Math. Struct. in Comp. Science (2003), *vol.* 13, *pp.* 857–890.
DOI: 10.1017/S0960129503003979 Printed in the United Kingdom

Diagrammatic specifications

DOMINIQUE DUVAL

Université Joseph Fourier, Laboratoire de Modélisation et Calcul,
B. P. 53, 38041 Grenoble Cedex 9, France
Email: Dominique.Duval@imag.fr

Received 27 February 2002; revised 17 October 2002

This paper presents a simple and powerful diagrammatic framework for dealing with specifications in computer science. Following a classical line, we define diagrammatic specifications as a kind of generalised sketch. In addition, the specifications themselves are defined as the realisations of projective sketches. This meta level provides adjunction properties: this is due to a well-known result of Ehresmann. Moreover, we prove in this paper that this meta level also provides an efficient definition of deduction.
This work results from a collaboration with Christian Lair.

1. Introduction

Many issues in computer science and logic can be considered from a diagrammatic point of view. Indeed, following Lawvere (1963), a theory can be considered as a category with additional structure. Then, a specification can be considered as a directed graph with additional structure.

A *sketch*, as introduced in Ehresmann (1966), is a directed graph with additional features for dealing with projective and inductive limits. Sketches have the same expressive power as infinitary first-order logic (Makkai and Reyes 1977; Guitart and Lair 1982). In order to specify higher-order structures diagrammatically, several generalisations of sketches have been introduced, amongst which are *trames* (Lair 1987), *forms* (Wells 1990), *generalised sketches* (Makkai 1997a; 1997b; 1997c), *nested sketches* (Reichel 1999) and just *sketches* (Kinoshita *et al.* 1999).

Our diagrammatic specifications differ from these by making systematic use of projective sketches at the meta level. A *projective sketch* is a directed graph with additional features for dealing with projective limits only. It has been proved that sketches, as well as trames, are projectively sketchable: this means that the categories of sketches and of trames are *locally presentable categories* (Gabriel and Ulmer 1971). Roughly speaking, a diagrammatic specification is a realisation of any given projective sketch $\mathcal{E}$. In this way, we are able to define and handle specifications very easily. In addition, our *decomposition theorem* (Theorem 3.13) proves that this point of view gives rise to a simple notion of deduction.

Freely generated structures play a fundamental role in mathematics and computer science. For instance, words are freely generated, by concatenation, from an alphabet. Precisely, each set (or alphabet) X freely generates a monoid (the monoid of words over this alphabet) X^*. More technically, there is an omitting functor U from monoids to sets, which takes each monoid to its underlying set, and this functor has a left adjoint F, which takes each set to its freely generated monoid. At the meta level, there is a projective sketch $\mathcal{E}_{\mathcal{S}\mathrm{et}}$ for sets: a set X can be considered as a realisation of $\mathcal{E}_{\mathcal{S}\mathrm{et}}$; there is also a projective sketch $\mathcal{E}_{\mathcal{M}\mathrm{on}}$ for monoids and a homomorphism $P : \mathcal{E}_{\mathcal{S}\mathrm{et}} \to \mathcal{E}_{\mathcal{M}\mathrm{on}}$; the functor U is the omitting functor, which is associated to P.

More generally, freely generated structures occur as soon as there is, at the meta level, a *propagator*, that is, a homomorphism of projective sketches $P : \mathcal{E} \to \overline{\mathcal{E}}$. Indeed, in such a situation, it has been proved in Ehresmann (1967a; 1967b) that the omitting functor U associated to P has a left adjoint F. For instance, for dealing with equational logic, the projective sketch $\mathcal{E}$ describes the syntax of sorts, operation symbols and equations, so a realisation S of $\mathcal{E}$ is an equational specification; the propagator $P : \mathcal{E} \to \overline{\mathcal{E}}$ adds the requirement that the set of equations should form a congruence; so, the freely generated structure $F(S)$ is made of the whole equational theory generated by S. In this context, that is, with respect to this propagator P, it can be said that the theorems are freely generated by the axioms.

In addition, freely generated structures are often built in some *progressive* way. For instance, in order to generate progressively the words on the alphabet $X = \{a, b\}$, we might first generate one word ab, then add it to X, getting $X_1 = \{a, b, ab\}$, and repeat a similar process from X_1. However, for this process to result in the construction of X^*, we have to remember that the string ab in X_1 stands for the concatenation of a and b. This means that X_1 has to be considered not just as a set, but as a partial monoid, with a partially defined concatenation operation that maps the pair (a, b) to the string ab. In this example we have to deal with three different structures: the sets, the monoids, and the partial monoids. In order to describe the progressive construction of X^* from X, we need the adjunction between partial monoids and monoids: the adjunction between sets and monoids is not sharp enough. At the meta level, the propagator $P : \mathcal{E}_{\mathcal{S}\mathrm{et}} \to \mathcal{E}_{\mathcal{M}\mathrm{on}}$ can be decomposed as $J : \mathcal{E}_{\mathcal{S}\mathrm{et}} \to \mathcal{E}_{\mathcal{P}\mathrm{mon}}$ followed by $K : \mathcal{E}_{\mathcal{P}\mathrm{mon}} \to \mathcal{E}_{\mathcal{M}\mathrm{on}}$, where $\mathcal{E}_{\mathcal{P}\mathrm{mon}}$ is a sketch of partial monoids. Clearly, the sets are the partial monoids where the operation is nowhere defined, and the monoids are the partial monoids where the operation is everywhere defined. This means that the omitting functor U_K associated to K is full and faithful, as well as the freely generating functor F_J associated to J. This is an illustration of the main result of this paper, Theorem 3.13, about a decomposition of propagators.

Some knowledge of category theory is assumed in this paper, it can be found in Mac Lane (1971). The paper is organised as follows:

— Section 2 reviews some basic definitions and results for projective sketches;
— Section 3 is devoted to the study of fractioning and filling propagators and to the decomposition theorem;
— Section 4 gives definitions of the notions of specification, domain and model, as well as syntactic entailment, semantic consequence, inference rules and deduction steps;
— Section 5, looks at equational diagrammatic specifications and outlines some links between diagrammatic specifications and institutions;
— Section 6 concludes with a short summary of the main notions of diagrammatic specifications.

From the point of view of terminology, we have made some choices: *point* rather than *object*; *source* and *target* rather than *domain* and *codomain*; and so on. For technical issues, including the size issues, refer to the reference manual Duval and Lair (2001). So, for instance, we speak without taking care about *the category of categories*.

Moreover, in order to maintain the distinction between the specification level and the meta-specification level, we speak on the one hand about *morphisms* and *models* of specifications, but on the other about *propagators* and *realisations* of projective sketches.

All this work results from a collaboration with Christian Lair.

2. Projective sketches, propagators and realisations

This section presents basic notions about *projective sketches*. They stem from Ehresmann's pioneering work (Ehresmann 1966) and can be found, for instance, in Coppey and Lair (1984; 1988), Barr and Wells (1990), and Duval and Lair (2001). The fundamental Theorem (Theorem 2.12), which generalises the *associated sheaf theorem*, appears in Ehresmann (1967a; 1967b).

2.1. *Graphs*

A (*directed*) *graph* is made of points and arrows. A *graph homomorphism* $H : \mathcal{G} \to \mathcal{G}'$ is made of two maps, both denoted H, from the points (respectively, the arrows) of $\mathcal{G}$ to the points (respectively, the arrows) of $\mathcal{G}'$, such that if $g : G_1 \to G_2$, then $H(g) : H(G_1) \to H(G_2)$. A *contravariant graph homomorphism* $H : \mathcal{G} \twoheadrightarrow\!\!\!\!\!\!\times\;\, \mathcal{G}'$ is defined in a similar way, except for the direction of arrows: $H(g) : H(G_2) \to H(G_1)$. An *inclusion* $\mathcal{G} \subseteq \mathcal{G}'$ is a graph homomorphism that is an inclusion both on the sets of points and on the sets of arrows.

A *compositive graph* is a directed graph together with an identity arrow $\mathrm{id}_G : G \to G$ for *some* points G and a composite arrow $g_2 \circ g_1 : G_1 \to G_3$ for *some* pairs of consecutive arrows $(g_1 : G_1 \to G_2, g_2 : G_2 \to G_3)$. A *functor* $H : \mathcal{G} \to \mathcal{G}'$ is a graph homomorphism that preserves identities and composites. A *contravariant functor* $H : \mathcal{G} \twoheadrightarrow\!\!\!\!\!\!\times\;\, \mathcal{G}'$ is a contravariant graph homomorphism that preserves identities and composites. An *inclusion* of compositive graphs is a functor such that its underlying graph homomorphism is an inclusion.

So, a category can be identified with a compositive graph where there is an identity at each point, a composite for each consecutive pair of arrows, and that satisfies the unitarity and associativity properties.

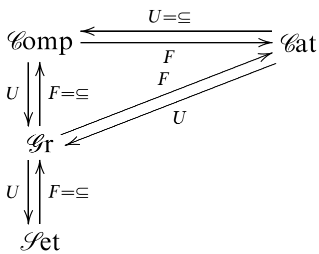
There could be quite a lot of variation in the definition of compositive graphs, which would not greatly influence the rest of the paper. Our choice is 'minimal', in the sense that for any arrow $g : G_1 \to G_2$, even if the identity id_{G_1} exists, it is not assumed that the composite $g \circ \mathrm{id}_{G_1}$ is defined, and even if the composite $g \circ \mathrm{id}_{G_1}$ exists, it is not assumed that it is equal to g.

Example 2.1. Up to some care about size issues, we have the following categories and functors; the functors are described only on points, their value on arrows follows easily.

— $\mathcal{S}$et is the category of sets and maps.

— $\mathcal{G}\mathrm{r}$ is the category of directed graphs and graph homomorphisms.
— Comp is the category of compositive graphs and functors (between compositive graphs).
— $\mathcal{C}\mathrm{at}$ is the category of categories and functors (between categories).
— The functor $U_{\mathcal{S}\mathrm{et},\mathcal{G}\mathrm{r}} = \mathrm{Pt} : \mathcal{G}\mathrm{r} \to \mathcal{S}\mathrm{et}$ maps each graph to its set of points.
— The functor $F_{\mathcal{S}\mathrm{et},\mathcal{G}\mathrm{r}} : \mathcal{S}\mathrm{et} \to \mathcal{G}\mathrm{r}$ maps a set X to the graph with X as its set of points and with no arrow; this functor identifies $\mathcal{S}\mathrm{et}$ with a full subcategory of $\mathcal{G}\mathrm{r}$.
— The functor $U_{\mathcal{G}\mathrm{r},\mathcal{C}\mathrm{omp}} : \mathcal{C}\mathrm{omp} \to \mathcal{G}\mathrm{r}$ maps a compositive graph to its underlying graph.
— The inclusion $F_{\mathcal{G}\mathrm{r},\mathcal{C}\mathrm{omp}} : \mathcal{G}\mathrm{r} \subseteq \mathcal{C}\mathrm{omp}$ maps a graph $\mathcal{G}$ to the compositive graph with $\mathcal{G}$ as its underlying graph and with no identity and no composite; this functor identifies $\mathcal{G}\mathrm{r}$ with a full subcategory of $\mathcal{C}\mathrm{omp}$.
— There is an inclusion $U_{\mathcal{C}\mathrm{omp},\mathcal{C}\mathrm{at}} : \mathcal{C}\mathrm{at} \subseteq \mathcal{C}\mathrm{omp}$, since a category can be considered as a compositive graph with an identity arrow for *each* point and a composite arrow for *each* pair of consecutive arrows, which satisfies the unitarity and associativity properties; this functor identifies $\mathcal{C}\mathrm{at}$ with a full subcategory of $\mathcal{C}\mathrm{omp}$.
— The functor $F_{\mathcal{C}\mathrm{omp},\mathcal{C}\mathrm{at}} : \mathcal{C}\mathrm{omp} \to \mathcal{C}\mathrm{at}$ maps a compositive graph to the category that is obtained by adding the missing identities and composites, and by performing identifications in such a way that the unitarity and associativity properties are satisfied.
— The functor $U_{\mathcal{G}\mathrm{r},\mathcal{C}\mathrm{at}} = U_{\mathcal{G}\mathrm{r},\mathcal{C}\mathrm{omp}} \circ U_{\mathcal{C}\mathrm{omp},\mathcal{C}\mathrm{at}} : \mathcal{C}\mathrm{at} \to \mathcal{G}\mathrm{r}$ maps a category to its underlying graph.
— The functor $F_{\mathcal{G}\mathrm{r},\mathcal{C}\mathrm{at}} = F_{\mathcal{C}\mathrm{omp},\mathcal{C}\mathrm{at}} \circ F_{\mathcal{G}\mathrm{r},\mathcal{C}\mathrm{omp}} : \mathcal{G}\mathrm{r} \to \mathcal{C}\mathrm{at}$ maps a graph $\mathcal{G}$ to the category with the same points as $\mathcal{G}$ and with the paths of $\mathcal{G}$ as arrows (including the empty paths of $\mathcal{G}$, which are the identity arrows of $F_{\mathcal{G}\mathrm{r},\mathcal{C}\mathrm{at}}(\mathcal{G})$).

Each of these four pairs (F, U) is an adjunction. In addition, the inclusion functors $F_{\mathcal{S}\mathrm{et},\mathcal{G}\mathrm{r}}$, $F_{\mathcal{G}\mathrm{r},\mathcal{C}\mathrm{omp}}$ and $U_{\mathcal{C}\mathrm{omp},\mathcal{C}\mathrm{at}}$ are full and faithful.



2.2. *Projective sketches*

Let $\mathcal{I}$ be a compositive graph. An *$\mathcal{I}$-projective cone* C in a compositive graph $\mathcal{G}$ is made up of:

— a functor $B : \mathcal{I} \to \mathcal{G}$ called the *base* of C;
— a point V of $\mathcal{G}$ called the *vertex* of C; and
— arrows $\mathrm{pr}_I : V \to B(I)$ for all point I of $\mathcal{I}$, called the *projections* of C,

such that $i \circ \mathrm{pr}_I = \mathrm{pr}_{I'}$ for all arrows $i : I \to I'$ of $\mathcal{I}$.

Definition 2.2. A *projective sketch* $\mathscr{E}$ consists of a compositive graph $\mathrm{Supp}(\mathscr{E})$, called the *support* of $\mathscr{E}$, together with a set of projective cones in $\mathrm{Supp}(\mathscr{E})$, called the *distinguished projective cones* (or DPCs) of $\mathscr{E}$. A *propagator* $P : \mathscr{E} \to \mathscr{E}'$ is a functor $\mathrm{Supp}(P) : \mathrm{Supp}(\mathscr{E}) \to \mathrm{Supp}(\mathscr{E}')$ that takes the distinguished projective cones of $\mathscr{E}$ to those of $\mathscr{E}'$. An *inclusion* of projective sketches is a propagator such that its support is an inclusion of compositive graphs.

It follows from the definition of the functors of compositive graphs that propagators preserve identities and composites.

Obviously, up to size issues, the projective sketches and their propagators form a category $\mathscr{S}$ketch.

Let $\mathscr{E}$ be a projective sketch.

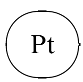
— A *potential isomorphism* is an arrow $e_1 : E_1 \to E_2$ with a potential inverse, that is, such that there are an arrow $e_2 : E_2 \to E_1$, two identities id_{E_1} and id_{E_2}, and two composites $e_1 \circ e_2 = \mathrm{id}_{E_2}$ and $e_2 \circ e_1 = \mathrm{id}_{E_1}$.
— A *potential monomorphism* is an arrow $e_1 : E_1 \to E_2$ such that there is a distinguished projective cone with base $E_1 \xrightarrow{e_1} E_2 \xleftarrow{e_1} E_1$, vertex E_1 and projections $\mathrm{id}_{E_1}, e_1, \mathrm{id}_{E_1}$.
— A *potential factorisation arrow* is an arrow $f : C \to L$ where C and L are projective cones with the same base and L is distinguished, such that $\mathrm{pr}_{C,I} = \mathrm{pr}_{L,I} \circ f$ for all points I of $\mathscr{I}$. It may be denoted $\mathrm{fact}(C, L)$, though it is not uniquely determined by C and L.
— A *potential terminal point* is a point U together with a distinguished projective cone with empty base and vertex U; this is denoted $U = \mathbb{1}$. Then, for each point E of $\mathscr{E}$, there may be potential factorisation arrows $\mathrm{fact}(E, U) : E \to U$.

By adding *distinguished inductive cones*, in a dual way, we get *mixed sketches*, which will not play any important role in this paper. In mixed sketches, we could define potential epimorphisms and potential initial points. The generalisation of this paper to mixed sketches would be far from trivial. It should use results from Guitart and Lair (1980) in order to generalise the freely generated realisation theorem (Theorem 2.12).

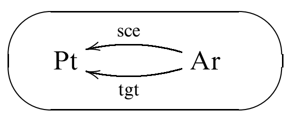
In this paper, we illustrate a projective sketch as its underlying compositive graph, together with the symbols $\cdots\cdots\!>$ for the projections and $\rightarrowtail$ for the potential monomorphisms. There is a lot of ambiguity in such an illustration, which has to come with some additional information about the distinguished projective cones. The representation of composite projections may be omitted.

Example 2.3. Below are two projective sketches $\mathscr{E}_{\mathscr{S}\mathrm{et}}$ and $\mathscr{E}_{\mathscr{G}\mathrm{r}}$ without any distinguished projective cone:

$\mathscr{E}_{\mathscr{S}\mathrm{et}}$:



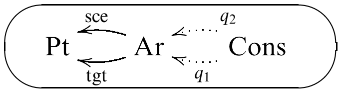
$\mathscr{E}_{\mathscr{G}\mathrm{r}}$:



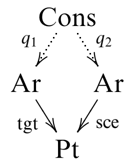
As will be seen in Example 2.7, the names Pt, Ar, sce and tgt stand for *points*, *arrows*, *source* and *target*, respectively.

Below is a projective sketch $\mathscr{E}'_{\mathscr{G}\mathrm{r}}$ with one distinguished projective cone:

$\mathscr{E}'_{\mathscr{G}\mathrm{r}}$:



DPC:



As will be seen in Example 2.7, the name Cons stands for *consecutive arrows*. Clearly, there are the following inclusions:

$$\mathscr{E}_{\mathscr{S}\mathrm{et}} \subseteq \mathscr{E}_{\mathscr{G}\mathrm{r}} \subseteq \mathscr{E}'_{\mathscr{G}\mathrm{r}}.$$

2.3. *Realisations*

Definition 2.4. Let $\mathscr{E}$ be a projective sketch and $\mathscr{A}$ a category. A *realisation* $R : \mathscr{E} \to \mathscr{A}$ *of* $\mathscr{E}$ *with values in* $\mathscr{A}$ is a functor $\mathrm{Supp}(R) : \mathrm{Supp}(\mathscr{E}) \to \mathscr{A}$ that maps each distinguished projective cone in $\mathscr{E}$ to a limit projective cone in $\mathscr{A}$.

So, a realisation of $\mathscr{E}$ maps a potential isomorphism of $\mathscr{E}$ to a (real) isomorphism of $\mathscr{A}$, and a potential monomorphism of $\mathscr{E}$ to a (real) monomorphism of $\mathscr{A}$.

The category $\mathscr{A}$ can be considered as a projective sketch: its support is the underlying compositive graph, and its DPCs are all its projective limit cones (with some care about the size and shape of the indexations of the cones). Thus, a realisation of $\mathscr{E}$ with values in $\mathscr{A}$ is a propagator from $\mathscr{E}$ to the projective sketch $\mathscr{A}$.

Definition 2.5. Let R_1 and R_2 be two realisations of $\mathscr{E}$ with values in $\mathscr{A}$. A *morphism* $\rho : R_1 \to R_2$ is a natural transformation between the underlying functors.

Obviously, the realisations of $\mathscr{E}$ with values in $\mathscr{A}$ and their morphisms form a category $\mathscr{R}\mathrm{eal}(\mathscr{E},\mathscr{A})$. Such a category is a *locally presentable category* (Gabriel and Ulmer 1971). In addition, for each point E of $\mathscr{E}$, there is a functor $\mathrm{ev}_E : \mathscr{R}\mathrm{eal}(\mathscr{E},\mathscr{A}) \to \mathscr{A}$, called the *evaluation at* E, such that $\mathrm{ev}_E(R) = R(E)$ for all realisations, and $\mathrm{ev}_E(\rho) = \rho(E)$ for all morphisms of realisations.

For all propagators $P : \mathscr{E} \to \mathscr{E}'$ there is a functor $\mathscr{R}\mathrm{eal}(P,\mathscr{A}) : \mathscr{R}\mathrm{eal}(\mathscr{E}',\mathscr{A}) \to \mathscr{R}\mathrm{eal}(\mathscr{E},\mathscr{A})$ that maps all realisations R' of $\mathscr{E}'$ to the realisation $R' \circ P$ of $\mathscr{E}$, and all morphisms of realisations $\rho' : R'_1 \to R'_2$ of $\mathscr{E}'$ to the morphism of realisations $\rho' \circ P : R'_1 \circ P \to R'_2 \circ P$ of $\mathscr{E}$. Altogether, we get the following contravariant functor:

$$\mathscr{R}\mathrm{eal}(-,\mathscr{A}) : \mathscr{S}\mathrm{ketch} \twoheadrightarrow\!\!\!\!\!\times\, \mathscr{C}\mathrm{at}.$$

Proposition 2.6. The functor $\mathscr{R}\mathrm{eal}(-,\mathscr{A})$ maps inductive limits to projective limits.

A *contravariant realisation* $Z : \mathscr{E} \twoheadrightarrow\!\!\!\!\!\times\, \mathscr{A}$ *of* $\mathscr{E}$ *with values in* a category $\mathscr{A}$ is a contravariant functor $\mathrm{Supp}(Z) : \mathrm{Supp}(\mathscr{E}) \twoheadrightarrow\!\!\!\!\!\times\, \mathscr{A}$ that maps each distinguished projective cone in $\mathscr{E}$ to a limit inductive cone in $\mathscr{A}$.

Example 2.7. A realisation R of $\mathscr{E}_{\mathscr{S}\mathrm{et}}$ is a set $R(\mathrm{Pt})$, and a morphism $\rho : R_1 \to R_2$ is a map $\rho(\mathrm{Pt}) : R_1(\mathrm{Pt}) \to R_2(\mathrm{Pt})$. So, there is an isomorphim $\mathscr{R}\mathrm{eal}(\mathscr{E}_{\mathscr{S}\mathrm{et}}) \cong \mathscr{S}\mathrm{et}$, where $\mathscr{S}\mathrm{et}$ denotes the category of sets.

A realisation R of $\mathscr{E}_{\mathscr{G}\mathrm{r}}$ is made of two sets $R(\mathrm{Pt})$ and $R(\mathrm{Ar})$, and two maps $R(\mathrm{sce})$ and $R(\mathrm{tgt}) : R(\mathrm{Ar}) \to R(\mathrm{Pt})$: it is a directed graph. And, indeed, there is an isomorphim $\mathscr{R}\mathrm{eal}(\mathscr{E}_{\mathscr{G}\mathrm{r}}) \cong \mathscr{G}\mathrm{r}$, where $\mathscr{G}\mathrm{r}$ denotes the category of directed graphs.

There is an equivalence $\mathscr{R}\mathrm{eal}(\mathscr{E}'_{\mathscr{G}\mathrm{r}}) \simeq \mathscr{G}\mathrm{r}$. Indeed, a realisation R of $\mathscr{E}'_{\mathscr{G}\mathrm{r}}$ is a directed graph, together with a set $R(\mathrm{Cons})$ that is, because of the distinguished projective cones, isomorphic to the set of consecutive arrows of this directed graph.

2.4. *Adjunction between categories*

In this section, we briefly recall the definition and some basic results about adjunction, which are well known and can be found in Mac Lane (1971).

Definition 2.8. Let $\mathscr{A}$ and $\mathscr{A}'$ be categories. An *adjunction from $\mathscr{A}$ to $\mathscr{A}'$* is a pair of functors,

$$(\mathscr{A} \xrightarrow{F} \mathscr{A}' \,,\; \mathscr{A} \xleftarrow{U} \mathscr{A}'),$$

together with, for all points A of $\mathscr{A}$ and A' of $\mathscr{A}'$, a bijection that is natural in A and A':

$$\mathrm{Hom}_{\mathscr{A}}(A, U(A')) \;\cong\; \mathrm{Hom}_{\mathscr{A}'}(F(A), A').$$

This bijection is denoted

$$a \longmapsto a^\star$$
$$a'_\star \longleftarrow\!\!\dashv a'$$

Theorem 2.9 (Adjunction). An adjunction (F, U) from $\mathscr{A}$ to $\mathscr{A}'$ determines two natural transformations,

$$\eta : \mathrm{id}_{\mathscr{A}} \Rightarrow U \circ F : \mathscr{A} \to \mathscr{A} \quad \text{and} \quad \varepsilon : F \circ U \Rightarrow \mathrm{id}_{\mathscr{A}'} : \mathscr{A}' \to \mathscr{A}',$$

such that for all points A of $\mathscr{A}$ and A' of $\mathscr{A}'$, the adjunction bijection maps $a : A \to U(A')$ to $a^\star = \varepsilon_{A'} \circ F(a) : F(A) \to A'$ and $a' : F(A) \to A'$ to $a'_\star = U(a') \circ \eta_A : A \to U(A')$.

Then the natural transformations $\eta : \mathrm{id}_{\mathscr{A}} \Rightarrow U \circ F$ and $\varepsilon : F \circ U \Rightarrow \mathrm{id}_{\mathscr{A}'}$ are the *unit* and the *counit* of the adjunction. The *monad* associated to the adjunction is the triple (M, η, μ) where $M = U \circ F : \mathscr{A} \to \mathscr{A}$ and $\mu = U \circ \varepsilon \circ F : M^2 \Rightarrow M$. Then M is the *endofunctor* and μ is the *multiplication* of the monad.

Theorem 2.10 (Full and faithful functors in adjunctions). Let (F, U) be an adjunction. Then:

— The functor U is full and faithful if and only if ε is a natural isomorphism.
— The functor F is full and faithful if and only if η is a natural isomorphism.

Corollary 2.11. Let (F, U) be an adjunction. If either U or F is full and faithful, then the following natural transformations are natural isomorphisms:

— $\eta \circ U : U \stackrel{\simeq}{\Longrightarrow} U \circ F \circ U$, with inverse $U \circ \varepsilon$;
— $\varepsilon \circ F : F \circ U \circ F \stackrel{\simeq}{\Longrightarrow} F$, with inverse $F \circ \eta$;
— $\mu : M^2 \stackrel{\simeq}{\Longrightarrow} M$, with inverse $\eta \circ M = M \circ \eta$ (this means that the monad (M, η, μ) is idempotent).

2.5. *Adjunction between categories of realisations*

The category of *set-valued realisations* of $\mathscr{E}$ (or just *realisations* of $\mathscr{E}$) is

$$\mathscr{R}\text{eal}(\mathscr{E}) = \mathscr{R}\text{eal}(\mathscr{E}, \mathscr{S}\text{et}) .$$

Up to some care about size issues, the category $\mathscr{R}\text{eal}(\mathscr{E})$ is both complete and cocomplete.

To each propagator $P : \mathscr{E} \to \mathscr{E}'$ is associated the *omitting functor*,

$$U_P = \mathscr{R}\text{eal}(P) : \mathscr{R}\text{eal}(\mathscr{E}') \to \mathscr{R}\text{eal}(\mathscr{E}),$$

which maps a realisation R' of $\mathscr{E}'$ to the underlying realisation $U_P(R') = R' \circ P$ of $\mathscr{E}'$.

The following fundamental result (Ehresmann 1967a; Ehresmann 1967b) generalises the *associated sheaf theorem*; a proof can be found in Duval and Lair (2001). A generalisation of this result to mixed sketches, which is far from trivial, is shown in Guitart and Lair (1980).

Theorem 2.12 (Freely generated realisation). Let $P : \mathscr{E} \to \mathscr{E}'$ be a propagator. The functor $U_P : \mathscr{R}\text{eal}(\mathscr{E}') \to \mathscr{R}\text{eal}(\mathscr{E})$ has a left adjoint

$$F_P : \mathscr{R}\text{eal}(\mathscr{E}) \to \mathscr{R}\text{eal}(\mathscr{E}').$$

The functor F_P is the *freely generating functor* associated to P. From the definition of an adjunction, it follows that, for all realisations R of $\mathscr{E}$ and R' of $\mathscr{E}'$, there is a bijection, which is natural in R and R':

$$\text{Hom}_{\mathscr{R}\text{eal}(\mathscr{E})}(R, U_P(R')) \;\cong\; \text{Hom}_{\mathscr{R}\text{eal}(\mathscr{E}')}(F_P(R), R').$$

The corresponding monad and counit are denoted

$$(M_P : \mathscr{R}\text{eal}(\mathscr{E}) \to \mathscr{R}\text{eal}(\mathscr{E}) \;,\; \eta_P : \text{id}_{\mathscr{R}\text{eal}(\mathscr{E})} \Rightarrow M_P \;,\; \mu_P : M_P^2 \Rightarrow M_P),$$

$$\varepsilon_P : F_P \circ U_P \Rightarrow \text{id}_{\mathscr{R}\text{eal}(\mathscr{E}')} \;,$$

respectively (the subscript P may be omitted).

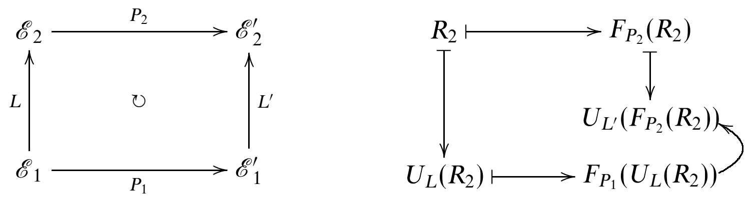
Proposition 2.13. Let $P_1 : \mathscr{E}_1 \to \mathscr{E}'_1$, $P_2 : \mathscr{E}_2 \to \mathscr{E}'_2$, $L : \mathscr{E}_1 \to \mathscr{E}_2$ and $L' : \mathscr{E}'_1 \to \mathscr{E}'_2$ be a commutative square in the category of projective sketches. Then, there is a natural transformation

$$F_{P_1} \circ U_L \Rightarrow U_{L'} \circ F_{P_2} : \mathscr{R}\text{eal}(\mathscr{E}_2) \to \mathscr{R}\text{eal}(\mathscr{E}'_1).$$

This natural transformation is not, in general, a natural isomorphism.

Proof. From the counit $\varepsilon_L : F_L \circ U_L \Rightarrow \text{id}_{\mathscr{R}\text{eal}(\mathscr{E}_2)}$, we get the natural transformation $F_{P_2} \circ \varepsilon_L : F_{P_2} \circ F_L \circ U_L \Rightarrow F_{P_2} : \mathscr{R}\text{eal}(\mathscr{E}_2) \to \mathscr{R}\text{eal}(\mathscr{E}'_2)$. Since $P_2 \circ L = L' \circ P_1$, this can be written as $F_{P_2} \circ \varepsilon_L : F_{L'} \circ F_{P_1} \circ U_L \Rightarrow F_{P_2}$. The result follows by adjunction. □

So, for all realisations R_2 of $\mathscr{E}_2$, there is a morphism $F_{P_1}(U_L(R_2)) \to U_{L'}(F_{P_2}(R_2))$ in $\mathscr{R}\mathrm{eal}(\mathscr{E}'_1)$:



Example 2.14. Let P denote the inclusion $P : \mathscr{E}_{\mathscr{S}\mathrm{et}} \subseteq \mathscr{E}_{\mathscr{G}\mathrm{r}}$. The omitting functor $U_P : \mathscr{R}\mathrm{eal}(\mathscr{E}_{\mathscr{G}\mathrm{r}}) \to \mathscr{R}\mathrm{eal}(\mathscr{E}_{\mathscr{S}\mathrm{et}})$ forgets the arrows. The freely generating functor $F_P : \mathscr{R}\mathrm{eal}(\mathscr{E}_{\mathscr{S}\mathrm{et}}) \to \mathscr{R}\mathrm{eal}(\mathscr{E}_{\mathscr{G}\mathrm{r}})$ is the inclusion functor $\mathscr{S}\mathrm{et} \subseteq \mathscr{G}\mathrm{r}$.

2.6. *Equivalence of sketches.*

The following definition of conservative propagators is semantic: it is relative to the set-valued realisations of the sketches involved.

Definition 2.15. A propagator $Q : \mathscr{E} \to \mathscr{E}'$ is *conservative* if both functors F_Q and U_Q are full and faithful.

From Theorem 2.10, Q is conservative if and only if the unit η_Q and the counit ε_Q are natural isomorphisms.

Definition 2.16. The *equivalence* of projective sketches is the equivalence relation generated by:

— $\mathscr{E} \equiv \mathscr{E}'$ as soon as there is a conservative propagator from $\mathscr{E}$ to $\mathscr{E}'$.

A *zigzag* of propagators $(P_1, \ldots, P_n)$ from $\mathscr{E}$ to $\mathscr{E}'$ is made up of projective sketches $\mathscr{E}_0, \mathscr{E}_1, \ldots, \mathscr{E}_n$ such that $\mathscr{E}_0 = \mathscr{E}$ and $\mathscr{E}_n = \mathscr{E}'$, and of propagators $P_1, \ldots, P_n$ with, for each k from 1 to n, either $P_k : \mathscr{E}_{k-1} \to \mathscr{E}_k$ or $P_k : \mathscr{E}_k \to \mathscr{E}_{k-1}$. Then, clearly, two projective sketches $\mathscr{E}$ and $\mathscr{E}'$ are equivalent if there is a zigzag of conservative propagators from $\mathscr{E}$ to $\mathscr{E}'$.

From Theorem 2.10, if two projective sketches $\mathscr{E}$ and $\mathscr{E}'$ are equivalent, the categories $\mathscr{R}\mathrm{eal}(\mathscr{E})$ and $\mathscr{R}\mathrm{eal}(\mathscr{E}')$ are equivalent: if $\mathscr{E} \equiv \mathscr{E}'$, then $\mathscr{R}\mathrm{eal}(\mathscr{E}) \simeq \mathscr{R}\mathrm{eal}(\mathscr{E}')$.

The following result lists some families of conservative propagators, which can be composed or used in zigzag in order to get equivalences of projective sketches. There are many other ways to get conservative propagators and equivalences of projective sketches.

Proposition 2.17 (Construction of conservative propagators). Let $Q : \mathscr{E} \to \mathscr{E}'$ be a propagator that consists (only) of one of:

— adding an identity loop at a point of $\mathscr{E}$;
— adding a composite for a pair of consecutive arrows of $\mathscr{E}$;
— adding a distinguished projective cone for a base in $\mathscr{E}$;

— adding a potential factorisation arrow, or identifying two potential factorisation arrows, between a projective cone and a distinguished projective cone with the same base, both in $\mathscr{E}$;
— stating that an invertible arrow or an identity arrow is a monomorphic arrow;
— adding a new point E', the identities id_E (if it is not yet in $\mathscr{E}$) and $\mathrm{id}_{E'}$, two arrows $e'_1 : E \to E'$ and $e'_2 : E' \to E$ with the composites $e'_2 \circ e'_1 = \mathrm{id}_E$ and $e'_1 \circ e'_2 = \mathrm{id}_{E'}$;
— mapping a potential isomorphism $e : E_1 \to E_2$, with $E_1 \neq E_2$, to an identity arrow.

Then Q is a conservative propagator.

Proof. This result is easily derived from the properties of the complete category $\mathscr{S}$et. For instance, the image of a point of $\mathscr{E}$ is a point in $\mathscr{S}$et, so it has one identity arrow, and so on. □

On the other hand, a propagator that maps a potential isomorphism $e : E \to E$ to an identity arrow is not conservative, in general. Indeed, let $\mathscr{E}$ be made up of one point E, the identity id_E, and two arrows $e_1, e_2 : E \to E$ with the composites $e_2 \circ e_1 = \mathrm{id}_E$ and $e_1 \circ e_2 = \mathrm{id}_E$. Let $\mathscr{E}'$ be made up of one point E' and the identity $\mathrm{id}_{E'}$, and let $P : \mathscr{E} \to \mathscr{E}'$ be the unique propagator from $\mathscr{E}$ to $\mathscr{E}'$. Now let R be a realisation of $\mathscr{E}$ such that $R(E)$ has two elements x and y, and $R(e_1) = R(e_2)$ permutes x and y. Then $F_P(R)$ identifies x and y, so $M_P(R)(E)$ is made of only one element, and $\eta_{P,R}$ cannot be an isomorphism.

Definition 2.18. The *equivalence* of propagators is the equivalence relation $P \equiv P'$ (where $P : \mathscr{E}_1 \to \mathscr{E}_2$ and $P' : \mathscr{E}'_1 \to \mathscr{E}'_2$) generated by:

— $P \equiv P'$ as soon as $\mathscr{E}_2 = \mathscr{E}'_2$ and there is a conservative propagator $Q_1 : \mathscr{E}_1 \to \mathscr{E}'_1$ such that $P' \circ Q_1 = P$,
— $P \equiv P'$ as soon as $\mathscr{E}_1 = \mathscr{E}'_1$ and there is a conservative propagator $Q_2 : \mathscr{E}_2 \to \mathscr{E}'_2$ such that $Q_2 \circ P = P'$.

If $P \equiv P'$, then, clearly, $\mathscr{E}_1 \equiv \mathscr{E}'_1$ and $\mathscr{E}_2 \equiv \mathscr{E}'_2$.

Example 2.19. The inclusion of $\mathscr{E}_{\mathscr{G}\mathrm{r}}$ in $\mathscr{E}'_{\mathscr{G}\mathrm{r}}$ (from Example 2.3) is a conservative propagator: indeed, it consists of the addition of a distinguished projective cone for a given base. In this way, from the isomorphism $\mathscr{R}\mathrm{eal}(\mathscr{E}_{\mathscr{G}\mathrm{r}}) \cong \mathscr{G}\mathrm{r}$, we get another proof of the equivalence $\mathscr{R}\mathrm{eal}(\mathscr{E}'_{\mathscr{G}\mathrm{r}}) \simeq \mathscr{G}\mathrm{r}$.

2.7. *Prototypes and types*

Definition 2.20. A *projective prototype* is a projective sketch such that its support is a category and its distinguished projective cones are limit cones.

It can be proved that each projective sketch $\mathscr{E}$ freely generates a projective prototype $\mathrm{Proto}(\mathscr{E})$. The unit propagator $\mathscr{E} \to \mathrm{Proto}(\mathscr{E})$ maps each distinguished projective cone of $\mathscr{E}$ to a distinguished limit projective cone of $\mathrm{Proto}(\mathscr{E})$. It follows that

$$\mathscr{R}\mathrm{eal}(\mathrm{Proto}(\mathscr{E})) \cong \mathscr{R}\mathrm{eal}(\mathscr{E}) .$$

Definition 2.21. With respect to some family of compositive graphs for indexations, a *projective type* is a category with chosen projective limit cones: this means that the category is complete, and that for each base, a limit cone is chosen.

A projective type can be considered as a projective prototype, by distinguishing all its chosen projective cones.

It can be proved that each projective sketch $\mathscr{E}$ freely generates a projective type Type($\mathscr{E}$). The unit propagator $\mathscr{E} \to \text{Type}(\mathscr{E})$ maps each distinguished projective cone of $\mathscr{E}$ to a chosen (hence distinguished) limit projective cone of Type($\mathscr{E}$).

The following remark will not be used in the paper. The categories $\mathscr{R}\text{eal}(\text{Type}(\mathscr{E}))$ and $\mathscr{R}\text{eal}(\mathscr{E})$ are not isomorphic, and not even equivalent, in general. However (with respect to some family of indexations), let us choose a projective limit for each base in the category of sets. Then a *strict* (*set-valued*) *realisation* of $\mathscr{E}$ can be defined as a functor from Supp($\mathscr{E}$) to $\mathscr{S}\text{et}$ that maps each distinguished projective cone in $\mathscr{E}$ to a chosen limit projective cone in $\mathscr{S}\text{et}$. The category $\mathscr{R}\text{eal}_{st}(\mathscr{E})$ of strict realisations of $\mathscr{E}$ is then defined in a straightforward way, and, indeed,

$$\mathscr{R}\text{eal}(\text{Type}(\mathscr{E})) \cong \mathscr{R}\text{eal}_{st}(\mathscr{E}) \, .$$

A *regular* projective sketch, in the sense of Ehresmann, is a projective sketch $\mathscr{E}$ such that $\mathscr{R}\text{eal}_{st}(\mathscr{E}) \simeq \mathscr{R}\text{eal}(\mathscr{E})$; then, clearly,

$$\mathscr{R}\text{eal}(\text{Type}(\mathscr{E})) \simeq \mathscr{R}\text{eal}_{st}(\mathscr{E}) \, .$$

Usually, the same notation is used for the points and arrows of $\mathscr{E}$ and their images in Proto($\mathscr{E}$) and in Type($\mathscr{E}$), although the unit propagators $\mathscr{E} \to \text{Proto}(\mathscr{E})$ and $\mathscr{E} \to \text{Type}(\mathscr{E})$ need not be injections.

2.8. *Yoneda lemma for projective sketches*

For all categories $\mathscr{A}$ and $\mathscr{A}'$, up to relevant assumptions about size (the category $\mathscr{A}$ has to be locally small), $\mathscr{F}\text{unc}(\mathscr{A}, \mathscr{A}')$ denotes the category of functors from $\mathscr{A}$ to $\mathscr{A}'$ and natural transformations. The *Yoneda contravariant functor*,

$$Y_{\mathscr{A}} : \mathscr{A} \twoheadrightarrow\!\!\!\!\leftarrow \mathscr{F}\text{unc}(\mathscr{A}, \mathscr{S}\text{et}),$$

which is associated to every category $\mathscr{A}$, is such that:

— $Y_{\mathscr{A}}(A) = \text{Hom}_{\mathscr{A}}(A, -) : \mathscr{A} \to \mathscr{S}\text{et}$ for all points A of $\mathscr{A}$;
— $Y_{\mathscr{A}}(a) = \text{Hom}_{\mathscr{A}}(a, -) : Y_{\mathscr{A}}(A_2) \Rightarrow Y_{\mathscr{A}}(A_1) : \mathscr{A} \to \mathscr{S}\text{et}$ for all arrows $a : A_1 \to A_2$ of $\mathscr{A}$.

Let $\mathbb{I}$ denote a one-element set. Then $X = \text{Hom}_{\mathscr{S}\text{et}}(\mathbb{I}, X)$ for each set X, so the bijection in the Yoneda lemma can be stated as the property of a freely generated structure:

$$\text{Hom}_{\mathscr{S}\text{et}}(\mathbb{I}, \text{ev}_A(H)) \cong \text{Hom}_{\mathscr{F}\text{unc}(\mathscr{A},\mathscr{S}\text{et})}(Y_{\mathscr{A}}(A), H),$$

naturally in H. So, $Y_{\mathscr{A}}(A)$ is free over $\mathbb{I}$ with respect to the functor ev_A (Ehresmann 1965).

Let $\mathscr{E}$ be a projective sketch. Then there is a Yoneda contravariant functor

$$Y_{\text{Proto}(\mathscr{E})} : \text{Proto}(\mathscr{E}) \twoheadrightarrow\!\!\!\!\leftarrow \mathscr{F}\text{unc}(\text{Proto}(\mathscr{E}), \mathscr{S}\text{et}).$$

For all points E of $\mathscr{E}$, the functor $\mathrm{Hom}_{\mathrm{Proto}(\mathscr{E})}(E,-) : \mathrm{Proto}(\mathscr{E}) \to \mathscr{S}\mathrm{et}$ maps projective limits to projective limits. So, the functor $Y_{\mathrm{Proto}(\mathscr{E})}(E) : \mathrm{Proto}(\mathscr{E}) \to \mathscr{S}\mathrm{et}$ takes the projective limit cones of Proto($\mathscr{E}$) to projective limit cones of $\mathscr{S}$et, which means that the image of $Y_{\mathrm{Proto}(\mathscr{E})}$ is contained in $\mathscr{R}$eal(Proto($\mathscr{E}$)):

$$Y_{\mathrm{Proto}(\mathscr{E})} : \mathrm{Proto}(\mathscr{E}) \twoheadrightarrow\!\!\!\!\!\to \mathscr{R}\mathrm{eal}(\mathrm{Proto}(\mathscr{E})).$$

In addition, since $\mathscr{R}$eal(Proto($\mathscr{E}$)) is isomorphic to $\mathscr{R}$eal($\mathscr{E}$), by composition of $Y_{\mathrm{Proto}(\mathscr{E})}$ with the unit propagator $\mathscr{E} \to \mathrm{Proto}(\mathscr{E})$, we get a contravariant functor

$$Y_{\mathscr{E}} : \mathscr{E} \twoheadrightarrow\!\!\!\!\!\to \mathscr{R}\mathrm{eal}(\mathscr{E}).$$

Theorem 2.22 (Yoneda lemma for projective sketches). The Yoneda contravariant functor $Y_{\mathscr{E}} : \mathscr{E} \twoheadrightarrow\!\!\!\!\!\to \mathscr{R}\mathrm{eal}(\mathscr{E})$ is such that, for each point E of $\mathscr{E}$ and each realisation R of $\mathscr{E}$, naturally in E and in R, the map $\rho \mapsto \rho_E(\mathrm{id}_E)$ is a bijection

$$\mathrm{Hom}_{\mathscr{R}\mathrm{eal}(\mathscr{E})}(Y_{\mathscr{E}}(E), R) \xrightarrow{\simeq} R(E).$$

For all set-valued realisations R of $\mathscr{E}$, the contravariant functor $\mathrm{Hom}_{\mathscr{R}\mathrm{eal}(\mathscr{E})}(-,R)$ from $\mathscr{R}$eal($\mathscr{E}$) to $\mathscr{S}$et maps inductive limits to projective limits. Hence, it follows from Theorem 2.22 that the functor $Y_{\mathscr{E}}$ maps distinguished projective cones to limit inductive cones, which leads to the following corollary.

Corollary 2.23. The Yoneda contravariant functor of $\mathscr{E}$ is a contravariant realisation of $\mathscr{E}$.

A consequence of Theorem 2.22 is the *density* result of Corollary 2.23 below: any set-valued realisation of $\mathscr{E}$ is the vertex of an inductive limit cone that has its base in $Y_{\mathscr{E}}(\mathscr{E})$. The description of this cone makes use of a new compositive graph, denoted $\mathrm{Supp}(\mathscr{E})\backslash\mathrm{Supp}(R)$. This compositive graph is built according to the *Grothendieck construction*, as explained below.

Definition 2.24. Let $\mathscr{G}$ be a compositive graph and $H : \mathscr{G} \to \mathscr{S}\mathrm{et}$ a functor. The compositive graph $\mathscr{G}\backslash H$ consists of:

— a point $[G,x]$ for all points G of $\mathscr{G}$ and all $x \in H(G)$;
— an arrow $[g,x] : [G,x] \to [G',x']$ for all arrows $g : G \to G'$ of $\mathscr{G}$ and all $x \in H(G)$, where $x' = H(g)(x)$;
— an identity $\mathrm{id}_{[G,x]} = [\mathrm{id}_G, x]$ for all identities id_G of $\mathscr{G}$ and all $x \in H(G)$;
— a composite $[g_2 \circ g_1, x_1] = [g_2, x_2] \circ [g_1, x_1]$ for all composites $g_2 \circ g_1$ of $\mathscr{G}$ and all x_1 in the source of g_1, where $x_2 = H(g_1)(x_1)$.

Let us write Y for $Y_{\mathscr{E}}$. Let R be a set-valued realisation of $\mathscr{E}$, and let $\mathscr{I}$ denote the compositive graph $(\mathrm{Supp}(\mathscr{E})\backslash\mathrm{Supp}(R))^{op}$. Let C_R denote the $\mathscr{I}$-inductive cone in $\mathscr{R}$eal($\mathscr{E}$) with:

— vertex R;
— base $B : \mathscr{I} \to \mathscr{R}\mathrm{eal}(\mathscr{E})$ such that $B([E,x]) = Y(E)$ for all points $[E,x]$ of $\mathscr{I}$ and $B([e,x]) = Y(e)$ for all arrows $[e,x]$ of $\mathscr{I}$;
— inductions (also called coprojections) $\mathrm{in}_{[E,x]} : Y(E) \to R$ such that for each point E' in $\mathscr{E}$ the map $\mathrm{in}_{[E,x]}(E') : \mathrm{Hom}_{\mathrm{Proto}(\mathscr{E})}(E,E') \to R(E')$ maps e to $R(e)(x)$.

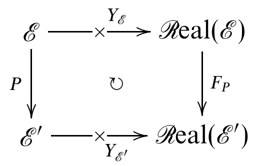
It is easy to check that this is indeed an inductive cone. The density of Yoneda realisation states that it is an inductive limit cone.

Corollary 2.25 (Density of Yoneda realisation). Let R be a realisation of $\mathscr{E}$. Then the inductive cone C_R in $\mathscr{R}\text{eal}(\mathscr{E})$ is a limit cone:

$$R \cong \text{indlim}_{\mathscr{E}\backslash R}(Y_{\mathscr{E}}(E)) \,.$$

The next result relates the freely generated functor and Yoneda contravariant realisations.

Proposition 2.26. Let $P : \mathscr{E} \to \mathscr{E}'$ be a propagator. Then there is an isomorphism of contravariant models of $\mathscr{E}$ with values in $\mathscr{R}\text{eal}(\mathscr{E}')$: $F_P \circ Y_{\mathscr{E}} \cong Y_{\mathscr{E}'} \circ P$.



Proof. Let E be a point of $\mathscr{E}$, and R' be a realisation of $\mathscr{E}'$. Then, from the Yoneda lemma applied to $\mathscr{E}$, $\text{Hom}_{\mathscr{R}\text{eal}(\mathscr{E})}(Y_{\mathscr{E}}(E), U_P(R')) \cong U_P(R')(E) = R'(P(E))$. On the other hand, from the Yoneda lemma applied to $\mathscr{E}'$, $\text{Hom}_{\mathscr{R}\text{eal}(\mathscr{E}')}(Y_{\mathscr{E}'}(P(E)), R') \cong R'(P(E))$. So,

$$\text{Hom}_{\mathscr{R}\text{eal}(\mathscr{E})}(Y_{\mathscr{E}}(E), U_P(R')) \cong \text{Hom}_{\mathscr{R}\text{eal}(\mathscr{E}')}(Y_{\mathscr{E}'} \circ P(E), R') \,,$$

naturally in E and R', which means that $Y_{\mathscr{E}'} \circ P$ is isomorphic to $F_P \circ Y_{\mathscr{E}}$. □

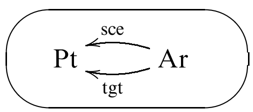
3. Fractioning and filling propagators

In this section, we focus on two families of propagators, the *fractioning* propagators and the *filling* propagators; these words stem from Theorems 3.2 and 3.8, respectively. We prove that any propagator P can be decomposed as $P \equiv K \circ J$ with K fractioning and J filling.

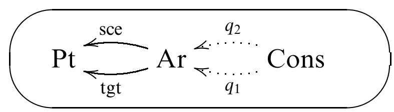
3.1. *A basic example*

Directed graphs. Let $\mathscr{E}_{\mathscr{G}\text{r}}$ and $\mathscr{E}'_{\mathscr{G}\text{r}}$ denote the projective sketches from Example 2.3, such that the inclusion of $\mathscr{E}_{\mathscr{G}\text{r}}$ in $\mathscr{E}'_{\mathscr{G}\text{r}}$ is conservative, $\mathscr{R}\text{eal}(\mathscr{E}_{\mathscr{G}\text{r}}) \cong \mathscr{G}\text{r}$ and $\mathscr{R}\text{eal}(\mathscr{E}'_{\mathscr{G}\text{r}}) \simeq \mathscr{G}\text{r}$.

$\mathscr{E}_{\mathscr{G}\text{r}}$:



$\mathscr{E}'_{\mathscr{G}\text{r}}$:

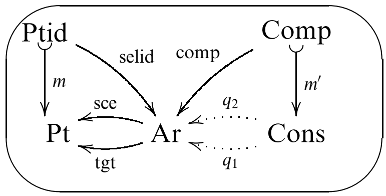


Compositive graphs. Add to $\mathscr{E}'_{\mathscr{G}r}$:

— the two points Comp for the consecutive arrows with a composite and Ptid for the points with an identity;
— the two arrows m : Ptid → Pt and m' : Comp → Cons, which are potential monomorphisms;
— the two arrows selid : Ptid → Ar for the selection of identitie, and comp : Comp → Ar, for the composition; and
— the composites $\text{sce} \circ \text{selid} = m$, $\text{tgt} \circ \text{selid} = m$, $\text{sce} \circ \text{comp} = \text{sce} \circ q_1 \circ m'$, $\text{tgt} \circ \text{comp} = \text{tgt} \circ q_2 \circ m'$ (the required intermediate composites are omitted).

It is easy to check that the resulting projective sketch $\mathscr{E}_{\mathscr{C}omp}$ is such that $\mathscr{R}eal(\mathscr{E}_{\mathscr{C}omp}) \simeq \mathscr{C}omp$.

$\mathscr{E}_{\mathscr{C}omp}$:

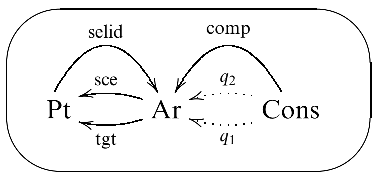


Categories. Add to $\mathscr{E}'_{\mathscr{G}r}$:

— the two arrows selid : Pt → Ar for the selection of identities and comp : Cons → Ar for the composition;
— the composites $\text{sce} \circ \text{selid} = \text{id}_{\text{Pt}}$, $\text{tgt} \circ \text{selid} = \text{id}_{\text{Pt}}$, $\text{sce} \circ \text{comp} = \text{sce} \circ q_1$, $\text{tgt} \circ \text{comp} = \text{tgt} \circ q_2$; and
— whatever is needed to express the unitarity and associativity of categories.

It is easy to check that the resulting projective sketch $\mathscr{E}_{\mathscr{C}at}$ is such that $\mathscr{R}eal(\mathscr{E}_{\mathscr{C}at}) \simeq \mathscr{C}at$. The following illustration does not represent the unitarity and associativity properties:

$\mathscr{E}_{\mathscr{C}at}$:



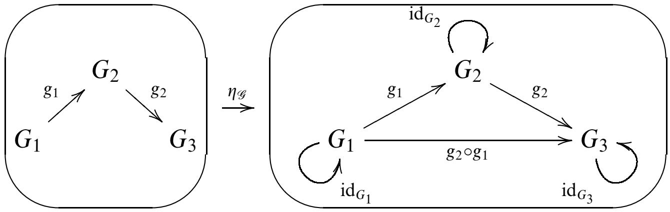
Add to $\mathscr{E}_{\mathscr{C}at}$:

— the two identities id_{Comp} and id_{Pt}; and
— two DPCs such that the identity arrows id_{Comp} and id_{Pt} are potential monomorphisms.

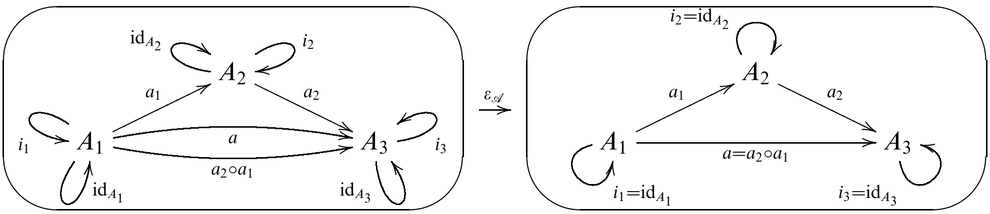
Then the inclusion of $\mathscr{E}_{\mathscr{C}at}$ in $\mathscr{E}'_{\mathscr{C}at}$ is conservative, so $\mathscr{E}_{\mathscr{C}at} \equiv \mathscr{E}'_{\mathscr{C}at}$ and $\mathscr{R}eal(\mathscr{E}'_{\mathscr{C}at}) \simeq \mathscr{C}at$.

Decomposition of P. Let $P : \mathscr{E}_{\mathscr{G}r} \to \mathscr{E}_{\mathscr{C}at}$ be the inclusion, so U_P maps each category to its underlying graph.

Let $\mathscr{G}$ be a graph. Then the graph $U_P(F_P(\mathscr{G}))$ is not isomorphic to $\mathscr{G}$. Indeed, the functor F_P adds the required identities and composites, which are not removed by the functor U_P. So, the unit $\eta_{\mathcal{G}} : \mathcal{G} \to U_P(F_P(\mathcal{G}))$ is far from an isomorphism. For instance:



Let $\mathcal{A}$ be a category. Then the category $F_P(U_P(\mathcal{A}))$ is not isomorphic to $\mathcal{A}$. Indeed, the functor U_P forgets that some arrows are identities or composites. Then, the functor F_P adds to the graph $U_P(\mathcal{A})$ a new copy of these identities or composites. So, the counit $\varepsilon_{\mathcal{A}} : F(U(\mathcal{A})) \to \mathcal{A}$ is far from an isomorphism. For instance:



The propagator $P : \mathcal{E}_{\mathcal{G}\mathrm{r}} \to \mathcal{E}_{\mathcal{C}\mathrm{at}}$ can be composed with the inclusion $\mathcal{E}_{\mathcal{C}\mathrm{at}} \subseteq \mathcal{E}'_{\mathcal{C}\mathrm{at}}$. The resulting propagator $P' : \mathcal{E}_{\mathcal{G}\mathrm{r}} \to \mathcal{E}'_{\mathcal{C}\mathrm{at}}$ is equivalent to P. In addition, it can be decomposed as $P' = K' \circ J$, where $J : \mathcal{E}_{\mathcal{G}\mathrm{r}} \to \mathcal{E}_{\mathcal{C}\mathrm{omp}}$ is the inclusion and $K' : \mathcal{E}_{\mathcal{C}\mathrm{omp}} \to \mathcal{E}'_{\mathcal{C}\mathrm{at}}$ is such that m and m' are mapped to $\mathrm{id}_{\mathrm{Comp}}$ and $\mathrm{id}_{\mathrm{Pt}}$, respectively.

Let $\mathcal{G}$ be a graph. Then the graph $U_J(F_J(\mathcal{G}))$ is isomorphic to $\mathcal{G}$, because the compositive graph $F_J(\mathcal{G})$ has neither identities nor composites.

Let $\mathcal{A}$ be a category. Then, clearly, the category $F_{K'}(U_{K'}(\mathcal{A}))$ is isomorphic to $\mathcal{A}$.

Now, let us return to the propagator $P : \mathcal{E}_{\mathcal{G}\mathrm{r}} \to \mathcal{E}_{\mathcal{C}\mathrm{at}}$ and to the construction of the category $F_P(\mathcal{G})$ that is freely generated by some given graph $\mathcal{G}$. Up to equivalence, we can consider the propagator $P' : \mathcal{E}_{\mathcal{G}\mathrm{r}} \to \mathcal{E}'_{\mathcal{C}\mathrm{at}}$ and build the category $F_{P'}(\mathcal{G})$. The intermediate sketch $\mathcal{E}_{\mathcal{C}\mathrm{omp}}$ can be used in order to get a progressive construction of $F_{P'}(\mathcal{G})$. First, $F_{P'}(\mathcal{G}) = F_{K'}(F_J(\mathcal{G}))$, where $F_J(\mathcal{G})$ is easily obtained: it is $\mathcal{G}$ together with no identity and no composite. So, we can assume that $\mathcal{G}$ is a compositive graph, and look for a progressive construction of $F_{K'}(\mathcal{G})$. If G is a point in $\mathcal{G}$ without an identity, we can build a compositive graph by adding $\mathrm{id}_G : G \to G$. If $g_1 : G_1 \to G_2$ and $g_2 : G_2 \to G_3$ are consecutive arrows in $\mathcal{G}$ without a composite, we can build a compositive graph by adding $g_2 \circ g_1 : G_1 \to G_3$. In both cases, the resulting compositive graph $\mathcal{G}'$ is such that $F_{K'}(\mathcal{G}) = F_{K'}(\mathcal{G}')$, so the construction may start again from $\mathcal{G}'$.

Thus, the composites and identities can be built little by little from a directed graph (where they are nowhere defined) to a category (where they are everywhere defined), thanks to intermediate compositive graphs (where they are partially defined). In the following, we prove that this property of $P : \mathcal{E}_{\mathcal{G}\mathrm{r}} \to \mathcal{E}_{\mathcal{C}\mathrm{at}}$ can be generalised to any propagator.

3.2. *Fractioning propagators*

Definition 3.1. A propagator $K : \mathcal{E} \to \mathcal{E}'$ is *fractioning* if the omitting functor U_K is full and faithful.

From Theorem 2.10, K is fractioning if and only if the counit ε_K is a natural isomorphism:

$$\varepsilon_K : F_K \circ U_K \stackrel{\simeq}{\Longrightarrow} \mathrm{id}_{\mathcal{R}\mathrm{eal}(\mathcal{E}')}.$$

Then, the multiplication μ_K is a natural isomorphism, that is, the monad associated to K is idempotent:

$$\mu_K : M_K^2 \stackrel{\simeq}{\Longrightarrow} M_K.$$

Obviously, a conservative propagator is fractioning, the composite of fractioning propagators is fractioning, and a propagator that is equivalent to a fractioning one is also fractioning.

On the other hand, we say that a propagator $K : \mathcal{E} \to \mathcal{E}'$ adds an inverse to an arrow $e : E_1 \to E_2$ of $\mathcal{E}$ if it adds an arrow $e^{-1} : E_2 \to E_1$, two identities id_{E_1} and id_{E_2}, if they are needed, and two composites $e^{-1} \circ e = \mathrm{id}_{E_1}$ and $e \circ e^{-1} = \mathrm{id}_{E_2}$.

Theorem 3.2 (Fractioning propagators). A propagator is fractioning if and only if, up to equivalence, it consists of adding inverses to arrows.

Proof (*partial*). We only prove here the easy part of this result. A complete proof can be found in Hébert *et al.* (2001), and a similar result in Gabriel and Zisman (1967).

Assume that K adds an inverse to an arrow $e : E_1 \to E_2$ of $\mathcal{E}$. Let R' be a realisation of $\mathcal{E}'$, so the map $R'(e^{-1})$ is the inverse of $R'(e)$. The map $U(R')(e)$ is equal to $R'(e)$, so it is invertible. Hence, $F(U(R'))$ only gives a name to the inverse of $U(R')(e)$, so $\varepsilon(R') : F \circ U(R') \to R'$ is an isomorphism. It follows that K is fractioning, so any propagator that adds inverses to arrows is fractioning. □

Theorem 3.3. A propagator is fractioning if and only if, up to equivalence, it consists in the distinction of projective cones.

Proof. We prove that, up to equivalence, a propagator K consists in adding inverses to arrows if and only if it consists of distinguishing projective cones. So, Theorems 3.2 and 3.3 are equivalent.

Let $e : E_1 \to E_2$ be an arrow in $\mathcal{E}$, and let us distinguish the projective cone with vertex E_1, base E_2 and projection e. Then, up to equivalence, we can add the identity id_{E_2} and a potential factorisation arrow $f = \mathrm{fact}(\mathrm{id}_{E_2}, e) : E_2 \to E_1$, that is, an arrow $f : E_2 \to E_1$ together with the composite $e \circ f = \mathrm{id}_{E_2}$. It follows that, up to adding some composites and identities, $e \circ (f \circ e) = (e \circ f) \circ e = e$, which means that $f \circ e = \mathrm{fact}(e, e)$, and, clearly, $\mathrm{id}_{E_1} = \mathrm{fact}(e, e)$ also, so the identification of $f \circ e$ and id_{E_1} is conservative. So, up to equivalence, f is an inverse of e.

Let C be a projective cone in $\mathcal{E}$ with base B and vertex E_1. Then, up to equivalence, we can add a distinguished projective cone C' with the same base B and some vertex E_2

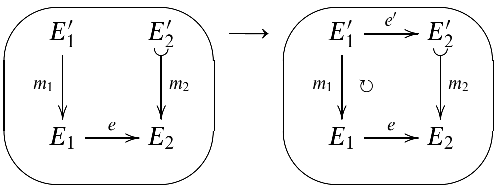
(a new point), and a potential factorisation arrow $e = \mathrm{fact}(C, C') : E_1 \to E_2$. Now we add an inverse e^{-1} to e, and, up to equivalence, we can distinguish the cone C. □

It is also possible to give a direct proof of Theorem 3.3.

Proposition 3.4. A propagator that consists of mapping an arrow to an identity is fractioning.

Proof. Let us assume that $K : \mathscr{E} \to \mathscr{E}'$ maps an arrow $e : E_1 \to E_2$ of $\mathscr{E}$ to an identity $\mathrm{id}_{E'} : E' \to E'$ of $\mathscr{E}'$. Let R' be a realisation of $\mathscr{E}'$, so the map $R'(K(e))$ is the identity of $R'(E')$. The sets $U(R')(E_1)$ and $U(R')(E_2)$ are both equal to $R'(E')$, and the map $U(R')(e)$ is the identity. So, $\varepsilon(R') : F \circ U(R') \to R'$ is an isomorphism. It follows that K is fractioning. □

Let $e : E_1 \to E_2$ be an arrow in a projective sketch $\mathscr{E}$. A propagator $P : \mathscr{E} \to \mathscr{E}'$ adds a *restriction* to e with respect to m_1 and m_2, where $m_1 : E_1' \to E_1$ and $m_2 : E_2' \to E_2$ are arrows of $\mathscr{E}$ and m_2 is a potential monomorphism, if it adds an arrow $e' : E_1' \to E_2'$ with a commutative square $e \circ m_1 = m_2 \circ e'$.



Proposition 3.5. A propagator that consists of adding a restriction to an arrow is fractioning.

Proof. Let us assume that $K : \mathscr{E} \to \mathscr{E}'$ adds a restriction $e' : E_1' \to E_2'$ to an arrow $e : E_1 \to E_2$ with respect to m_1 and m_2. Let R' be a realisation of $\mathscr{E}'$, so the map $R'(K(e')) : R'(K(E_1') \to R'(K(E_2'))$ is the restriction of $R'(K(e))$. It remains true that $U(R')(e) \circ U(R')(m_1) = U(R')(m_2) \circ f$ for some map f. Since the map $U(R')(m_2)$ is injective, the map f is characterised by this equality. So, $F(U(R'))$ only gives the name $F(U(R'))(e')$ to the map f, hence $\varepsilon(R') : F \circ U(R') \to R'$ is an isomorphism. It follows that K is fractioning. □

Example 3.6. In Section 3.1, the propagator $P : \mathscr{E}_{\mathscr{G}\mathrm{r}} \to \mathscr{E}_{\mathscr{C}\mathrm{at}}$ is not fractioning, whereas the propagator $K : \mathscr{E}_{\mathscr{C}\mathrm{omp}} \to \mathscr{E}_{\mathscr{C}\mathrm{at}}$ is fractioning.

3.3. *Filling propagators*

Definition 3.7. A propagator $J : \mathscr{E} \to \mathscr{E}'$ is *filling* if the freely generating functor F_J is full and faithful.

From Theorem 2.10, J is a filling propagator if and only if the unit η_J is a natural isomorphism:

$$\eta_J : \mathrm{id}_{\mathscr{R}\mathrm{eal}(\mathscr{E})} \overset{\simeq}{\Longrightarrow} U_J \circ F_J \; (= M_J).$$

Obviously, a conservative propagator is filling, the composite of filling propagators is filling, and a propagator that is equivalent to a filling one is also filling.

The next result gives a characterisation of filling propagators in terms of their types, as defined in Section 2.7. This result will not be used, or proved, in this paper.

Theorem 3.8 (Filling propagators). A propagator J is filling if and only if the functor that underlies the morphism of projective types $\mathrm{Type}(J)$ is full and faithful.

We now define a notion of *distributor*, which is a variant of the idea defined originally in Bénabou (1973).

Definition 3.9. In this paper, a *distributor* is a propagator $J : \mathscr{E} \to \mathscr{E}'$ that is an inclusion and adds to $\mathscr{E}$:

— a copy of a projective sketch $\widetilde{\mathscr{E}}$ that has no distinguished projective cone with empty base;
— some *transition* arrows from $\widetilde{\mathscr{E}}$ to $\mathscr{E}$, that is, some arrows with their source in $\widetilde{\mathscr{E}}$ and their target in $\mathscr{E}$;
— some *transverse* commutative squares, that is, some commutative squares $\mathrm{tr}' \circ \widetilde{e} = e \circ \mathrm{tr}$, where tr and tr′ are transition arrows, $\widetilde{e}$ is in $\widetilde{\mathscr{E}}$ and e in $\mathscr{E}$; and
— some distinguished *transverse* projective cones, where a transverse projective cone has its vertex in $\widetilde{\mathscr{E}}$, at least a point of its base in $\widetilde{\mathscr{E}}$, and at least a point of its base in $\mathscr{E}$.

Proposition 3.10. A propagator that is equivalent to a distributor is filling.

Proof. Let $J : \mathscr{E} \to \mathscr{E}'$ be a distributor. For each realisation R of $\mathscr{E}$, the realisation $F_J(R)$ of $\mathscr{E}'$ is easy to compute: it coincides with R on $\mathscr{E}$, and $F_J(R)(E') = \emptyset$ for all points E' of $\mathscr{E}'$ that is not in $\mathscr{E}$. It follows immediately that $U_J \circ F_J(R) \xrightarrow{\simeq} R$, so F_J is full and faithful. This proves that a distributor is a filling propagator, and hence the proposition follows. □

In a distributor, the base of a transverse projective cone can be $\widetilde{E} \xrightarrow{\mathrm{tr}} E \xleftarrow{\mathrm{tr}} \widetilde{E}$ for some transition arrow tr, so it is possible to state that some transition arrows are potential monomorphisms.

Proposition 3.11. Let J be a distributor with at least one potential monomorphic transition arrow with source $\widetilde{E}$ for each point $\widetilde{E}$ of $\widetilde{\mathscr{E}}$. Then the omitting functor $U_J : \mathscr{R}\mathrm{eal}(\mathscr{E}') \to \mathscr{R}\mathrm{eal}(\mathscr{E})$ is faithful.

Proof. Let $\rho', \tau' : R'_1 \to R'_2$ be two morphisms of realisations of $\mathscr{E}'$ such that $U(\rho') = U(\tau') : U(R'_1) \to U(R'_2)$. We have to prove that $\rho'(E') = \tau'(E')$ for all points E' of $\mathscr{E}'$.

If E' is a point of $\mathscr{E}$, then $\rho'(E') = U(\rho')(E')$ and $\tau'(E') = U(\tau')(E')$, so $\rho'(E') = \tau'(E')$.

Otherwise, E' is a point of $\widetilde{\mathscr{E}}$, and there is a monomorphic transition arrow $\mathrm{tr} : E' \to E$ for some point E of $\mathscr{E}$. From the naturality of ρ' and τ', we get $R'_2(\mathrm{tr}) \circ \rho'(E') = \rho'(E) \circ R'_1(\mathrm{tr})$ and $R'_2(\mathrm{tr}) \circ \tau'(E') = \tau'(E) \circ R'_1(\mathrm{tr})$. Since $\rho'(E) = \tau'(E)$, we get $R'_2(\mathrm{tr}) \circ \rho'(E') = R'_2(\mathrm{tr}) \circ \tau'(E')$. But $R'_2(\mathrm{tr})$ is a monomorphism, so $\rho'(E') = \tau'(E')$. □

Example 3.12. In Section 3.1, the propagator $P : \mathscr{E}_{\mathscr{G}\mathrm{r}} \to \mathscr{E}_{\mathscr{C}\mathrm{at}}$ is not filling, whereas the propagator $J : \mathscr{E}_{\mathscr{G}\mathrm{r}} \to \mathscr{E}_{\mathscr{C}\mathrm{omp}}$ is filling. Indeed, it is equivalent to $J' : \mathscr{E}'_{\mathscr{G}\mathrm{r}} \to \mathscr{E}_{\mathscr{C}\mathrm{omp}}$, and it is easily checked that J' is a distributor.

3.4. *Decomposition of propagators*

A propagator is, in general, neither fractioning nor filling. The following theorem proves that, up to equivalence, it can be decomposed as a filling propagator followed by a fractioning one. Actually, there are several ways to achieve such a decomposition. One systematic way stems from the proof of the theorem.

Theorem 3.13 (Decomposition of propagators). Let $P : \mathscr{E} \to \overline{\mathscr{E}}$ be a propagator. There are a projective sketch $\mathscr{E}'$, a fractioning propagator $K : \mathscr{E}' \to \overline{\mathscr{E}}$ and a filling propagator $J : \mathscr{E} \to \mathscr{E}'$ such that

$$P \equiv K \circ J.$$

In addition, it can be assumed that J is a distributor.

Proof. Let $J : \mathscr{E} \to \mathscr{E}'$ be the distributor that adds to $\mathscr{E}$:

— a copy of the support $\widetilde{\mathscr{E}} = \mathrm{Supp}(\overline{\mathscr{E}})$ of $\overline{\mathscr{E}}$ (so, $\widetilde{\mathscr{E}}$ is a projective sketch without any distinguished projective cone);
— the transition arrows $\mathrm{tr}_E : \widetilde{E} \to E$ for all points $\widetilde{E}$ of $\widetilde{\mathscr{E}}$ and E of $\mathscr{E}$ such that $P(E) = \widetilde{E}$;
— the transverse commutative squares $\mathrm{tr}_{E_2} \circ \widetilde{e} = e \circ \mathrm{tr}_{E_1}$ for all arrows $\widetilde{e} : \widetilde{E}_1 \to \widetilde{E}_2$ of $\widetilde{\mathscr{E}}$ and $e : E_1 \to E_2$ of $\mathscr{E}$ such that $P(e) = \widetilde{e}$; and
— no distinguished transverse projective cone.

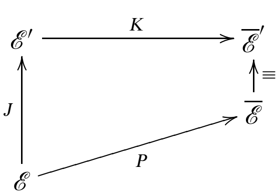
Now, let $\overline{\mathscr{E}}'$ be made up of $\overline{\mathscr{E}}$ together with one identity for each point, so the inclusion $\overline{\mathscr{E}} \subseteq \overline{\mathscr{E}}'$ is an equivalence. Let $K : \mathscr{E}' \to \overline{\mathscr{E}}'$ be the propagator such that:

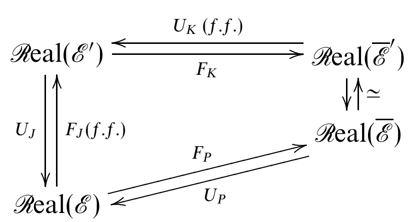
— on $\mathscr{E}$, it coincides with P;
— on $\widetilde{\mathscr{E}}$, it coincides with the inclusion $\mathrm{Supp}(\overline{\mathscr{E}}) \subseteq \overline{\mathscr{E}} \subseteq \overline{\mathscr{E}}'$;
— all transition arrows $\mathrm{tr}_E : \widetilde{E} \to E$ are mapped to $\mathrm{id}_{\widetilde{E}} : \widetilde{E} \to \widetilde{E}$: this is possible since $K(E) = P(E) = \widetilde{E}$ and $K(\widetilde{E}) = \widetilde{E}$.

Thus all transverse commutative squares $\mathrm{tr}_{E_2} \circ \widetilde{e} = e \circ \mathrm{tr}_{E_1}$ are preserved, since both $\mathrm{tr}_{E_2} \circ \widetilde{e}$ and $e \circ \mathrm{tr}_{E_1}$ are mapped to $\widetilde{e}$: indeed $K(e) = P(e) = \widetilde{e}$ and $K(\widetilde{e}) = \widetilde{e}$.

Thus, obviously, $P \equiv K \circ J$.

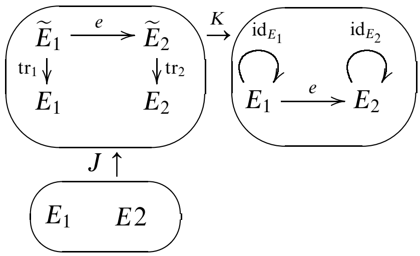
Finally, K can be decomposed as $K = K_2 \circ K_1$, where K_1 maps the transition arrows to identities and K_2 is the distinction of the projective cones of $\overline{\mathscr{E}}$. From Proposition 3.4 and Theorem 3.3, both K_1 and K_2 are fractioning, so K itself is fractioning:



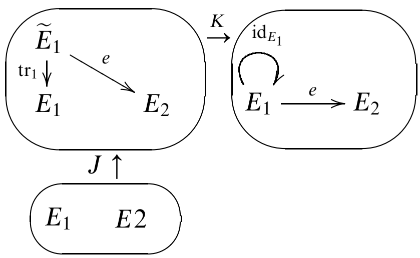


□

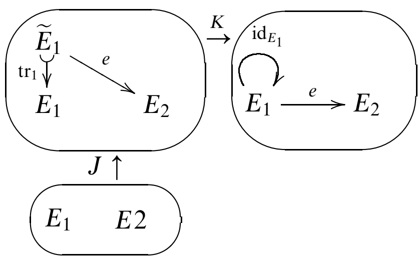
As a basic application of this decomposition theorem, consider the inclusion $P : \mathcal{E} \subseteq \overline{\mathcal{E}}$, where $\mathcal{E}$ is made of two points E_1 and E_2, and where P adds an arrow $e : E_1 \to E_2$. Neither U nor F is full and faithful. According to the proof of Theorem 3.13, the intermediate sketch $\mathcal{E}'$ consists of four points E_1, E_2, $\widetilde{E}_1$ and $\widetilde{E}_2$, an arrow $e : \widetilde{E}_1 \to \widetilde{E}_2$ and two transition arrows $\mathrm{tr}_1 : \widetilde{E}_1 \to E_1$ and $\mathrm{tr}_2 : \widetilde{E}_2 \to E_2$. Thus $P \equiv K \circ J$, where J is the inclusion $\mathcal{E} \subseteq \mathcal{E}'$ and K maps tr_1 and tr_2 to identity loops:



In this example, we could use the following variant. The intermediate sketch $\mathcal{E}'$ is made of three points E_1, $\widetilde{E}_1$ and E_2, two arrows $e : \widetilde{E}_1 \to E_2$ and $\mathrm{tr}_1 : \widetilde{E}_1 \to E_1$. Thus $P \equiv K \circ J$, where J is the inclusion $\mathcal{E} \subseteq \mathcal{E}'$ and K maps tr_1 to an identity loop:



In addition, the arrow tr_1 could be a potential monomorphism. This would mean that in $\mathcal{E}'$ the operation e is partial, and then in $\overline{\mathcal{E}}$ it becomes total:



Example 3.14. In Section 3.1, the propagator $P : \mathcal{E}_{\mathcal{G}\mathrm{r}} \to \mathcal{E}_{\mathcal{C}\mathrm{at}}$ was decomposed as $P \equiv K' \circ J$ with $J : \mathcal{E}_{\mathcal{G}\mathrm{r}} \to \mathcal{E}_{\mathcal{C}\mathrm{omp}}$ filling and $K' : \mathcal{E}_{\mathcal{C}\mathrm{omp}} \to \mathcal{E}'_{\mathcal{C}\mathrm{at}}$ fractioning. This decomposition of P corresponds to the last variant above: both operations comp and selid, which do not occur in $\mathcal{E}_{\mathcal{G}\mathrm{r}}$, are introduced as partial operations in $\mathcal{E}_{\mathcal{C}\mathrm{omp}}$, then they are made total in $\mathcal{E}'_{\mathcal{C}\mathrm{at}}$.

4. Diagrammatic specifications

In this section we define some basic notions related to logic, such as syntactic entailment and semantic consequence, in the general framework of propagators. Some fundamental notions and results are valid only when the propagator is fractioning.

4.1. *Specifications and entailment*

Specifications and entailment are now defined, with respect to any propagator

$$P : \mathscr{E} \to \overline{\mathscr{E}} \,.$$

Definition 4.1. The category of (*diagrammatic*) *specifications with respect to* P, or P-*specifications*, is the category of realisations of $\mathscr{E}$:

$$\mathscr{S}\mathrm{pec}(P) = \mathscr{R}\mathrm{eal}(\mathscr{E}).$$

A P-specification S is *saturated* if the morphism $\eta_{P,S} : S \to M_P(S)$ is an isomorphism.

Definition 4.2. A morphism $\sigma : S \to S'$ of P-specifications is a *syntactic entailment*, which is denoted $S \xrightarrow{\sigma}\!\!\!\!\!\rightarrow S'$, if the morphism $F_P(\sigma)$ is an isomorphism of realisations of $\overline{\mathscr{E}}$:

$$S \xrightarrow{\sigma}\!\!\!\!\!\rightarrow S' \quad \Longleftrightarrow \quad F_P(\sigma) : F_P(S) \xrightarrow{\simeq} F_P(S').$$

Clearly, since $M_P = U_P \circ F_P$, if σ is a syntactic entailment, $M_P(\sigma)$ is an isomorphism of P-specifications:

$$S \xrightarrow{\sigma}\!\!\!\!\!\rightarrow S' \quad \Longrightarrow \quad M_P(\sigma) : M_P(S) \xrightarrow{\simeq} M_P(S').$$

Of course, the definitions of specifications and entailment can be used when the propagator is fractioning. On the other hand, from the decomposition theorem (Theorem 3.13), up to equivalence, all propagators $P : \mathscr{E} \to \overline{\mathscr{E}}$ can be decomposed as $P = K \circ J$, with $K : \mathscr{E}' \to \overline{\mathscr{E}}$ fractioning and $J : \mathscr{E} \to \mathscr{E}'$ filling. Then, each P-specification S freely generates a K-specification $S' = F_J(S)$, which is such that $F_P(S) = F_K(S')$. In addition, from the proof of Theorem 3.13, J can be chosen in such a way that S' is essentially the same as S. Hence, we will now deal with a fractioning propagator

$$K : \mathscr{E} \to \overline{\mathscr{E}} \,.$$

Proposition 4.3. Let $\sigma : S \to S'$ be a morphism of K-specifications. Then $S \xrightarrow{\sigma}\!\!\!\!\!\rightarrow S'$ if and only if $M_K(\sigma)$ is an isomorphism of K-specifications:

$$S \xrightarrow{\sigma}\!\!\!\!\!\rightarrow S' \quad \Longleftrightarrow \quad M_K(\sigma) : M_K(S) \xrightarrow{\simeq} M_K(S').$$

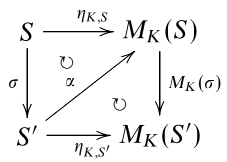
Proof. It has been noted that, for all propagators P, if $S \xrightarrow{\sigma}\!\!\!\!\!\rightarrow S'$, then $M_P(\sigma)$ is an isomorphism. We have to prove that, when K is a fractioning propagator, if $M_K(\sigma)$ is an isomorphism, $F_K(\sigma)$ is also an isomorphism. Since K is fractioning, the functor U_K is full and faithful. So, if a morphism $\delta : D \to D'$ of realisations of $\overline{\mathscr{E}}$ is such that $U_K(\delta)$ is an isomorphism, then δ itself is an isomorphism. This can be applied to $\delta = F_K(\sigma)$, proving the result. □

Proposition 4.4. For all K-specification S, the K-specification $M_K(S)$ is saturated and the morphism $\eta_{K,S}$ is an entailment: $S \overset{\eta_{K,S}}{\twoheadrightarrow} M_K(S)$.

Proof. Since U_K is full and faithful, the natural transformation $\mu_K : M_K^2 \to M_K$ is a natural isomorphism, with inverse $\eta_K \circ M_K = M_K \circ \eta_K : M_K \to M_K^2$. So, on the one hand, $\eta_{K,M_K(S)}$ is an isomorphism of K-specifications, which proves that $M_K(S)$ is saturated. But, on the other hand, $M_K(\eta_{K,S})$ is an isomorphism of K-specifications, which proves, because of Proposition 4.3, that $\eta_{K,S}$ is an entailment. □

Proposition 4.5. Let $\sigma : S \to S'$ be a morphism of K-specifications. Then $S \overset{\sigma}{\twoheadrightarrow} S'$ if and only if there is a morphism of K-specifications $\alpha : S' \to M_K(S)$ such that $\alpha \circ \sigma = \eta_{K,S}$ and $M_K(\sigma) \circ \alpha = \eta_{K,S'}$. In such a case, $\alpha = M_K(\sigma)^{-1} \circ \eta_{S'}$ and $M_K(\sigma)^{-1} = \mu_{K,S} \circ M_K(\alpha)$.

The condition in the proposition means that the commutative square $\eta_{K,S'} \circ \sigma = M_K(\sigma) \circ \eta_{K,S}$ is *split*:



Proof. Let $S \overset{\sigma}{\twoheadrightarrow} S'$, so $M_K(\sigma)$ is an isomorphism, by Proposition 4.3. Let $\alpha = M_K(\sigma)^{-1} \circ \eta_{K,S'} : S' \to M_K(S)$. Then

$$\alpha \circ \sigma = M_K(\sigma)^{-1} \circ \eta_{K,S'} \circ \sigma = M_K(\sigma)^{-1} \circ M_K(\sigma) \circ \eta_{K,S} = \eta_{K,S},$$

and

$$M(\sigma) \circ \alpha = M(\sigma) \circ (M(\sigma))^{-1} \circ \eta_{S'} = \eta_{S'}.$$

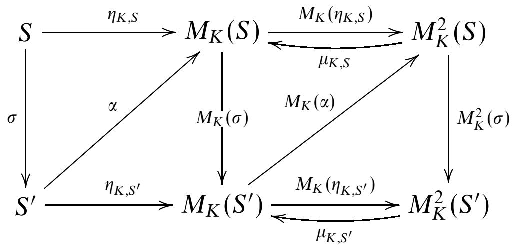
On the other hand, let $\alpha : S' \to M_K(S)$ be such that $\alpha \circ \sigma = \eta_{K,S}$ and $M_K(\sigma) \circ \alpha = \eta_{K,S'}$. We will prove that $\mu_{K,S} \circ M_K(\alpha) : M_K(S') \to M_K(S)$ is an inverse of $M_K(\sigma)$. Since K is fractioning, it follows from Corollary 2.11 that the monad M_K is idempotent, which means that μ_K is a natural isomorphism with inverse $M_K \circ \eta_K$. From $\alpha \circ \sigma = \eta_{K,S}$, we get

$$\mu_{K,S} \circ M_K(\alpha) \circ M_K(\sigma) = \mu_{K,S} \circ M_K(\alpha \circ \sigma) = \mu_{K,S} \circ M_K(\eta_S) = \mathrm{id}_{M_K(S)}.$$

From $M_K(\sigma) \circ \alpha = \eta_{K,S'}$, we get $M_K^2(\sigma) \circ M_K(\alpha) = M_K(\eta_{K,S'})$, so (thanks to the naturality of μ_K)

$$M_K(\sigma) \circ \mu_{K,S} \circ M_K(\alpha) = \mu_{K,S'} \circ M_K^2(\sigma) \circ M_K(\alpha) = \mu_{K,S'} \circ M_K(\eta_{K,S'}) = \mathrm{id}_{M_K(S')}.$$

So, $M_K(\sigma)$ is an isomorphism, with inverse $\mu_{K,S} \circ M_K(\alpha)$.



□

From Theorem 3.2, up to equivalence of sketches, we can assume that K adds inverses to arrows. So, any arrow of $\overline{\mathscr{E}}$ that is not in $\mathscr{E}$ is the inverse of an arrow of $\mathscr{E}$.

Definition 4.6. Let K add inverses to arrows. An *inference rule with respect to* K is an arrow $r : H \to C$ in $\overline{\mathscr{E}}$. The point H is the *hypothesis* of the rule r and the point C is its *conclusion.* An inference rule $r : H \to C$ is *passive* if r is an arrow of $\mathscr{E}$, otherwise it is *active.*

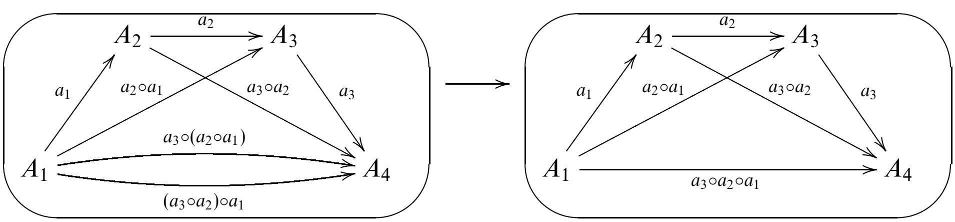
An inference rule $r : H \to C$ can be written as $\frac{H}{C}(r)$, or simply as $\frac{H}{C}$. Inference rules can be composed, as arrows in $\overline{\mathscr{E}}$.

The Yoneda contravariant realisation $Y_{\mathscr{E}}$ of $\mathscr{E}$ yields illustrations for active inference rules. Indeed, let $r : H \to C$ be an active inference rule, and let $e : C \to H$ be the arrow of $\mathscr{E}$ such that $r = e^{-1}$. The image of $e : C \to H$ by $Y_{\mathscr{E}}$ is a morphism of realisations of $\mathscr{E}$:

$$Y_{\mathscr{E}}(e) : Y_{\mathscr{E}}(H) \to Y_{\mathscr{E}}(C).$$

Since the Yoneda realisation is contravariant, the source and target of the morphism $Y_{\mathscr{E}}(e)$ are (the images of) the hypothesis and the conclusion, respectively, of the rule r. The image of the morphism $Y_{\mathscr{E}}(e)$ by F_K satisfies $F_K(Y_{\mathscr{E}}(e)) = Y_{\overline{\mathscr{E}}}(K(e)) = Y_{\overline{\mathscr{E}}}(e)$, by Proposition 2.26 and because K is an inclusion. Since e is invertible in $\overline{\mathscr{E}}$, this is an isomorphism. So, the rule $r : H \to C$ is such that $Y_{\mathscr{E}}(e) : Y_{\mathscr{E}}(H) \to Y_{\mathscr{E}}(C)$ becomes an isomorphism, by applying F_K.

Example 4.7. Consider the fractioning propagator $K' = \mathscr{E}_{\mathscr{C}\mathrm{omp}} \to \mathscr{E}'_{\mathscr{C}\mathrm{at}}$ from Section 3.1. The associativity property of the composition of arrows is one of the properties of categories that is not satisfied by compositive graphs. This corresponds, up to equivalence, to the inversion of an arrow $e : C \to H$ by the propagator K'. The inverse arrow $r = e^{-1} : H \to C$ is an active inference rule, which can be illustrated by the functor of compositive graphs $Y_{\mathscr{E}}(e) : Y_{\mathscr{E}}(H) \to Y_{\mathscr{E}}(C)$:



Of course, e can be described directly in $\mathscr{E}_{\mathscr{C}\mathrm{omp}}$ (more precisely in some $\mathscr{E}'_{\mathscr{C}\mathrm{omp}}$ equivalent to $\mathscr{E}_{\mathscr{C}\mathrm{omp}}$) without any use of the Yoneda contravariant realisation. We will outline this description now; it is more complicated than the illustration *via* Yoneda. The hypothesis H is the vertex of a distinguished projective cone with its base B_H in $\mathscr{E}_{\mathscr{C}\mathrm{omp}}$; the indexation $\mathscr{I}_H$ of this cone is a compositive graph made of fifteen points, and quite a lot of arrows; the base B_H maps the fifteen points of $\mathscr{I}_H$ to four copies of Pt, seven copies of Ar, four copies of Cons, and the arrows of $\mathscr{I}_H$ to copies of sce, tgt, comp, projections ... Similarly, the conclusion C is the vertex of a distinguished projective cone with its indexation made of fourteen points and many arrows, which are mapped to four copies of Pt, six copies of Ar, four copies of Cons, and copies of sce, tgt, comp, projections ... The arrow $e : C \to H$ is the obvious factorisation arrow. The interpretation $\mathscr{G}(H)$ of H in a compositive graph $\mathscr{G}$ is the set of consecutive triples of arrows (a_1, a_2, a_3) in $\mathscr{G}$ such that $(a_3 \circ a_2) \circ a_1$ and $a_3 \circ (a_2 \circ a_1)$ exist in $\mathscr{G}$. The interpretation $\mathscr{G}(C)$ of C in $\mathscr{G}$ is the set of consecutive triples of arrows (a_1, a_2, a_3) such that $(a_3 \circ a_2) \circ a_1$ and $a_3 \circ (a_2 \circ a_1)$ exist in $\mathscr{G}$ and are equal. The interpretation $\mathscr{G}(e)$ of e in $\mathscr{G}$ is the inclusion of $\mathscr{G}(C)$ in $\mathscr{G}(H)$. The associativity property holds whenever this inclusion is an equality.

4.2. *Syntactic deduction steps*

In this section, we define syntactic deduction steps, with respect to a fractioning propagator

$$K : \mathscr{E} \to \overline{\mathscr{E}} ,$$

and prove that deduction steps result in syntactic deductions.

Let $r = e^{-1} : H \to C$ be an active inference rule. Let S be a K-specification and $x \in S(H)$. The inverse image of x by $S(e)$ can be any subset $(S(e))^{-1}(x)$ of $S(C)$. However, when S is saturated, $(S(e))^{-1}(x)$ consists of exactly one element y of $S(C)$. We now define the 'simplest' morphism $\sigma : S \to S'$ with source S such that, if $x' = \sigma(H)(x)$, the inverse image of x' by $S'(e)$ is made of exactly *one* element y' of $S'(C)$.

To this end, let $P : \mathscr{E} \to \mathscr{E}_1$ be the inclusion that adds points H_1 and C_1, arrows $h : H_1 \to H$, $c : C_1 \to C$ and $e_1 : C_1 \to H_1$, and two distinguished projective cones; the first means that H_1 is a potential terminal point, that is, $H_1 = \mathbb{1}$, and the second is a pullback:

$$\begin{array}{ccc} H_1 & \qquad & C_1 \\ & & c \swarrow \qquad \searrow e_1 \\ \text{(empty base)} & & C \xrightarrow{e} H \xleftarrow{h} H_1 \end{array}$$

The set-valued realisations of $\mathscr{E}_1$ are, up to isomorphism, the pairs $S_1 = (S, x)$ where S is a set-valued realisation of $\mathscr{E}$ and x is an element of $S(H)$. Then $S_1(e) = S(e)$, $S_1(H_1) = \{x\}$ and $S_1(C_1) = (S(e))^{-1}(x)$. Moreover, $S = U_P(S_1)$.

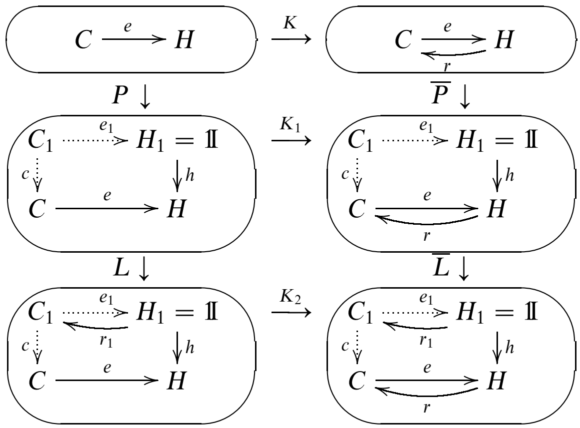
Now, let $L : \mathscr{E}_1 \to \mathscr{E}_2$ denote the fractioning propagator that adds an inverse $r_1 : H_1 \to C_1$ to e_1.

Any set-valued realisation S_2 of $\mathscr{E}_2$ is such that, up to isomorphism, $S_2(H_1) = \{x\}$ and $S_2(C_1) = \{y\}$ for some $x \in S_2(H)$ and $y \in S_2(C)$ with $(S_2(e))^{-1}(x) = \{y\}$.

Let $\overline{P} : \overline{\mathscr{E}} \to \overline{\mathscr{E}}_1$ and $\overline{L} : \overline{\mathscr{E}}_1 \to \overline{\mathscr{E}}_2$ be obtained by similar constructions from $\overline{\mathscr{E}}$. Then clearly $\overline{L}$ is conservative, the inclusions $K_1 : \mathscr{E}_1 \to \overline{\mathscr{E}}_1$ and $K_2 : \mathscr{E}_2 \to \overline{\mathscr{E}}_2$ are fractioning and the following squares are pushouts:

$$\begin{array}{ccc}
\mathscr{E} & \xrightarrow{K} & \overline{\mathscr{E}} \\
{\scriptstyle P}\downarrow & & \downarrow{\scriptstyle \overline{P}} \\
\mathscr{E}_1 & \xrightarrow{K_1} & \overline{\mathscr{E}}_1 \\
{\scriptstyle L}\downarrow & & \downarrow{\scriptstyle \overline{L}} \\
\mathscr{E}_2 & \xrightarrow{K_2} & \overline{\mathscr{E}}_2
\end{array}$$

When $\mathscr{E}$ contains only $C \xrightarrow{e} H$, this can be illustrated as follows:



Let $S_1 = (S, x)$ be a K_1-specification, and let $S_1' = M_L(S_1)$ and $S' = U_P(S_1')$. Then, from $\sigma_1 = \eta_{L,S_1} : S_1 \to S_1'$, we get a morphism of K-specifications $\sigma = U_P(\sigma_1) : S \to S'$.

Let $x' = \sigma(H)(x) \in S'(H)$. Then the inverse image of x' by $S'(e)$ consists of one point: namely, $(S'(e))^{-1}(x') = \{y'\}$ where $y' = S'(r_1)(x') \in S'(C)$. On the other hand, if $(S(e))^{-1}(x)$ consists of one point, we have $\sigma = \mathrm{id}_S : S \to S$.

Definition 4.8. Let $r : H \to C$ be an inference rule with respect to K, S a K-specification and x an element of $S(H)$. The *deduction step with respect to* K associated to r, S and x is the following morphism of K-specifications with source S:

— If r is a passive inference rule, it is the identity morphism $\mathrm{id}_S : S \to S$.
— If $r = e^{-1}$ is an active inference rule, with the notations above, it is the morphism

$$\sigma : S \to S' \quad = \quad U_P(\eta_{L,(S,x)}) : S \to U_P(M_L(S,x)) \, .$$

Theorem 4.9. Let $\sigma : S \to S'$ be a deduction step. Then it is a syntactic entailment: $S \overset{\sigma}{\longrightarrow\!\!\!\!\!\!\!\rightarrow} S'$.

Proof. Let $\sigma : S \to S'$ be the deduction step associated to the rule $r : H \to C$, the K-specification S and $x \in S(H)$. Let us prove that $F_K(\sigma) : F_K(S) \to F_K(S')$ is an isomorphism. If r is passive, σ is the identity, so $F_K(\sigma)$ is the identity. Now, let us assume that r is active.

With the notation as above, $\sigma = U_P(\sigma_1)$ where $\sigma_1 = \eta_{L,S_1} : S_1 \to M_L(S_1)$, and we have to prove that $F_K(U_P(\sigma_1))$ is an isomorphism.

Since L is fractioning, according to Corollary 2.11, $F_L(\sigma_1) = F_L(\eta_{L,S_1})$ is an isomorphism. It follows that $F_{K_2}(F_L(\sigma_1))$ is also an isomorphism. Since $K_2 \circ L = \overline{L} \circ K_1$, this means that $F_{\overline{L}}(F_{K_1}(\sigma_1))$ is an isomorphism. And since $\overline{L}$ is conservative, it follows that $F_{K_1}(\sigma_1)$ is an isomorphism, so $U_{\overline{P}}(F_{K_1}(\sigma_1))$ is also an isomorphism.

In this situation, the natural transformation $F_K \circ U_P \Rightarrow U_{\overline{P}} \circ F_{K_1}$ from Proposition 2.13 is a natural isomorphism. Indeed, naturally in S_1, if $S_1 = (S, x)$, then $F_{K_1}(S_1) \cong (F_K(S), \widetilde{x})$, where $\widetilde{x} = \eta_{K,S}(x) \in M_K(S)(H)$, and $M_K(S)(H) = F_K(S)(H)$, since K is an inclusion. So, $F_K(U_P(\sigma_1))$ is indeed an isomorphism.

$$
\begin{array}{ccccc}
\sigma = U_P(\sigma_1) & \overset{F_K}{\longmapsto} & F_K(U_P(\sigma_1)) & \cong & U_{\overline{P}}(F_{K_1}(\sigma_1)) \\
\uparrow{\scriptstyle U_P} & & & & \uparrow{\scriptstyle U_{\overline{P}}} \\
\sigma_1 & & \overset{F_{K_1}}{\longmapsto} & & F_{K_1}(\sigma_1) \\
\downarrow{\scriptstyle F_L} & & & & \uparrow{\scriptstyle F_{\overline{L}}^{-1}} \\
F_L(\sigma_1) & \overset{F_{K_2}}{\longmapsto} & F_{K_2}(F_L(\sigma_1)) & \cong & F_{\overline{L}}(F_{K_1}(\sigma_1))
\end{array}
$$

□

It follows that any finite composition of deduction steps is a syntactic entailment. In the opposite direction, it could be proved that, under some 'reasonable' assumptions (essentially, finiteness assumptions) about K, S and S', all syntactic entailment can be obtained from deduction steps by a countable induction process.

4.3. *Domains, models and consequence*

Domains, models and consequence are now defined with respect to any propagator:

$$P : \mathcal{E} \to \overline{\mathcal{E}} .$$

Definition 4.10. The category of (*diagrammatic*) *domains with respect to* P, or P-*domains*, is the category of realisations of $\overline{\mathcal{E}}$:

$$\mathcal{D}\text{om}(P) = \mathcal{R}\text{eal}(\overline{\mathcal{E}}).$$

Definition 4.11. Let S be a P-specification and D a P-domain. The *models of* S *with values in* D are the morphisms from $F_P(S)$ to D in $\mathcal{R}\text{eal}(\overline{\mathcal{E}})$:

$$\text{Mod}_P(S, D) = \text{Hom}_{\mathcal{R}\text{eal}(\overline{\mathcal{E}})}(F_P(S), D).$$

This definition is natural in both S (in a contravariant way) and D. Clearly, Kleisli categories could be invoked here (Mac Lane 1971). It follows from the generated realisation theorem (Theorem 2.12) that the models of S with values in D can be identified with the morphisms from S to $U_P(D)$ in $\mathcal{R}\text{eal}(\mathcal{E})$:

$$\text{Mod}_P(S, D) \cong \text{Hom}_{\mathcal{R}\text{eal}(\mathcal{E})}(S, U_P(D)).$$

So, from the definition of morphisms in Section 2.3, a model ω of S with values in D can be identified with a natural transformation between the functors that underlie S and $U_P(D)$: it consists of a map $\omega_E : S(E) \to D(P(E))$ for each point E of $\mathscr{E}$, naturally in E.

Let $\sigma : S \to S'$ be a morphism of P-specifications. Then, for all models ω' of S' with values in D, the morphism $\omega' \circ F_P(\sigma) : F_P(S) \to D$ is a model of S with values in D.

In such a general setting, there is no canonical notion of morphism of models, hence no category of models of S with values in D. However, in many important special cases, there is such a category of models; and then the contravariant functor of models is denoted

$$\mathscr{M}\mathrm{od}_P(-, D) : \mathscr{S}\mathrm{pec}(P) \twoheadrightarrow \mathscr{C}\mathrm{at}.$$

Definition 4.12. Let D be a P-domain. A morphism $\sigma : S \to S'$ of P-specifications is a *semantic consequence with respect to* D, which is denoted $S \xrightarrow{\sigma}\!\!\!\!\twoheadrightarrow_D S'$, if the map $\mathrm{Mod}_P(\sigma, D)$ is a bijection:

$$S \xrightarrow{\sigma}\!\!\!\!\twoheadrightarrow_D S' \quad \text{if and only if} \quad \mathrm{Mod}_P(\sigma, D) : \mathrm{Mod}_P(S', D) \xrightarrow{\simeq} \mathrm{Mod}_P(S, D).$$

Now let us assume that the propagator is fractioning:

$$K : \mathscr{E} \to \overline{\mathscr{E}}.$$

Then the following result can be expressed as 'the models of a theory are the models of its axioms' (Lalement 1990).

Proposition 4.13. For all K-specification S and all K-domain D, the morphism $\eta_{K,S}$ is a consequence: $S \xrightarrow{\eta_{K,S}}\!\!\!\!\twoheadrightarrow_D M_K(S)$.

Proof. We have to prove that $\mathrm{Mod}_K(\eta_{K,S}, D)$ is a bijection from $\mathrm{Mod}_K(M_K(S), D)$ to $\mathrm{Mod}_K(S, D)$. From Corollary 2.11, since U_K is full and faithful, $F_K \circ \eta_K$ is a natural isomorphism $F_K \circ \eta_K : F_K \stackrel{\simeq}{\Longrightarrow} F_K \circ U_K \circ F_K$. So, the map $\mathrm{Hom}_{\mathscr{R}\mathrm{eal}(\overline{\mathscr{E}})}(F_K(\eta_{K,S}), D)$, that is, the map $\mathrm{Mod}_K(\eta_{K,S}, D)$, is a bijection. □

So, altogether, when K is a fractioning propagator, the morphism $\eta_{K,S} : S \to M_K(S)$ is both an entailment (Proposition 4.4) and a consequence with respect to any K-domain (Proposition 4.13).

4.4. *Soundness*

Let $P : \mathscr{E} \to \overline{\mathscr{E}}$ be any propagator. Entailment and consequence are related by the following result, which is easily derived from the properties of adjunction.

Theorem 4.14 (Soundness). A morphism of P-specifications $\sigma : S \to S'$ is a syntactic entailment if and only if it is a semantic consequence with respect to all P-domains D:

$$S \xrightarrow{\sigma}\!\!\!\twoheadrightarrow S' \quad \Longleftrightarrow \quad \text{for all } D,\ S \xrightarrow{\sigma}\!\!\!\!\twoheadrightarrow_D S'.$$

Proof. First, let us assume that $S \xrightarrow{\sigma}\!\!\!\twoheadrightarrow S'$. This means that $F_P(\sigma)$ is an isomorphism, hence for all domains D the map $\mathrm{Hom}_{\overline{\mathscr{E}}}(F_P(\sigma), D)$ is a bijection. But this map is equal to $\mathrm{Mod}_P(\sigma, D)$, so $S \xrightarrow{\sigma}\!\!\!\!\twoheadrightarrow_D S'$.

Now let us assume that $S \xrightarrow{\sigma}\!\!\!\!\!\twoheadrightarrow_D S'$ for all P-domains D. This means that the map $\mathrm{Mod}_P(\sigma, D) : \mathrm{Mod}_P(S', D) \to \mathrm{Mod}_P(S, D)$ is a bijection for all P-domains D. Let Hom stand for $\mathrm{Hom}_{\mathscr{R}\mathrm{eal}(\overline{\mathscr{E}})}$. From the definition of models, this means that the map $\mathrm{Hom}(F_P(\sigma), D)$, such that $\delta \mapsto \delta \circ F_P(\sigma)$, is a bijection for all domains D.

So, when $D = F_P(S)$, the map $\delta \mapsto \delta \circ F_P(\sigma)$ is a bijection; hence, there is a unique morphism $\tau : F_P(S') \to F_P(S)$ such that $\tau \circ F_P(\sigma) = \mathrm{id}_{F_P(S)}$.

Now, when $D = F_P(S')$, the map $\delta \mapsto \delta \circ F_P(\sigma)$ is a bijection. This map is such that $F_P(\sigma) \circ \tau \mapsto F_P(\sigma) \circ \tau \circ F_P(\sigma)$, which is equal to $F_P(\sigma)$, since $\tau \circ F_P(\sigma) = \mathrm{id}_{F_P(S)}$. But, clearly, $\mathrm{id}_{F_P(S')} \mapsto F_P(\sigma)$, so $F_P(\sigma) \circ \tau = \mathrm{id}_{F_P(S')}$.

Altogether, $F_P(\sigma)$ is an isomorphism with inverse τ, so $S \xrightarrow{\sigma}\!\!\!\!\twoheadrightarrow S'$. □

The direct part of this theorem is the *soundness* property. The inverse part is not the *completeness* property: indeed, the completeness would mean that a consequence with respect to one (well chosen) P-domain is an entailment. Quite often, for instance, this P-domain is some 'domain of sets', or some 'domain of objects of a topos'. Our point of view might help to determine such a domain.

4.5. *Satisfaction*

The relation of semantic consequence between two specifications can also be obtained from a relation of satisfaction between a model and a specification. However, the satisfaction only makes sense when there is some notion of *signature* of a specification. More precisely, let $P : \mathscr{E} \to \overline{\mathscr{E}}$ and $P_0 : \mathscr{E}_0 \to \overline{\mathscr{E}}_0$ be two propagators, together with a pair of propagators $(L, \overline{L})$ such that the functor and $\overline{L} \circ P_0 = P \circ L$:

$$\begin{array}{ccc} \mathscr{E} & \xrightarrow{\;P\;} & \overline{\mathscr{E}} \\ {\scriptstyle L}\uparrow & \circlearrowleft & \uparrow{\scriptstyle \overline{L}} \\ \mathscr{E}_0 & \xrightarrow[\;P_0\;]{} & \overline{\mathscr{E}}_0 \end{array}$$

Let us also assume that U_L is faithful, that is satisfied as soon as L is a filling propagator that satisfies the condition of Proposition 3.11. Then the *signature functor* with respect to $(L, \overline{L})$ is $U_L : \mathscr{S}\mathrm{pec}(P) \to \mathscr{S}\mathrm{pec}(P_0)$.

Let S_0 be a P_0-specification and D_0 a P_0-domain. For all P-specifications S such that $U_L(S) = S_0$ and all P-domain D such that $U_{\overline{L}}(D) = D_0$, the signature functor determines a map:

$$(U_L)_{S, U_P(D)} : \mathrm{Hom}_{\mathscr{R}\mathrm{eal}(\mathscr{E})}(S, U_P(D)) \to \mathrm{Hom}_{\mathscr{R}\mathrm{eal}(\mathscr{E}_0)}(S_0, U_L(U_P(D))).$$

But $U_L \circ U_P = U_{P_0} \circ U_{\overline{L}}$, so $U_L(U_P(D)) = U_{P_0}(D_0)$, and by adjunction we get the map

$$(U_{L,\overline{L}})_{S,D} : \mathrm{Mod}_P(S, D) \to \mathrm{Mod}_{P_0}(S_0, D_0),$$

such that

$$\omega \mapsto (U_L(\omega_\star))^\star.$$

This map is natural in both S and D.

Definition 4.15. For all P-specifications S such that $U_L(S) = S_0$ and all P-domains D such that $U_{\overline{L}}(D) = D_0$, the *underlying model map* with respect to $(L, \overline{L})$ is the map $\omega \mapsto (U_L(\omega_\star))^\star$:

$$(U_{L,\overline{L}})_{S,D} : \mathrm{Mod}_P(S, D) \to \mathrm{Mod}_{P_0}(S_0, D_0).$$

A model ω_0 of S_0 with values in D_0 *satisfies S with respect to D* if ω_0 is in the image of $\mathrm{Mod}(S, D)$ by $(U_{L,\overline{L}})_{S,D}$. This is denoted

$$\omega_0 \twoheadrightarrow_D S.$$

For all P-specifications S and all P-domains D, the map $(U_L)_{S,U_P(D)}$ is injective, so the map $(U_{L,\overline{L}})_{S,D}$ is injective also. Hence, this map can be used for identifying $\mathrm{Mod}_P(S, D)$ and its image in $\mathrm{Mod}_{P_0}(S_0, D_0)$, so we can say that a model ω_0 of S_0 with values in D_0 satisfies S with respect to D if and only if it 'is' a model of S with values in D.

When S_0, D_0 and D are given, the following result proves that there is a consequence $S \twoheadrightarrow_D S'$ if and only if each ω_0 that satisfies S also satisfies S'.

Theorem 4.16 (Satisfaction and consequence). Let $\sigma : S \to S'$ be a morphism of P-specifications such that $U_L(S) = U_L(S') = S_0$ and $U_L(\sigma) = \mathrm{id}_{S_0}$, and let D be a P-domain such that $U_{\overline{L}}(D) = D_0$. Then $S \overset{\sigma}{\twoheadrightarrow}_D S'$ if and only if, for all models ω_0 of S_0 with values in D_0, if $\omega_0 \twoheadrightarrow_D S$, then $\omega_0 \twoheadrightarrow_D S'$.

Proof. Let U stand for $U_{L,\overline{L}}$. Because of the naturality of the map $U_{S,D}$ with respect to S, the following triangle T is commutative:

$$\begin{array}{ccc} \mathrm{Mod}_P(S,D) & \xrightarrow{U_{S,D}} & \\ \uparrow{\scriptstyle \mathrm{Mod}_P(\sigma,D)} & & \mathrm{Mod}_{P_0}(S_0,D_0) \\ \mathrm{Mod}_P(S',D) & \xrightarrow{U_{S',D}} & \end{array}$$

Let us assume that $S \overset{\sigma}{\twoheadrightarrow}_D S'$, that is, that $\mathrm{Mod}_P(\sigma, D)$ is bijective. Let ω_0 be a model of S_0 with values in D_0 such that $\omega_0 \twoheadrightarrow_D S$, which means that ω_0 is in the image of the map $U_{S,D}$. Then, since $\mathrm{Mod}_P(\sigma, D)$ is surjective, ω_0 is in the image of the map $U_{S,D} \circ \mathrm{Mod}_P(\sigma, D)$. Because of the commutativity of T, this map is equal to $U_{S',D}$, so $\omega_0 \twoheadrightarrow_D S'$.

On the other hand, since the map $U_{S',D}$ is injective, the commutativity of T proves that the map $\mathrm{Mod}_P(\sigma, D)$ is injective also. Now let us assume that for all models ω_0 of S_0 with values in D_0, if $\omega_0 \twoheadrightarrow_D S$ then $\omega_0 \twoheadrightarrow_D S'$. For all models ω of S with values in D, let $\omega_0 = U_{S,D}(\omega)$, so $\omega_0 \twoheadrightarrow_D S$. Then $\omega_0 \twoheadrightarrow_D S'$, hence there is some model ω' of S' with values in D such that $\omega_0 = U_{S',D}(\omega')$. Since the map $U_{S',D}$ is injective, this ω' is uniquely determined. In this way we get a map $f : \mathrm{Mod}_P(S, D) \to \mathrm{Mod}_P(S', D)$, defined by $f(\omega) = \omega'$, which is such that $U_{S',D} \circ f = U_{S,D}$. It follows, because of the commutativity of T, that $U_{S',D} \circ f \circ \mathrm{Mod}_P(\sigma, D) = U_{S,D} \circ \mathrm{Mod}_P(\sigma, D) = U_{S',D}$ and $U_{S,D} \circ \mathrm{Mod}_P(\sigma, D) \circ f = U_{S',D} \circ f = U_{S,D}$. Finally, because of the injectivity of the maps $U_{S,D}$ and $U_{S',D}$, this prove that f is an inverse to $\mathrm{Mod}_P(\sigma, D)$. Hence $\mathrm{Mod}_P(\sigma, D)$ is bijective, so $S \overset{\sigma}{\twoheadrightarrow}_D S'$. □

5. About logic

In this section, we outline a few links between our diagrammatic specification techniques and some issues in logic. First, we look at equational diagrammatic specifications, and then, more generally, at institutions.

5.1. *About equational logic*

In the context of algebraic specifications, as for instance in Goguen *et al.* (1976), an equational specification is defined in three steps: first a set of sorts, then a signature (that is, a structured set of operators) on this set of sorts, and, finally, a set of equations on this signature. Some strings of sorts are used for introducing the operators, and some terms (composed from operators) are used for introducing the equations.

For example, an equational specification S_{nat} of naturals can be defined as follows:

— Sort: N;
— Operators: $s : N \to N$, $z : \lambda \to N$, $a : NN \to N$, with the strings of sorts NN and λ (empty string);
— Equations: $a(x, z) = x$ and $a(x, s(y)) = s(a(x, y))$ where x and y are variables of sort N.

These equations can be written without variables, as relations between composed arrows. For instance, the second equation can be written as $a \circ \mathrm{fact}(\mathrm{id}_N, s) \equiv s \circ a : NN \to N$, with one identity arrow $\mathrm{id}_N : N \to N$, one factorisation arrow $\mathrm{fact}(\mathrm{id}_N, s) : N \to NN$ and two composed arrows.

The construction of an equational specification makes use of three successive propagators: P_s for sorts, P_o for operators and P_e for equations.

Sorts. The propagator $P_s : \mathscr{E}_s \to \overline{\mathscr{E}}_s$ is the usual one from a projective sketch of sets to projective sketch of monoids.

The projective sketch $\mathscr{E}_s$ is the simplest sketch of sets: it consists of one point Sort (similar to Pt). So, a P_s-specification S_s is a set of sorts.

The projective sketch $\overline{\mathscr{E}}_s$ is a sketch of monoids: it contains the points Sort^0, Sort, Sort^2, two arrows $p_1, p_2 : \mathrm{Sort}^2 \to \mathrm{Sort}$ and two DPCs: one with vertex Sort^0 and empty base, the other with vertex Sort^2, base $\{\mathrm{Sort}, \mathrm{Sort}\}$ (discrete, that is, without any arrow) and projections p_1, p_2. The point Sort will be interpreted as the set of strings of sorts, Sort^0 as a one-element set, and Sort^2 as the set of pairs of strings of sorts. In $\overline{\mathscr{E}}_s$, two more arrows $\lambda : \mathrm{Sort}^0 \to \mathrm{Sort}$ and $\kappa : \mathrm{Sort}^2 \to \mathrm{Sort}$ stand for the empty string of sorts and the concatenation of strings of sorts, respectively. There are additional features in $\overline{\mathscr{E}}_s$ that ensure that κ will be interpreted as an associative operation and λ as its unit. So, the functor F_{P_s} freely generates the strings of sorts.

The propagator P_s can be decomposed as $P_s = K_s \circ J_s$, with an intermediate projective sketch $\mathscr{E}'_s$ of partial monoids.

Operators. The propagator $P_o : \mathscr{E}_o \to \overline{\mathscr{E}}_o$ is similar to the propagator, considered in the previous sections, from a projective sketch of directed graphs to a projective sketch of categories. There is a point Op (similar to Ar) that stands for the set of operators in $\mathscr{E}_o$ and for the set of terms in $\overline{\mathscr{E}}_o$. However, because of arities, P_o is somewhat larger than that.

The sketch $\mathscr{E}_o$ contains $\mathscr{E}'_s$, not only $\mathscr{E}_s$, in order to allow the definition of multivariate operators and constant operators. So, a P_o-specification S_o is a signature, in the equational meaning.

The inclusion propagator $J_{s,o} : \mathscr{E}'_s \to \mathscr{E}_o$ is filling. Let S_s be a P_s-specification. Then S_o is a S_s-sorted signature if $U_{J_{s,o}}(S_o)$ can be deduced from S_s, which means that $F_{J_s}(S_s) \longrightarrow\!\!\!\!\!\rightarrow U_{J_{s,o}}(S_o)$ as K_s-specifications.

The sketch $\overline{\mathscr{E}}_o$, besides identities and composed arrows, also takes care of projection and factorisation arrows. So, the functor F_{P_o} freely generates the terms, in their categorical version, that is, without variables.

The propagator P_o can be decomposed as $P_o = K_o \circ J_o$, with an intermediate projective sketch $\mathscr{E}'_o$ that contains the sketch of compositive graphs.

Equations. The propagator $P_e : \mathscr{E}_e \to \overline{\mathscr{E}}_e$ is the propagator for equational specifications.

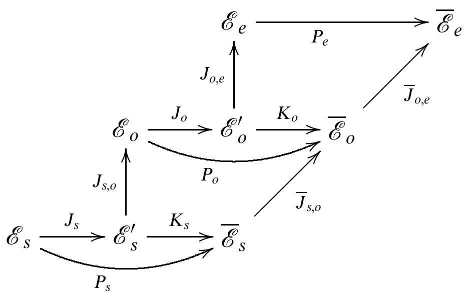
The sketch $\mathscr{E}_e$ contains $\mathscr{E}'_o$ and a point Eq for equations, with a potential monomorphism from Eq to a point Sst that stands (thanks to a DPC) for the set of pairs of terms with the same source and target. So, a P_e-specification S_e is an equational specification.

The inclusion propagator $J_{o,e} : \mathscr{E}'_o \to \mathscr{E}_e$ is filling. Let S_o be a P_o-specification. Then the signature of S_e is S_o if $U_{J_{o,e}}(S_e)$ can be deduced from S_o, which means that $F_{J_o}(S_o) \longrightarrow\!\!\!\!\!\rightarrow U_{J_{o,e}}(S_e)$ as K_e-specifications.

The sketch $\overline{\mathscr{E}}_e$ adds deduction rules in such a way that the interpretation of Eq in a realisation of $\overline{\mathscr{E}}_e$ is a congruence, that is, an equivalence relation that is compatible with the composition of terms. So, the functor F_{P_e} freely generates the congruence from the equations, that is, the theorems from the axioms.

It can be checked that the propagator P_e is fractioning.

To sum up, the definition of equational specifications makes use of the following commutative diagram of projective sketches and propagators:

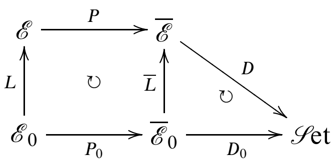


The domain of values is the realisation D_{set} of $\overline{\mathscr{E}}_e$ that interprets the sorts as sets, the operations as maps, and the equations as identities between maps.

5.2. *About institutions*

The theory of *institutions* (Goguen and Burstall 1992) defines some notions of logic in a very general setting. Diagrammatic specifications can easily be related to institutions, more precisely to *chartered institutions*.

The idea is to consider a propagator $P_0 : \mathcal{E}_0 \to \overline{\mathcal{E}}_0$, together with a point Sen in $\mathcal{E}_0$ and a P_0-domain D_0, such that the interpretation of the point $P_0(\mathrm{Sen})$ by D_0 is the set $\{\mathit{true}, \mathit{false}\}$ of booleans. Then a filling propagator $L : \mathcal{E}_0 \to \mathcal{E}$, such that U_L is faithful, is built by adding to $\mathcal{E}_0$ a point Ax and a potential monomorphism $m : \mathrm{Ax} \to \mathrm{Sen}$. This may be completed by a propagator $P : \mathcal{E} \to \overline{\mathcal{E}}$ and a propagator $\overline{L}$ such that $\overline{L} \circ P_0 = P \circ L$, together with a P-domain D such that $D_0 = U_{\overline{L}}(D)$:



The point $P_0(\mathrm{Sen})$ of $\overline{\mathcal{E}}_0$ stands for the set of *sentences*, the point Ax of $\mathcal{E}$ for the set of *axioms*, and the point $P(\mathrm{Ax})$ of $\overline{\mathcal{E}}$ for the set of *valid sentences*.

Let S be a P-specification and $S_0 = U_L(S)$ be its signature. Then $S(\mathrm{Sen})$ is equal to $S_0(\mathrm{Sen})$, and the image of $S(\mathrm{Ax})$ by $S(m)$ is a subset of $S_0(\mathrm{Sen})$. Clearly, in this way the category of P-specifications (up to isomorphisms) can be identified with the category of pairs (S_0, V) where S_0 is a P_0-specification, V is a subset of $S_0(\mathrm{Sen})$, and the morphisms are straightforward.

This gives rise to an institution I as follows:

— $\mathscr{R}\mathrm{eal}(\mathcal{E}_0)$ is the category of *signatures* of I;

— $\mathrm{Mod}_{P_0}(-, D_0) : \mathscr{R}\mathrm{eal}(\mathcal{E}_0) \twoheadrightarrow\!\!\!\!\!\!\times\;\; \mathscr{S}\mathrm{et}$ is the contravariant functor of *models* of I;

— $\mathrm{ev}_{\mathrm{Sen}} \circ F_{P_0} : \mathscr{R}\mathrm{eal}(\mathcal{E}_0) \to \mathscr{S}\mathrm{et}$ is the functor of *sentences* of I; and

— for all signatures S_0, all models ω of S_0 with values in D_0 and all sentences s of S_0, the *satisfaction relation* between ω and s holds if and only if ω satisfies (in the sense of diagrammatic specifications) the P-specification S with signature S_0 and s as its unique axiom.

Then the required *satisfaction condition* is easily checked.

In addition, such an institution, together with the notion of syntactic entailment, in the sense of diagrammatic specifications, gives rise to a *logic*, in the sense of Martí-Oliet and Meseguer (1994).

In this context, we can clarify the relations between the diagrammatic notions of entailment $\longrightarrow\!\!\!\!\!\!\!+\;$ and consequence $\longrightarrow\!\!\!\!\!\!\!\!++\;$ on the one hand, and the usual logical notions of entailment $\vdash$ and consequence $\models$ on the other.

Let S_0 be some fixed signature, and let $\varphi_1, \varphi_2, \ldots, \varphi_k$ and ψ be sentences of S_0. Let $S_{\varphi_1,\varphi_2,\ldots,\varphi_k}$ be the specification with signature S_0 such that $S_{\varphi_1,\varphi_2,\ldots,\varphi_k}(\mathrm{Ax}) = \{\varphi_1, \varphi_2, \ldots, \varphi_k\}$. Let $S_{\varphi_1,\varphi_2,\ldots,\varphi_k,\psi}$ be the specification with signature S_0 such that $S_{\varphi_1,\varphi_2,\ldots,\varphi_k,\psi}(\mathrm{Ax}) = \{\varphi_1, \varphi_2, \ldots, \varphi_k, \psi\}$. Let $\sigma : S_{\varphi_1,\varphi_2,\ldots,\varphi_k} \to S_{\varphi_1,\varphi_2,\ldots,\varphi_k,\psi}$ be the inclusion. Then, clearly,

$$S_{\varphi_1,\varphi_2,\ldots,\varphi_k} \xrightarrow{\sigma} S_{\varphi_1,\varphi_2,\ldots,\varphi_k,\psi} \quad \text{if and only if} \quad \varphi_1,\varphi_2,\ldots,\varphi_k \vdash \psi \ ,$$

$$S_{\varphi_1,\varphi_2,\ldots,\varphi_k} \xrightarrow{\sigma}_D S_{\varphi_1,\varphi_2,\ldots,\varphi_k,\psi} \quad \text{if and only if} \quad \varphi_1,\varphi_2,\ldots,\varphi_k \vDash \psi \ .$$

6. Conclusion

Thanks to the use of projective sketches at the meta level, the theory of diagrammatic specifications is quite powerful and effective. Our main definitions and results are so simple that they can now be summed up in a few lines.

Let $P : \mathscr{E} \to \overline{\mathscr{E}}$ be a *propagator*, that is, a homomorphism of projective sketches. Then, with respect to P:

— The category of *specifications* is the category of set-valued realisations of $\mathscr{E}$.
— The category of *domains* is the category of set-valued realisations of $\overline{\mathscr{E}}$.
— The *models* of a specification S with *values* in a domain D are the morphisms from $F_P(S)$ to D, that is, by adjunction, the morphisms from S to $U_P(D)$.
— A morphism of specifications $\sigma : S \to S'$ is a *syntactic entailment* if $F_P(\sigma)$ is an isomorphism.
— A morphism of specifications $\sigma : S \to S'$ is a *semantic consequence with respect to* a domain D if $\mathrm{Mod}_P(\sigma, D)$ is a bijection.
— Theorem 4.14 states that a morphism of specifications is a syntactic entailment if and only if it is a semantic consequence with respect to all domains.
— An *inference rule* is an arrow in $\overline{\mathscr{E}}$; thanks to the decomposition theorem (Theorem 3.13), it can be assumed that P consists in adding inverses to arrows, so the non-trivial inference rules (that is, the *active* ones) are the inverses of arrows of $\mathscr{E}$.
— A *deduction* is an arrow in the type of $\overline{\mathscr{E}}$, so it is composed from inference rules.

One of the applications of this framework is in the study of some features of computer languages, which should be the subject of forthcoming papers.

Acknowledgments

We would like to thank Catherine Oriat, Jean-Claude Reynaud, and many others.

References

Barr, M. and Wells, C. (1990) *Category Theory for Computing Science*, Prentice Hall.
Bénabou, J. (1973) Les Distributeurs. Rapport 33, Université Catholique de Louvain, Institut de Mathématique Pure et Appliquée.
Coppey, L. and Lair, C. (1984) Leçons de Théorie des esquisses (I). *Diagrammes* **12**.
Coppey, L. and Lair, C. (1988) Leçons de Théorie des esquisses (II). *Diagrammes* **19**.
Duval, D. and Lair, C. (2001) Esquisses et spécifications. Manuel de référence, 4ème partie: Fibrations et Eclatements, Lemmes de Yoneda et Modèles Engendrés. Rapport de recherche du LACO 2001-03, Université de Limoges. (Available at http://www.unilim.fr/laco/rapports/.)
Ehresmann, C. (1965) *Catgories et structures*, Dunod.

Ehresmann, C. (1966) Introduction to the theory of structured categories. Report 10, University of Kansas, Lawrence.

Ehresmann, C. (1967a) Problèmes universels relatifs aux catgories *n*-aires. *Comptes-Rendus de l'Acadmie des Sciences* **264** 273–276.

Ehresmann, C. (1967b) Sur les structures algébriques. *Comptes-Rendus de l'Acadmie des Sciences* **264** 840–843.

Gabriel, P. and Ulmer, F. (1971) Lokal Präsentierbare Kategorien. *Springer-Verlag Lecture Notes in Mathematics* **221**.

Gabriel, P. and Zisman, M. (1967) *Calculus of fractions and homotopy theory*, Springer-Verlag.

Goguen, J. A. and Burstall, R. M. (1992) Institutions: abstract model theory for specification and programming. *Journal of the ACM* **39** 95–146.

Goguen, J. A., Thatcher, J. W. and Wagner, E. G. (1976) An initial algebra approach to the specification, correctness and implementation of abstract data types. Technical Report RC 6487, IBM T.J. Watson Research Center. (Reprinted in: Yeh, R. (ed.) (1978) *Current trends in programming methodology IV*, Prentice Hall 80–149.)

Guitart, R. and Lair, C. (1980) Calcul syntaxique des modèles et calcul des formules internes. *Diagrammes* **4** 1–106.

Guitart, R. and Lair, C. (1982) Limites et colimites pour représenter les formules. *Diagrammes* **7** 1–24.

Hébert, M., Adámek, J. and Rosický, J. (2001) More on orthogonality in locally presentable categories. *Cahiers de Topologie et Géométrie Différentielle Catégoriques* **XLII-1** 51–80.

Kinoshita, Y., Power, A. J. and Takeyama, M. (1999) Sketches. *Journal of Pure and Applied Algebra* **143** 275–291.

Lair, C. (1987) Trames et sémantiques catégoriques des systèmes de trames. *Diagrammes* **18** 1–47.

Lalement, R. (1990) *Logique, réduction, résolution*, Masson.

Lawvere, F. W. (1963) Functorial semantics of algebraic theories. *Proceedings of the National Academy of Science USA* **50** 869–872.

Mac Lane, S. (1971) *Categories for the working mathematician*, Springer-Verlag.

Makkai, M. (1997a) Generalized sketches as a framework for completeness theorems (I). *Journal of Pure and Applied Algebra* **115** 49–79.

Makkai, M. (1997b) Generalized sketches as a framework for completeness theorems (II). *Journal of Pure and Applied Algebra* **115** 179–212.

Makkai, M. (1997c) Generalized sketches as a framework for completeness theorems (III). *Journal of Pure and Applied Algebra* **115** 241–274.

Makkai, M. and Reyes, G. E. (1977) First order categorical logic. *Springer-Verlag Lecture Notes in Mathematics* **611**.

Martí-Oliet, N. and Meseguer, J. (1994) General logics and logical framework. In: Gabbay, D. M. (ed.) *What is a logical system?* Oxford University Press.

Reichel, H. (1999) A Uniform Model Theory for the Specification of Data and Process Types In: Bert, D. (ed.) Recent Trends in Algebraic Development Techniques. *Springer-Verlag Lecture Notes in Computer Science* **1827**.

Wells, C. (1990) A generalization of the concept of sketch. *Theoretical Computer Science* **70** 159–178.