is preserved, since both $tr_{E_1',E_0'} \circ e_1$ and $e_0 \circ tr_{E_1,E_0}$ are mapped to e_1: indeed $K(e_0) = P(e_0) = e_1$ and $K(e_1) = e_1$.

Then obviously $P \equiv K \circ J$.

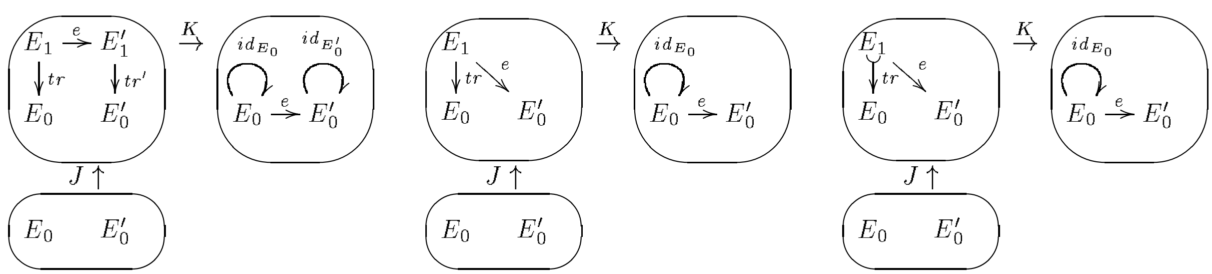
Finally, K can be decomposed as $K = K_2 \circ K_1$, where K_1 maps the transition arrows to identities and K_2 is the distinction of the projective cones of $\overline{\mathcal{E}}$. From proposition 4.2.4 and theorem 4.2.3, both K_1 and K_2 are fractioning, so that K itself is fractioning.





□

As a basic application of this decomposition theorem, let us consider the inclusion $P : \mathcal{E}_0 \subseteq \overline{\mathcal{E}}$ where $\mathcal{E}_0$ is made of two points E_0 and E_0' and P adds an arrow $e : E_0 \to E_0'$. Neither U nor F is full and faithful. According to the proof of theorem 4.4.1, the intermediate sketch $\mathcal{E}$ is made of four points E_0, E_0', E_1 and E_1', an arrow $e : E_1 \to E_1'$ and two transition arrows $tr : E_1 \to E_0$ and $tr' : E_1' \to E_0'$. Then $P \equiv K \circ J$ where J is the inclusion $\mathcal{E}_0 \subseteq \mathcal{E}$ and K maps tr and tr' to identity loops.
In this example, we could use the following variant. The intermediate sketch $\mathcal{E}$ is made of three points E_0, E_0' and E_1, two arrows $e : E_1 \to E_0'$ and $tr : E_1 \to E_0$. Then $P \equiv K \circ J$ where J is the inclusion $\mathcal{E}_0 \subseteq \mathcal{E}$ and K maps tr to an identity loop.
In addition, the arrow tr could be a potential monomorphism. This would mean that in $\mathcal{E}$ the operation e is partial, and then in $\overline{\mathcal{E}}$ it becomes total.
These three variants can be illustrated as follows:



Example 4.4.2
In section 4.1, the propagator $P : \mathcal{E}_{\mathcal{G}r} \to \mathcal{E}_{\mathcal{C}at}$ has been decomposed as $P \equiv K' \circ J$ with $J : \mathcal{E}_{\mathcal{G}r} \to \mathcal{E}_{\mathcal{C}omp}$ filling and $K' : \mathcal{E}_{\mathcal{C}omp} \to \mathcal{E}'_{\mathcal{C}at}$ fractioning. This decomposition of P corresponds to the last variant above: both operations *comp* and *selid*, which do not occur in $\mathcal{E}_{\mathcal{G}r}$, are introduced as partial operations in $\mathcal{E}_{\mathcal{C}omp}$, then they are made total in $\mathcal{E}'_{\mathcal{C}at}$.

5 Diagrammatic specifications

In this section we define some basic notions related to logic, like syntactic entailment and semantic consequence, in the general framework of fractioning propagators.

5.1 Specifications and domains

Definition 5.1.1
Let $P : \mathcal{E}_0 \to \overline{\mathcal{E}}$ be a propagator. The category of *(diagrammatic) specifications with respect to* P, or P*-specifications*, is the category of realizations of $\mathcal{E}_0$, and the category of *(diagrammatic) domains with respect to* P, or P*-domains*, is the category of realizations of $\overline{\mathcal{E}}$:

$$\mathcal{S}pec(P) = \mathcal{R}eal(\mathcal{E}_0) \quad \text{and} \quad \mathcal{D}om(P) = \mathcal{R}eal(\overline{\mathcal{E}}).$$

Of course, this definition can be used when the propagator is fractioning. On the other hand, from the decomposition theorem 4.4.1, all propagator $P : \mathcal{E}_0 \to \overline{\mathcal{E}}$ can be decomposed as $P = K \circ J$, with $K : \mathcal{E}_0 \to \mathcal{E}$ fractioning and $J : \mathcal{E}_0 \to \mathcal{E}$ filling. Then, the K-domains are the P-domains, and each P-specification S_0 freely generates a K-specification $S = F_J(S_0)$, which is such that $F_P(S_0) = F_K(S)$. In addition, from the proof of theorem 4.4.1, J can be chosen in such a way that S is essentially the same as S_0.

Hence, from now on, $K : \mathcal{E} \to \overline{\mathcal{E}}$ is a fractioning propagator.

Definition 5.1.2
A K-specification S is *saturated* if the morphism $\eta_{K,S} : S \to M_K(S)$ is an isomorphism.

Proposition 5.1.3 (saturated specifications)
Let S be a K-specification. Then the K-specification $M_K(S)$ is saturated.

Proof. Since $\eta \circ M$ is a natural isomorphism, the morphism $\eta \circ M(S) = \eta_{M(S)} : M(S) \to M(M(S))$ is an isomorphism. □

5.2 Syntactic entailment

Definition 5.2.1
A morphism $\sigma : S \to S'$ of K-specifications is a *syntactic entailment* if the morphism of K-domains $F_K(\sigma)$ is an isomorphism:

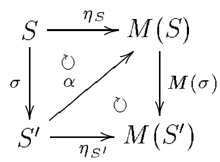
$$S \overset{\sigma}{\twoheadrightarrow} S' \quad \text{if and only if} \quad F_K(\sigma) : F_K(S) \xrightarrow{\simeq} F_K(S').$$

Proposition 5.2.2
Let $\sigma : S \to S'$ be a morphism of K-specifications. Then $S \overset{\sigma}{\twoheadrightarrow} S'$ if and only if $M_K(\sigma)$ is an isomorphism.

Proof. If $F(\sigma)$ is an isomorphism, then clearly $M(\sigma)$ is an isomorphism, because $M = U \circ F$ (and this is true for any propagator K).
On the other hand, since K is fractioning, the functor U is full and faithful. So, if a morphism $\delta : D \to D'$ is such that $U(\delta)$ is an isomorphism, then δ itself is an isomorphism. This can be applied to $\delta = F(\sigma)$: if $M(\sigma)$ is an isomorphism, then $F(\sigma)$ is an isomorphism. □

Theorem 5.2.3
Let $\sigma : S \to S'$ be a morphism of K-specifications. Then $S \overset{\sigma}{\twoheadrightarrow} S'$ if and only if there is a morphism of K-specifications $\alpha : S' \to M_K(S)$ such that $\alpha \circ \sigma = \eta_{K,S}$ and $M_K(\sigma) \circ \alpha = \eta_{K,S'}$. In such a case, $\alpha = (M_K(\sigma))^{-1} \circ \eta_{S'}$.

The condition in the theorem means that the commutative square $\eta_{K,S'} \circ \sigma = M_K(\sigma) \circ \eta_{K,S}$ is *splitted*:



Proof. Let $S \overset{\sigma}{\twoheadrightarrow} S'$, so that $M(\sigma)$ is an isomorphism, by proposition 5.2.2. Let $\alpha = (M(\sigma))^{-1} \circ \eta_{S'} : S' \to M(S)$. Then $\alpha \circ \sigma = (M(\sigma))^{-1} \circ \eta_{S'} \circ \sigma = (M(\sigma))^{-1} \circ M(\sigma) \circ \eta_S = \eta_S$, and $M(\sigma) \circ \alpha = M(\sigma) \circ (M(\sigma))^{-1} \circ \eta_{S'} = \eta_{S'}$.

Now, let $\alpha : S' \to M_K(S)$ be such that $\alpha \circ \sigma = \eta_{K,S}$ and $M_K(\sigma) \circ \alpha = \eta_{K,S'}$. Let us prove that $\mu_S \circ M(\alpha) : M(S') \to M(S)$ is an inverse of $M(\sigma)$. Since K is fractioning, the monad M is idempotent, which means that μ is a natural isomorphism, with inverse $M \circ \eta$. (see corollary 2.3.5).
On one hand, from $\alpha \circ \sigma = \eta_S$, we get:
$\mu_S \circ M(\alpha) \circ M(\sigma) = \mu_S \circ M(\alpha \circ \sigma) = \mu_S \circ M(\eta_S) = id_{M(S)}$.
On the other hand, from $M(\sigma) \circ \alpha = \eta_{S'}$, we get $M^2(\sigma) \circ M(\alpha) = M(\eta_{S'})$, so that (thanks to the naturality of μ):
$M(\sigma) \circ \mu_S \circ M(\alpha) = \mu_{S'} \circ M^2(\sigma) \circ M(\alpha) = \mu_{S'} \circ M(\eta_{S'}) = id_{M(S')}$.
So, $M(\sigma)$ is an isomorphism, with inverse $\mu_S \circ M(\alpha)$. □

5.3 Syntactic deduction steps

Definition 5.3.1
A *deduction rule with respect to* K is an arrow $r : H \to C$ in $\overline{\mathcal{E}}$. The point H is the *hypothesis* and the point C is the *conclusion* of the rule r.

A deduction rule $r : H \to C$ can be written as $\frac{H}{C}(r)$, or simply as $\frac{H}{C}$.

From theorem 4.2.2, up to equivalence of sketches, the hypothesis and conclusion of a rule are points of $\mathcal{E}$, and there are two kind of rules:
– a deduction rule $r : H \to C$ is *passive* if r is an arrow of $\mathcal{E}$,
– a deduction rule $r : H \to C$ is *active* if r is the inverse of an arrow $e : C \to H$ of $\mathcal{E}$.

Deduction rules can be composed, as arrows in $\overline{\mathcal{E}}$.

The Yoneda contravariant realization $Y_{\mathcal{E}}$ of $\mathcal{E}$ yields illustrations for active deduction rules. Indeed, let $e : C \to H$ be an arrow of $\mathcal{E}$, and let $r = e^{-1} : H \to C$ be the corresponding active deduction rule. The image of $e : C \to H$ by $Y_{\mathcal{E}}$ is a morphism of realizations of $\mathcal{E}$:

$$Y_{\mathcal{E}}(e) : Y_{\mathcal{E}}(H) \to Y_{\mathcal{E}}(C).$$

Since the Yoneda realization is contravariant, the source and target of the morphism $Y_{\mathcal{E}}(e)$ are respectively (the images of) the hypothesis and the conclusion of the rule r. The morphism $Y_{\mathcal{E}}(e)$ becomes an isomorphism in $S(\overline{\mathcal{E}})$. In this way, $Y_{\mathcal{E}}(e)$ illustrates the deduction rule $r : H \to C$. For instance, this is the way the definition of categories is illustrated by functors of compositive graphs in section 2.2.

Now, let $r = e^{-1} : H \to C$ be an active deduction rule.

Let S be a K-specification and $x \in S(H)$. The inverse image of x by $S(e)$ can be any subset $(S(e))^{-1}(x)$ of $S(C)$:

$$(S(e))^{-1}(x) \subseteq S(C).$$

When S is saturated, $(S(e))^{-1}(x)$ is made of exactly one element y of $S(C)$.

We now define the "simplest" morphism $\sigma : S \to S'$ with source S such that, if $x' = \sigma(H)(x)$, the inverse image of x' by $S'(e)$ is made of exactly one element y' of $S'(C)$:

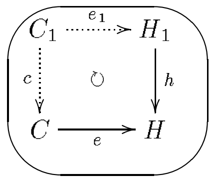
$$(S'(e))^{-1}(x') = \{y'\}.$$

For this purpose, let $\Phi : \mathcal{E} \to \mathcal{E}_1$ be the inclusion propagator which adds points H_1 and C_1, arrows $h : H_1 \to H$, $c : C_1 \to C$ and $e_1 : C_1 \to H_1$, and two distinguished projective cones: the first one with vertex H_1 and empty base, the second one with vertex C_1, base $C \xrightarrow{e} H \xleftarrow{h} H_1$ and projections c and e_1.

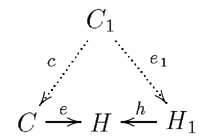
The set-valued realizations of $\mathcal{E}_1$ are, up to isomorphisms, the pairs $S_1 = (S, x)$ where S is a set-valued realization of $\mathcal{E}$ and x is an element of $S(H)$. Then clearly $S = U_\Phi(S_1)$.

When $\mathcal{E}$ contains only $C \xrightarrow{e} H$, then $\mathcal{E}_1$ is as follows:

$\mathcal{E}_1$:



with 2 *dpc*s: H_1 (empty base)



Let $\overline{\Phi} : \overline{\mathcal{E}} \to \overline{\mathcal{E}}_1$ be obtained by a similar construction from $\overline{\mathcal{E}}$. Then the inclusion $K_1 : \mathcal{E}_1 \to \overline{\mathcal{E}}_1$ is a fractioning propagator and the following square is a pushout:

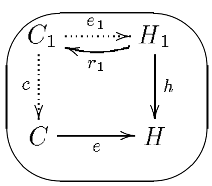
$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{K} & \overline{\mathcal{E}} \\ \Phi\downarrow & & \downarrow\overline{\Phi} \\ \mathcal{E}_1 & \xrightarrow{K_1} & \overline{\mathcal{E}}_1 \end{array}$$

Let S be a set-valued realization of $\mathcal{E}$ and $x \in S(H)$. Let $\widetilde{x}$ denote the image of x in $M_K(S)(H)$ by the map $\eta_{K,S}(H) : S(H) \to M_K(S)(H)$. Since K is an inclusion, $\widetilde{x} \in F_K(S)(H)$. The next result is easy to prove.

Lemma 5.3.2
With the above notations, if $S_1 = (S, x)$ *then* $F_{K_1}(S_1) \cong (F_K(S), \widetilde{x})$, *naturally in* S_1 *in* $\mathcal{R}eal(\overline{\mathcal{E}}_1)$.

Now, let $\Psi : \mathcal{E}_1 \to \mathcal{E}_2$ denote the fractioning propagator which adds an inverse $r_1 : H_1 \to C_1$ to e_1.
When $\mathcal{E}$ contains only $C \xrightarrow{e} H$, then $\mathcal{E}_2$ is as follows:

$\mathcal{E}_2$:



with 2 *dpc*s (like $\mathcal{E}_1$)

and with 2 composites:

$r_1 \circ e_1 = id_{C_1}$

$e_1 \circ r_1 = id_{H_1}$

Let $\overline{\Psi} : \overline{\mathcal{E}}_1 \to \overline{\mathcal{E}}_2$ be obtained by a similar construction from $\overline{\mathcal{E}}_1$. Then the inclusion $K_2 : \mathcal{E}_2 \to \overline{\mathcal{E}}_2$ is a fractioning propagator and the following square is a pushout:

$$\begin{array}{ccc} \mathcal{E}_1 & \xrightarrow{K_1} & \overline{\mathcal{E}}_1 \\ \Psi\downarrow & & \downarrow\overline{\Psi} \\ \mathcal{E}_2 & \xrightarrow{K_2} & \overline{\mathcal{E}}_2 \end{array}$$

Let $S_1 = (S, x)$ be a K_1-specification, so that $S = U_\Phi(S_1)$, and let $S'_1 = M_{K_1}(S_1)$ and $S' = U_\Phi(S'_1)$. Then, from $\eta_{K_1,S_1} : S_1 \to S'_1$, we get a morphism of K-specifications:

$$\sigma = U_\Phi(\eta_{K_1,S_1}) : S \to S'.$$

Let $x' = \sigma(H)(x) \in S'(H)$, then the inverse image of x' by $S'(e)$ is made of one point: namely, $(S'(e))^{-1}(x') = \{y'\}$ where $y' = S'(r_1)(x') \in S'(C)$. On the other hand, if $(S(e))^{-1}(x)$ is made of one point, then $\sigma = id_S : S \to S$.

Definition 5.3.3
Let $r : H \to C$ be a deduction rule with respect to K, and let x be an element in $S(H)$. The *deduction step with respect to* K associated to r and x is a morphism of K-specifications with source S. If r is a passive deduction rule, it is the identity morphism $id_S : S \to S$. If $r = e^{-1}$ is an active deduction rule, it is the morphism $\sigma : S \to S'$ as defined above.

Proposition 5.3.4 (syntactic deduction step)
Let $\sigma : S \to S'$ be a deduction step. Then it is a syntactic entailment: $S \overset{\sigma}{\longrightarrow\!\!\!\!\!\!\rightarrow} S'$.

Proof. Let $\sigma : S \to S'$ be the deduction step associated to the rule $r : H \to C$ and to $x \in S(H)$. Let us prove that $F_K(\sigma) : F_K(S) \to F_K(S')$ is an isomorphism.

If r is passive then σ is the identity, so that $F_K(\sigma)$ is the identity.

If r is active, lemma 5.3.2 proves that $F_K(\sigma) \cong U_{\overline{\Phi}}(F_{K_1}(\sigma_1))$. Since $\overline{\Psi}$ is conservative, it follows that, up to isomorphism, $F_K(\sigma)$ is the deduction step associated to the rule $r : H \to C$ and to $\widetilde{x} \in F_K(S)(H)$. Since $F_K(S)$ is saturated, this deduction step is the identity. □

It follows that any finite composition of deduction steps is a syntactic entailment.

In the opposite direction, it can be proven that all syntactic entailment (for instance $\eta_{K,S} : S \longrightarrow\!\!\!\!\!\!\rightarrow M_K(S)$) can be obtained from syntactic deduction steps by nested inductive limits of various types.

5.4 Models

Definition 5.4.1
Let D be a K-domain. The *contravariant functor of models* $\mathrm{Mod}_K(-, D) : \mathcal{S}pec(K) \twoheadrightarrow \mathcal{S}et$ is:

$$\mathrm{Mod}_K(-, D) = \mathrm{Hom}_{\mathcal{R}eal(\overline{\mathcal{E}})}(F_K(-), D).$$

It follows from the generated realization theorem 3.4.1 that:

$$\mathrm{Mod}_K(-, D) \cong \mathrm{Hom}_{\mathcal{R}eal(\mathcal{E})}(-, U_K(D)).$$

So, for each K-specification S, the *models of S with values in D* are the morphisms from $F_K(S)$ to D in $\mathcal{R}eal(\overline{\mathcal{E}})$, and they can be identified with the morphisms from S to $U_K(D)$ in $\mathcal{R}eal(\mathcal{E})$:

$$\mathrm{Mod}_K(S, D) = \mathrm{Hom}_{\mathcal{R}eal(\overline{\mathcal{E}})}(F_K(S), D) \cong \mathrm{Hom}_{\mathcal{R}eal(\mathcal{E})}(S, U_K(D)).$$

From the definition of morphisms in section 3.3, a model ω of S with values in D can be identified with a natural transformation between the functors underlying S and $U_K(D)$: it is made of a map $\omega_E : S(E) \to D(K(E))$ for each point E of $\mathcal{E}$, naturally in E.

In this paper, a model ω of S with values in D is illustrated as $\omega : S \dashrightarrow D$, so that (Kleisli categories could be invoked here, see [Mac Lane, 1971, p.143]):

$$(\omega : S \dashrightarrow D) \;=\; (\omega : F_K(S) \to D).$$

Let $\sigma : S \to S'$ be a morphism of K-specifications. The map $\mathrm{Mod}_K(\sigma, D) : \mathrm{Mod}_K(S', D) \to \mathrm{Mod}_K(S, D)$ maps $\omega' : S' \dashrightarrow D$, i.e. $\omega' : F_K(S') \to D$, to $\omega' \circ F_K(\sigma) : F_K(S) \to D$, which is denoted, in this paper, $\omega' \circledcirc \sigma : S \dashrightarrow D$, so that:

$$(\omega' \circledcirc \sigma : S \dashrightarrow D) \;=\; (\omega' \circ F_K(\sigma) : F_K(S) \to D).$$

Proposition 5.4.2
The map $\mathrm{Mod}_K(\eta_{K,S}, D)$ is a bijection: $\mathrm{Mod}_K(M_K(S), D) \xrightarrow{\simeq} \mathrm{Mod}_K(S, D)$.

So, the models of a K-specification S can be identified with the models of $M_K(S)$.

Proof. From theorem 2.3.4, since U is full and faithful, $F \circ \eta$ is a natural isomorphism $F \circ \eta : F \stackrel{\simeq}{\Longrightarrow} F \circ U \circ F$. So, the map $\mathrm{Hom}_{\mathcal{R}eal(\overline{\mathcal{E}})}(F_K(\eta_S), D)$, *i.e.* the map $\mathrm{Mod}_K(\eta_S, D)$, is a bijection. □

In such a general setting, there is no canonical notion of morphism of models, hence no category of models of S with values in D. However, in many important special cases, there is such a category of models; then the contravariant functor of models is:

$$\mathcal{M}od_K(-, D) : \mathcal{S}pec(K) \twoheadrightarrow \mathcal{C}at.$$

5.5 Semantic consequence

Definition 5.5.1
Let D be a K-domain. A morphism $\sigma : S \to S'$ of K-specifications is a *semantic consequence with respect to D* if the map $\mathrm{Mod}(\sigma, D)$ is a bijection:

$$S \stackrel{\sigma}{\twoheadrightarrow}_D S' \quad \text{if and only if} \quad \mathrm{Mod}(\sigma, D) : \mathrm{Mod}(S', D) \stackrel{\simeq}{\longrightarrow} \mathrm{Mod}(S, D).$$

5.6 Soundness

Entailment and consequence are related by the soundness property, which is easily derived from the properties of adjunction.

Theorem 5.6.1 (soundness)
Let D be a K-domain. For all morphism of K-specifications $\sigma : S \to S'$, if σ is a syntactic entailment, then it is a semantic consequence with respect to D:

$$\textit{if} \quad S \stackrel{\sigma}{\longrightarrow} S' \quad \textit{then} \quad S \stackrel{\sigma}{\twoheadrightarrow}_D S'.$$

This means that all fractioning propagators are *sound*.

Proof. Let $\sigma : S \to S'$ be a morphism of K-specifications. Since η is a natural transformation, we have $\eta_{S'} \circ \sigma = M_\sigma \circ \eta_S$. Hence $\mathrm{Mod}(\sigma, D) \circ \mathrm{Mod}(\eta_{S'}, D) = \mathrm{Mod}(\eta_S, D) \circ \mathrm{Mod}(M_\sigma, D)$. On one hand, since σ is a conservative morphism, M_σ is an isomorphism, and $\mathrm{Mod}(M_\sigma, D)$ is a bijection. On the other hand, from proposition 5.4.2, both $\mathrm{Mod}(\eta_S, D)$ and $\mathrm{Mod}(\eta_{S'}, D)$ are bijections. It follows that $\mathrm{Mod}(\sigma, D)$ is also a bijection. □

Proposition 5.6.2
Let $\sigma : S \to S'$ be a morphism of K-specifications. If σ is a semantic consequence with respect to all K-domain D, then it is a syntactic entailment:

$$\textit{if} \quad S \stackrel{\sigma}{\twoheadrightarrow}_D S' \ \textit{for all } D \quad \textit{then} \quad S \stackrel{\sigma}{\longrightarrow} S'.$$

Proof. The assumption means that the map $\mathrm{Mod}(\sigma, D) : \mathrm{Mod}(S', D) \to \mathrm{Mod}(S, D)$ is a bijection for all domain D. From the definition of models, this means that the map $\mathrm{Hom}(F(\sigma), D) : \mathrm{Hom}(F(S'), D) \to \mathrm{Hom}(F(S), D)$ is a bijection for all domain D.

So, when $D = F(S)$, the map $\delta \mapsto \delta \circ F(\sigma)$ is a bijection $\mathrm{Hom}(F(\sigma), F(S)) : \mathrm{Hom}(F(S'), F(S)) \stackrel{\simeq}{\longrightarrow} \mathrm{Hom}(F(S), F(S))$; hence, there is a unique morphism $\tau : F(S') \to F(S)$ such that $\tau \circ F(\sigma) = id_{F(S)}$.

Now, when $D = F(S')$, the map $\delta \mapsto \delta \circ F(\sigma)$ is a bijection $\mathrm{Hom}(F(\sigma), F(S')) : \mathrm{Hom}(F(S'), F(S')) \stackrel{\simeq}{\longrightarrow} \mathrm{Hom}(F(S), F(S'))$. This map is such that $F(\sigma) \circ \tau \mapsto F(\sigma) \circ \tau \circ F(\sigma)$, which is equal to $F(\sigma)$, since $\tau \circ F(\sigma) = id_{F(S)}$. But clearly $id_{F(S')} \mapsto F(\sigma)$, so that $F(\sigma) \circ \tau = id_{F(S')}$.

So, $F(\sigma)$ is an isomorphism, with inverse τ. □

5.7 Satisfaction

Here it is proven that the relation of semantic consequence, between two specifications, can also be obtained from a relation of satisfaction, between a model and a specification. The satisfaction only makes sense when there is some notion of signature of a specification. In our context, the *signatures* are the set-valued realizations of a projective sketch $\mathcal{E}_0$, such that there is a propagator $J : \mathcal{E}_0 \to \mathcal{E}$.

More precisely, let $K_0 : \mathcal{E}_0 \to \overline{\mathcal{E}}_0$ be a fractioning propagator, together with a homomorphism $\widetilde{J} = (J, \overline{J}) : K_0 \to K$.

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{K} & \overline{\mathcal{E}} \\ {\scriptstyle J}\uparrow & \circlearrowleft & \uparrow{\scriptstyle \overline{J}} \\ \mathcal{E}_0 & \xrightarrow{K_0} & \overline{\mathcal{E}}_0 \end{array}$$

Let S_0 be a K_0-specification and D_0 a K_0-domain.

For all K-specification S such that $U_J(S) = S_0$ and all K-domain D such that $U_{\overline{J}}(D) = D_0$, the underlying functor $U_J : \mathcal{R}eal(\mathcal{E}) \to \mathcal{R}eal(\mathcal{E}_0)$ determines a map:

$$(U_J)_{S,U_K(D)} : \mathrm{Hom}_{\mathcal{R}eal(\mathcal{E})}(S, U_K(D)) \to \mathrm{Hom}_{\mathcal{R}eal(\mathcal{E}_0)}(S_0, U_J(U_K(D)),$$

and since $U_J \circ U_K = U_{K_0} \circ U_{\overline{J}}$ this map is:

$$(U_J)_{S,U_K(D)} : \mathrm{Hom}_{\mathcal{R}eal(\mathcal{E})}(S, U_K(D)) \to \mathrm{Hom}_{\mathcal{R}eal(\mathcal{E}_0)}(S_0, U_{K_0}(D_0)).$$

So, by adjunction, we get a map:

$$(U_{\widetilde{J}})_{S,D} : \mathrm{Mod}_K(S, D) \to \mathrm{Mod}_{K_0}(S_0, D_0),$$

such that:

$$\omega \mapsto (U_J(\omega_\star))^\star.$$

This map is natural both in S and in D.

Definition 5.7.1 For all K-specification S such that $U_J(S) = S_0$ and all K-domain D such that $U_{\overline{J}}(D) = D_0$, the *underlying model map* with respect to $\widetilde{J}$ is the map $\omega \mapsto (U_J(\omega_\star))^\star$:

$$(U_{\widetilde{J}})_{S,D} : \mathrm{Mod}_K(S, D) \to \mathrm{Mod}_{K_0}(S_0, D_0).$$

Definition 5.7.2 Let S be a K-specification such that $U_J(S) = S_0$, and D a K-domain such that $U_{\overline{J}}(D) = D_0$. A model ω_0 of S_0 with values in D_0 *satisfies* S *with respect to* D if ω_0 is in the image of $\mathrm{Mod}(S, D)$ by $(U_{\widetilde{J}})_{S,D}$. This is denoted:

$$\omega_0 \twoheadrightarrow\!\!\!\!\!\!\!\!+\!\!\;\;\,_D\ S.$$

Let us now make the following assumption (INJ) :

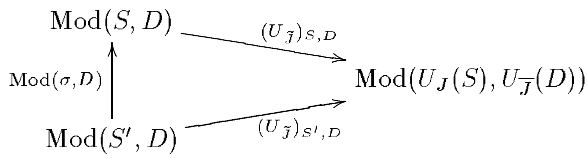
The map $(U_{\widetilde{J}})_{S,D}$ is injective, for all K-specification S and all K-domain D.

This happens when J is a filling propagator which satisfies the condition of proposition 4.3.5.

Under this assumption (INJ), the map $(U_{\widetilde{J}})_{S,D}$ can be used for identifying $\mathrm{Mod}(S, D)$ and its image in $\mathrm{Mod}(U_J(S), U_{\overline{J}}(D))$. Then, we can say that $\omega_0 : S_0 \dashrightarrow D_0$ satisfies S with respect to D if and only if it "is" a model of S with values in D.

Theorem 5.7.3
Under assumption (INJ), let $\sigma : S \to S'$ be a morphism of K-specifications such that $U_J(S) = U_J(S') = S_0$ and $U_J(\sigma) = id_{S_0}$, and let D be a K-domain such that $U_{\overline{J}}(D) = D_0$.
Then $S \overset{\sigma}{\twoheadrightarrow}_D S'$ if and only if, for all $\omega_0 : S_0 \dashrightarrow D_0$, if $\omega_0 \twoheadrightarrow_D S$ then $\omega_0 \twoheadrightarrow_D S'$.

Proof. Because of the naturality of the map $(U_{\overline{J}})_{S,D}$ with respect to S, the following triangle T is commutative:



Let us assume that $S \overset{\sigma}{\twoheadrightarrow}_D S'$, *i.e.* that $\mathrm{Mod}(\sigma, D)$ is bijective. Let $\omega_0 : S_0 \dashrightarrow D_0$ be such that $\omega_0 \twoheadrightarrow_D S$, which means that ω_0 is in the image of the map $(U_{\overline{J}})_{S,D}$. Then, since $\mathrm{Mod}(\sigma, D)$ is surjective, ω_0 is in the image of the map $(U_{\overline{J}})_{S,D} \circ \mathrm{Mod}(\sigma, D)$, which is equal to $(U_{\overline{J}})_{S',D}$ because of the commutativity of T, so that $\omega_0 \twoheadrightarrow_D S'$.

On the other hand, the commutativity of T together with the assumption (INJ) proves that the map $\mathrm{Mod}(\sigma, D)$ is injective. Now, let us assume that for all $\omega_0 : S_0 \dashrightarrow D_0$, if $\omega_0 \twoheadrightarrow_D S$ then $\omega_0 \twoheadrightarrow_D S'$. For all $\omega : S \dashrightarrow D$, let $\omega_0 = (U_{\overline{J}})_{S,D}(\omega)$, so that $\omega_0 \twoheadrightarrow_D S$. Then $\omega_0 \twoheadrightarrow_D S'$, hence there is some $\omega' : S' \dashrightarrow D$ such that $\omega' : S' \dashrightarrow D$, *i.e.* such that $\omega_0 = (U_{\overline{J}})_{S',D}(\omega')$. Because of the assumption (INJ), this ω' is uniquely determined. So, we get a map $f : \mathrm{Mod}(S, D) \to \mathrm{Mod}(S', D)$, and because of the assumption (INJ), f is an inverse to $\mathrm{Mod}(\sigma, D)$. It follows that $\mathrm{Mod}(\sigma, D)$ is bijective, so that $S \overset{\sigma}{\twoheadrightarrow}_D S'$. □

6 About logic

In this last section, we outline some basic links between our diagrammatic specification techniques and issues in logic. First, we look at equational diagrammatic specifications, and then, more generally, at institutions.

6.1 About equational logic

In the context of algebraic specifications, as for instance in [Goguen, Thatcher and Wagner, 1976], an equational specification is defined in three steps: first a set of sorts, then a signature (*i.e.* a structured set of operators) on this set of sorts, and finally a set of equations on this signature. Some strings of sorts are used for introducing the operators, and some terms (composed from operators) are used for introducing the equations.

For example, an equational specification of naturals S_{nat} can be defined as follows:
– sorts: N,
– operators: $s : N \to N$, $z : \lambda \to N$, $a : NN \to N$, with the strings of sorts NN and λ (empty string),
– equations: $a(x, z) = x$ and $a(x, s(y)) = s(a(x, y))$ where x and y are variables of sort N. These equations can be written without variables, as relations between composed arrows. For instance, the second equation can be written as $a \circ fact(id_N, s) \equiv s \circ a : NN \to N$, with one identity arrow $id_N : N \to N$, one factorization arrow $fact(id_N, s) : N \to NN$ and two composed arrows.

The construction of an equational specification makes use of three successive propagators: P_s for sorts, P_o for operators, and P_e for equations.

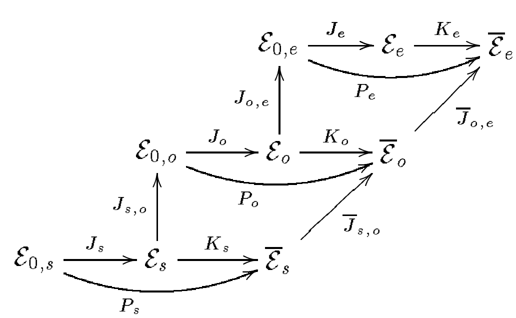
Sorts.
The propagator $P_s : \mathcal{E}_{0,s} \to \overline{\mathcal{E}}_s$ is the usual one from a projective sketch of sets to projective sketch of monoids.
The projective sketch $\mathcal{E}_{0,s}$ is the simplest sketch of sets: it is made of one point *Sort* (similar to *Pt*). So, a P_s-specification S_s is a set of sorts.
The projective sketch $\overline{\mathcal{E}}_s$ is a sketch of monoids: it contains the points $Sort^0$, $Sort$, $Sort^2$, two arrows $p_1, p_2 : Sort^2 \to Sort$ and two *dpc*'s: one with vertex $Sort^0$ and empty base, another one with vertex $Sort^2$, with two-points base $\{Sort, Sort\}$ (discrete, *i.e.* without any arrow) and projections p_1, p_2. The

point *Sort* will be interpreted as the set of strings of sorts, $Sort^0$ as a one-element set, and $Sort^2$ as the set of pairs of strings of sorts. In $\overline{\mathcal{E}}_s$, two more arrows $\lambda : Sort^0 \to Sort$ and $\kappa : Sort^2 \to Sort$ stand respectively for the empty string of sorts and the concatenation of strings of sorts. There are additional features in $\overline{\mathcal{E}}_s$ in order to ensure that κ will be interpreted as an associative operation and λ as its unit. So, the functor F_{P_s} freely generates the strings of sorts.
The propagator P_s is decomposed, according to theorem 4.4.1, as $P_s = K_s \circ J_s$, with an intermediate projective sketch $\mathcal{E}_s$ of partial monoids.

Operators.
The propagator $P_o : \mathcal{E}_{0,o} \to \overline{\mathcal{E}}_o$ is similar to the propagator which has been considered in the previous sections, from a projective sketch of directed graphs to a projective sketch of categories. There is a point *Op* (similar to *Ar*) which stands for the set of operators in $\mathcal{E}_{0,o}$ and for the set of terms in $\overline{\mathcal{E}}_o$. However, because of arities, P_o is somewhat larger than that.
The sketch $\mathcal{E}_{0,o}$ contains $\mathcal{E}_s$, not only $\mathcal{E}_{0,s}$, in order to allow the definition of multivariate operators and constant operators. So, a P_o-specification S_o is a signature, in the equational meaning.
The inclusion propagator $J_{s,o} : \mathcal{E}_s \to \mathcal{E}_{0,o}$ is filling. Let S_s be a P_s-specification. Then S_o is a S_s-sorted signature if $U_{J_{s,o}}(S_o)$ can be deduced from S_s, more precisely if $F_{J_s}(S_s) \twoheadrightarrow U_{J_{s,o}}(S_o)$, as K_s-specifications.
The sketch $\overline{\mathcal{E}}_o$, besides identity and composed arrows, also takes care of projection and factorization arrows. So, the functor F_{P_o} freely generates the terms, in their categorical version, *i.e.* without variables.
The propagator P_o is decomposed, according to theorem 4.4.1, as $P_o = K_o \circ J_o$, with an intermediate projective sketch $\mathcal{E}_o$ which contains the sketch of compositive graphs.

Equations.
The propagator $P_e : \mathcal{E}_{0,e} \to \overline{\mathcal{E}}_e$ is the propagator for equational specifications.
The sketch $\mathcal{E}_{0,e}$ contains $\mathcal{E}_o$ and a point *Eq* for equations, with a potential monomorphism from *Eq* to a point *Sst* which stands (thanks to a *dpc*) for the set of pairs of terms with the same source and target. So, a P_e-specification S_e is an equational specification.
The inclusion propagator $J_{o,e} : \mathcal{E}_o \to \mathcal{E}_{0,e}$ is filling. Let S_o be a P_o-specification. Then the signature of S_e is S_o if $U_{J_{o,e}}(S_e)$ can be deduced from S_o, more precisely if $F_{J_o}(S_o) \twoheadrightarrow U_{J_{o,e}}(S_e)$, as K_e-specifications.
The sketch $\overline{\mathcal{E}}_e$ adds deduction rules, in such a way that the interpretation of *Eq* in a realization of $\overline{\mathcal{E}}_e$ is a congruence, *i.e.* an equivalence relation which is compatible with the composition of terms. So, the functor F_{P_e} freely generates the congruence from the equations, *i.e.* the theorems from the axioms.
The propagator P_e is decomposed, according to theorem 4.4.1, as $P_e = K_e \circ J_e$, with an intermediate projective sketch $\mathcal{E}_e$ for derived equational specifications.

To sum up, the definition of equational specifications makes use of the following commutative diagram of projective sketches and propagators:



The domain of values is the realization D_{set} of $\overline{\mathcal{E}}_e$ which interprets the sorts as sets, the operations as maps, and the equations as identities between maps.

6.2 About institutions

The theory of *institutions* [Goguen and Burstall, 1992] defines some notions of logic in a very general setting. Diagrammatic specifications are quite different: they are restricted to projectively sketchable structures, in order to gain some effectiveness; and they do not assume any notion of formula or sentence, because diagrammatic specifications have been made for applications to computer languages which do not involve such notions. However, diagrammatic specifications can easily be related to institutions, more precisely to *chartered institutions*.

The idea is to consider a fractioning propagator $K_0 : \mathcal{E}_0 \to \overline{\mathcal{E}}_0$, together with a point Sen in $\mathcal{E}_0$ and with a K_0-domain D_0, such that the interpretation of the point $K_0(Sen)$ by D_0 is the set $\{true, false\}$ of booleans. Then a filling propagator $J : \mathcal{E}_0 \to \mathcal{E}$ is built by adding to $\mathcal{E}_0$ a point Ax and a potential monomorphism $m : Ax \to Sen$. This can be completed by a fractioning propagator $K : \mathcal{E} \to \overline{\mathcal{E}}$ and a filling propagator $\overline{J}$ such that $\widetilde{J} : K_0 \to K$ is a homomorphism of fractioning propagators, together with a K-domain D such that $D_0 = U_{\overline{J}}(D)$.

The point $K_0(Sen)$ of $\overline{\mathcal{E}}_0$ stands for the set of *sentences*, the point Ax of $\mathcal{E}$ for the set of *axioms*, and the point $K(Ax)$ of $\overline{\mathcal{E}}$ for the set of *valid sentences*.

Let S be a K-specification and $S_0 = U_J(S)$ its support. Then $S(Sen)$ is equal to $S_0(Sen)$, and the image of $S(Ax)$ by $S(m)$ is a subset of $S_0(Sen)$. Clearly, in this way, the category of K-specifications (up to isomorphisms) can be identified to the category of pairs (S_0, V) where S_0 is a K_0-specification, V is a subset of $S_0(Sen)$, and the morphisms are straightforward.

This gives rise to an institution I in the following way:
- $\mathcal{R}eal(\mathcal{E}_0)$ is the category of *signatures* of I,
- Mod : $\mathcal{R}eal(\mathcal{E}_0) \twoheadrightarrow \mathcal{S}et$ is the contravariant functor of *models* of I,
- $ev_{Sen} \circ F_{K_0} : \mathcal{R}eal(\mathcal{E}_0) \to \mathcal{S}et$ is the functor of *sentences* of I,
- and for all signature S_0, all model τ of S_0 with values in D_0 and all sentence s of S_0, the *satisfaction relation* between τ and s holds if and only if τ satisfies (in the sense of diagrammatic specifications) the K-specification S with support S_0 and with s as its unique axiom.

Then the required *satisfaction condition* is easily checked.

In addition, such an institution, together with the notion of syntactic entailment (in the sense of diagrammatic specifications) gives rise to a *logic* in the sense of [Martí-Oliet and Meseguer, 1994].

In this context, we can clarify the relations between the diagrammatic notions of entailment $\longrightarrow\!\!\!\!\!\!+\,$ and consequence $\longrightarrow\!\!\!\!\!\!\!\!++\,$, and the usual logical notions of entailment $\vdash$ and consequence $\models$.

Let S_0 be some fixed signature, and let $\varphi_1, \varphi_2, \ldots, \varphi_k$ and ψ be sentences of S_0. Let S be the specification with signature S_0 such that $S(Ax) = \{\varphi_1, \varphi_2, \ldots, \varphi_k\}$. Let S' be the specification with signature S_0 such that $S'(Ax) = \{\varphi_1, \varphi_2, \ldots, \varphi_k, \psi\}$. Let $\sigma : S \to S'$ be the inclusion. Then clearly:

$$S \overset{\sigma}{\longrightarrow\!\!\!\!\!\!+} S' \quad \text{if and only if} \quad \varphi_1, \varphi_2, \ldots, \varphi_k \vdash \psi \ ,$$

$$S \overset{\sigma}{\longrightarrow\!\!\!\!\!\!\!\!++}_D S' \quad \text{if and only if} \quad \varphi_1, \varphi_2, \ldots, \varphi_k \models \psi \ .$$

References

[Bénabou, 1973] J. Bénabou. *Les Distributeurs*. Univ. Cath. de Louvain, Instit. de Math. Pure et Appliquée, Rapport 33, 1973.

[Coppey and Lair, 1984] L. Coppey and C. Lair. Leçons de Théorie des esquisses (I). *Diagrammes* 12, Paris, 1984.

[Coppey and Lair, 1988] L. Coppey and C. Lair. Leçons de Théorie des esquisses (II). *Diagrammes* 19, Paris, 1988.

[Duval and Lair, 2001] D. Duval and C. Lair. *Esquisses et spécifications. Manuel de référence, 4ème partie : Fibrations et Eclatements, Lemmes de Yoneda et Modèles Engendrés.* Rapports de recherche du LACO 2001-03, 2001. http://www.unilim.fr/laco/rapports/

[Ehresmann, 1965] C. Ehresmann. *Catégories et structures.* Dunod, 1965.

[Ehresmann, 1966] C. Ehresmann. Introduction to the theory of structured categories. Report 10, University of Kansas, Lawrence, 1966.

[Gabriel and Ulmer, 1971] P. Gabriel and F. Ulmer. *Lokal Präsentierbare Kategorien.* Lecture Notes in Mathematics 221, Springer-Verlag, 1971.

[Gabriel and Zisman, 1967] P. Gabriel and M. Zisman. *Calculus of fractions and homotopy theory.* Springer-Verlag, 1967.

[Goguen and Burstall, 1992] J.A. Goguen and R.M. Burstall. Institutions: abstract model theory for specification and programming. *Journal of the ACM* 39, p. 95-146, 1992.

[Goguen, Thatcher and Wagner, 1976] J.A. Goguen, J.W. Thatcher and E.G. Wagner. *An initial algebra approach to the specification, correctness and implementation of abstract data types.* Technical Report RC 6487, IBM T.J. Watson Research Center, 1976. Reprinted in *Current trends in programming methodology IV*, ed. R. Yeh, Prentice Hall, p. 80-149, 1978.

[Guitart and Lair, 1980] R. Guitart and C. Lair. Calcul syntaxique des modèles et calcul des formules internes.
Diagrammes 4, Paris, p. GL1-GL106, 1980.

[Guitart and Lair, 1982] R. Guitart and C. Lair. Limites et colimites pour représenter les formules. *Diagrammes* 7, Paris, p. GL1-GL24, 1982.

[Hebert, Adamek and Rosicky, 2001] M. Hebert, J. Adamek and J. Rosicky. More on orthogonality in locally presentable categories. *Cahiers de Topologie et Géométrie Différentielle Catégoriques* XLII-1, p. 51-80, 2001.

[Lair, 1987] C. Lair, *Catégories qualifiables et catégories esquissables. Diagrammes* 17, Paris, 1987.

[Mac Lane, 1971] S. Mac Lane. *Categories for the working mathematician.* Springer-Verlag, 1971.

[Martí-Oliet and Meseguer, 1994] Martí-Oliet and Meseguer. General logics and logical framework. In: *What is a logical system?* ed. D.M. Gabbay, Oxford University Press, 1994.