UNIVERSITY OF Southampton
School of Electronics
and Computer Science

ADHESIVE CATEGORIES, VAN KAMPEN SQUARES AND BICOLIMITS

Pawel Sobocinski
(joint work with Steve Lack & Tobias Heindel)

Grenoble, 26/11/09

PLAN OF TALK

- **Categories with structure**
  - extensive categories, adhesive categories
  - Van Kampen colimits
- 2-categories, bicategories & bicolimits
- Van Kampen colimits as a universal property

EXTENSIVE CATEGORIES

(elementary definition)

- A category is **extensive** when it has
  - finite coproducts
  - pullbacks (along coproduct injections)
  - for illustrated commutative diagram, TFAE
    - top row is a coproduct diagram
    - two squares are pullbacks

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X \arrow[r] \arrow[d] & Z \arrow[d] & Y \arrow[l] \arrow[d] \\
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Slogan: finite coproducts exist and are "well-behaved"

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$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & B \arrow[d] & B \arrow[l] \arrow[d] \\ A \arrow[r] & A + B & B \arrow[l] \end{tikzcd}$$

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- **Lemma**: Extensive categories with products are distributive.

$$\begin{tikzcd}
A \times B \arrow[r, "A\times i_1"] \arrow[d, "\pi_2"'] & A \times (B+C) \arrow[d, "\pi_2"] & A \times C \arrow[l, "A\times i_2"'] \arrow[d, "\pi_2"] \\
B \arrow[r, "i_1"'] & B + C & C \arrow[l, "i_1"]
\end{tikzcd}$$

INITIAL OBJECTS IN EXTENSIVE CATS

- An initial object is **strict** if existence of arrow X→0 implies that X is initial

- **Lemma**: In extensive categories, the initial object is strict

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & X \arrow[d] & X \arrow[l] \arrow[d] \\
0 \arrow[r] & 0 & 0 \arrow[l]
\end{tikzcd}$$

EXAMPLES/ COUNTEREXAMPLES

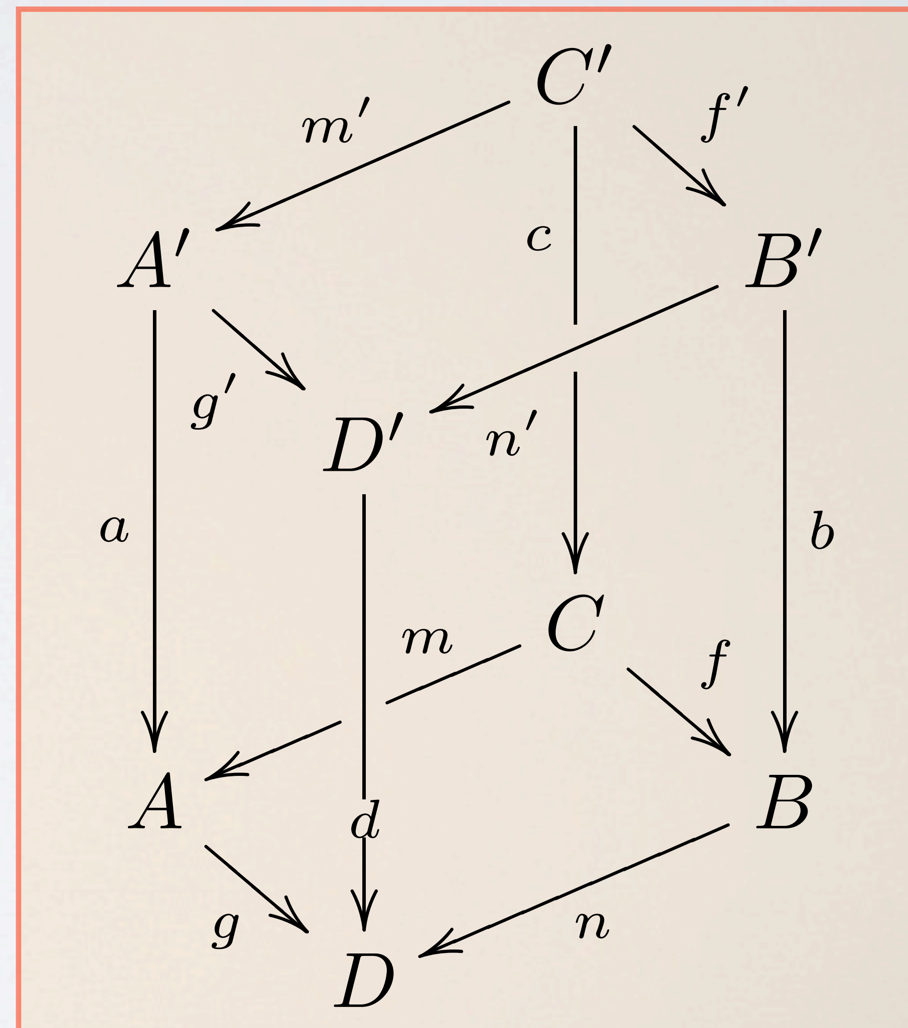
- **Set**, **Set**$_f$, **Graph**, toposes, **Cat**, **Top**, **C**/C for extensive **C**
  - note: in **Set** every mono is a coproduct injection

- Non examples:
  - powerset P(X) ordered by inclusion considered as a category (sums not disjoint)
  - C/**C** in general (no strict initial object)

VAN KAMPEN SQUARES

(elementary definition)

- Similar story to coproducts in extensive cats, for pushouts
    - pushout satisfies the **Van Kampen property** if when it is the bottom face of a commutative cube that has its rear faces pullbacks, tfae
        - the top face is a pushout
        - the front faces are pullbacks



NON VK PUSHOUT IN SET

$$\begin{tikzcd}
 & & \mathbb{Z}_2 \times \mathbb{Z}_2 \arrow[dll, "\pi_1"'] \arrow[dr, "\pi_2"] \arrow[ddd, "+"'] & \\
\mathbb{Z}_2 \arrow[dr] \arrow[ddd] & & & \mathbb{Z}_2 \arrow[dll] \arrow[ddd] \\
 & 1 \arrow[ddd] & & \\
 & & \mathbb{Z}_2 \arrow[dll] \arrow[dr] & \\
1 \arrow[dr] & & & 1 \arrow[dll] \\
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  - pushouts along monos
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- **Theorem**: **Set** is adhesive
  - proof relies on the fact that **Set** is extensive, monos in **Set** are coproduct injections and pushouts commute with coproducts

PROPERTIES OF ADHESIVE CATEGORIES

- **Lemma**
  - pushouts of monos are monos
  - pushouts along monos are pullbacks

$$\begin{tikzcd} & C \arrow[dl, "m"'] \arrow[dr, "f"] & \\ A \arrow[dr, "g"'] & & B \arrow[dl, "n"] \\ & D & \end{tikzcd}$$

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- **Lemma**
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- other properties:
  - unique pushout complements
  - effective unions
  - distributive lattices of subobjects

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EXAMPLES

- Set, Graph, toposes, C/$\mathbf{C}$, $\mathbf{C}$/C for adhesive $\mathbf{C}$

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- Non examples: **Pos**, **Cat**, **Top**

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$$[n] = \{0 \leq 1 \leq \cdots \leq n-1\}$$

$$\begin{tikzcd}
{[2]} \arrow[d] & {[1]} \arrow[l, "0"'] \arrow[d, "1"] \\
{[3]} & {[2]} \arrow[l]
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$$\begin{tikzcd} {[2]} \arrow[d] & {[1]} \arrow[l, "0"'] \arrow[d, "1"] \\ {[3]} & {[2]} \arrow[l] \end{tikzcd}$$

not stable under pullback with

$$[2] \to [3]$$
$$0 \mapsto 0$$
$$1 \mapsto 2$$

GENERALISING VAN KAMPEN CONDITION

Definition:

A colimit diagram $\kappa : \mathcal{F} \to C$ is Van Kampen when for all functors $\mathcal{F}' : \mathbf{J} \to \mathbf{C}$, cocones $\kappa' : \mathcal{F}' \to C'$ and cartesian nat. trans. $\gamma : \mathcal{F}' \to \mathcal{F}$

$$\begin{tikzcd}
\mathcal{F}'_i \arrow[r, "\mathcal{F}'_u"] \arrow[d, "\gamma_i"'] \arrow[rr, bend right, "\kappa'_i"] & \mathcal{F}'_j \arrow[r, "\kappa'_j"] \arrow[d, "\gamma_j"] & C' \arrow[d, "c"] \\
\mathcal{F}_i \arrow[r, "\mathcal{F}_u"'] \arrow[rr, bend right, "\kappa_i"'] & \mathcal{F}_j \arrow[r, "\kappa_j"] & C
\end{tikzcd}$$

TFAE

i. κ' is a colimit diagram

ii. $\mathcal{F}'_i C' C \mathcal{F}_i$ are all pullback diagrams

EXAMPLE - STRICT INITIAL OBJECT

- A colimit 0 of the empty diagram (initial object) is VK when for all arrows $X \to 0$, X is a colimit of the empty diagram
  - in other words:

VK initial object = strict initial object

EXAMPLE - VK COPRODUCT

- A coproduct diagram is VK when, given a commutative diagram:

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- TFAE
  - top row is a coproduct diagram
  - two squares are pullbacks
- Hence: coproducts in extensive categories are VK coproducts

ADHESIVE VS EXTENSIVE

- **Theorem**: An adhesive category is extensive iff it has a strict initial object

$$\begin{tikzcd}
 & & 0 \arrow[dll] \arrow[dr] \arrow[ddd] & \\
X \arrow[dr, "m"] \arrow[ddd, "r"'] & & & Y \arrow[dll, "n"] \arrow[ddd, "s"] \\
 & Z \arrow[ddd, "t"] & & \\
 & & 0 \arrow[dll] \arrow[dr] & \\
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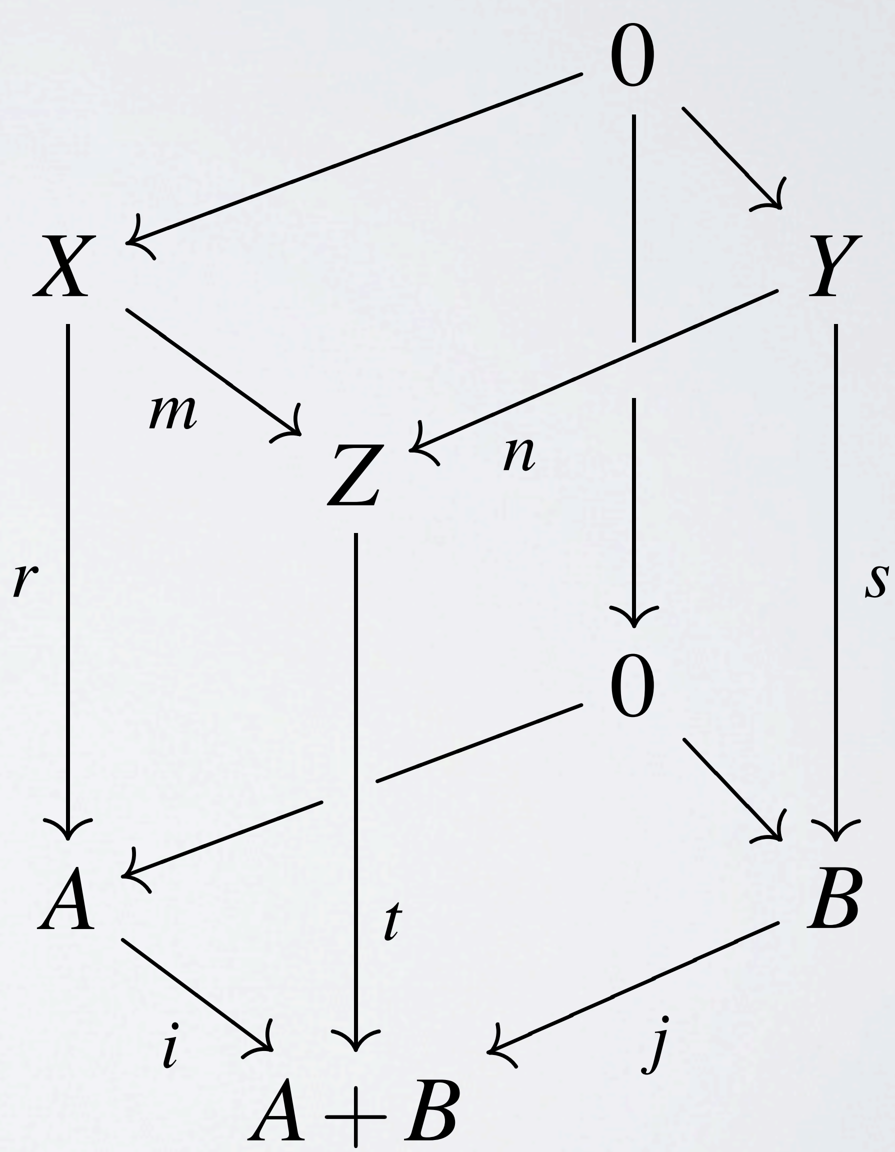
- is there a deeper meaning to being VK?



MAIN RESULT

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- ***A***. VK condition is an elementary characterisation (in **C**) of a universal property (in Span(**C**)) !

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2-CATEGORIES & BICATEGORIES

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- vertical composition

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$$\begin{tikzcd} X \arrow[r, bend left, "f"{name=F}] \arrow[r, bend right, "g"'{name=G}] \arrow[r, phantom, "\Downarrow\alpha"] & Y \end{tikzcd}$$

- vertical composition

$$\begin{tikzcd} X \arrow[r, bend left=60, "f"] \arrow[r, "g" description] \arrow[r, bend right=60, "h"'] \arrow[r, phantom, shift left=3ex, "\Downarrow\alpha"] \arrow[r, phantom, shift right=3ex, "\Downarrow\beta"] & Y \end{tikzcd} \quad \beta \bullet \alpha : f \Rightarrow h$$

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$$\begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\Downarrow\alpha"] & Y \arrow[r, bend left, "g"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "\Downarrow\beta"] & Z \end{tikzcd} \quad \beta\alpha : gf \Rightarrow g'f'$$

PROPERTIES OF COMPOSITIONS

- **C**(X,Y) is a category, so identities $1_f : f \Rightarrow f$ exist and vertical composition is associative

- **C**(Y, Z) × **C**(X,Y) → **C**(X, Z) is a functor, so

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\end{tikzcd}
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X \arrow[r, bend left, "gf"] \arrow[r, bend right, "gf"'] \arrow[r, phantom, "\Downarrow 1_{gf}"] & Z
\end{tikzcd}
$$

$$
\begin{tikzcd}
X \arrow[r, bend left=60, "f"] \arrow[r, "f'" description] \arrow[r, bend right=60, "f''"'] \arrow[r, phantom, shift left=3, "\Downarrow\alpha"] \arrow[r, phantom, shift right=3, "\Downarrow\alpha'"] & I_3 \arrow[r, bend left=60, "g"] \arrow[r, "g'" description] \arrow[r, bend right=60, "g''"'] \arrow[r, phantom, shift left=3, "\Downarrow\beta"] \arrow[r, phantom, shift right=3, "\Downarrow\beta'"] & Z
\end{tikzcd}
\quad \beta'\alpha' \bullet \beta\alpha = (\beta' \bullet \beta)(\alpha' \bullet \alpha)
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EXTRA ASSUMPTIONS

- 2-categories: horizontal composition has identities
  - bicategories: identity laws only up to coherent isomorphisms

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X \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\Downarrow\alpha"] & Y \arrow[r, bend left, "g"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "\Downarrow\beta"] & Z \arrow[r, bend left, "h"] \arrow[r, bend right, "h'"'] \arrow[r, phantom, "\Downarrow\gamma"] & W
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\Downarrow\alpha"] & Y \arrow[r, bend left, "hg"] \arrow[r, bend right, "h'g'"'] \arrow[r, phantom, "\Downarrow\gamma\beta"] & W
\end{tikzcd}
=
\begin{tikzcd}
X \arrow[r, bend left, "gf"] \arrow[r, bend right, "g'f'"'] \arrow[r, phantom, "\Downarrow\beta\alpha"] & Z \arrow[r, bend left, "h"] \arrow[r, bend right, "h'"'] \arrow[r, phantom, "\Downarrow\gamma"] & W
\end{tikzcd}
$$

BICATEGORY OF SPANS

- For any **C** with (chosen) pullbacks, Span(**C**) has
  - objects: those of **C**
  - arrows: spans of arrows in **C**
  - composition: by pullback
- Universal property of pullbacks gives associativity isomorphisms and implies coherence conditions
- There is an embedding $\Gamma : \mathbf{C} \to \mathrm{Span}(\mathbf{C})$

BICOLIMITS

(Kelly & Street)

- Span(**C**) is a bicategory: canonical notion of colimits are **bicolimit**

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BICOLIMITS
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- **categories**: usually equality of objects does not make sense
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- **2-categories and (especially) bicategories**: usually one does not talk about equality of arrows
  - bilimits, bicolimits are defined up to equivalence
  - mediating morphisms are "essentially unique"

MORE CONCRETELY

Let $\mathbf{J}$ be an ordinary category and $\mathcal{M}\colon \mathbf{J} \to \mathbb{B}$ a functor

- A bicolimit consists of the following data:
  - $\mathrm{bic}\,\mathcal{M} \in \mathbb{B}$
  - pseudo-cocone $\kappa\colon \mathcal{M} \to \mathrm{bic}\,\mathcal{M}$

$$
\begin{tikzcd}
\mathcal{M}_i \arrow[r, "\mathcal{M}_u"] \arrow[d, "\kappa_i"'] & \mathcal{M}_j \arrow[dl, bend left, "\kappa_j"] \\
\mathrm{bic}\,\mathcal{M} \arrow[ur, phantom, "\Nearrow\kappa_u"] &
\end{tikzcd}
\qquad
\begin{aligned}
\kappa_{\mathrm{id}_i} &= 1_{\kappa_i} \\
\kappa_{v \circ u} &= (\kappa_v \circ \mathcal{M}_u) \bullet \kappa_u
\end{aligned}
$$

UNIVERSAL PROPERTY

(existence)

- for all pseudo-cocones $\lambda\colon \mathcal{M} \to X$ there exists a pseudo mediating morphism that consists of:
  - an arrow $h\colon \operatorname{bic}\mathcal{M} \to X$
  - isomorphic 2-cells $\varphi_i : \lambda_i \Rightarrow (\Delta h) \circ \kappa$
  - satisfying:

$$
\begin{tikzcd}
\mathcal{M}_i \arrow[r, "\mathcal{M}_u"] \arrow[dr, "\kappa_i"'] \arrow[dd, "\lambda_i"'] & \mathcal{M}_j \arrow[d, bend left, "\kappa_j"] \arrow[dl, phantom, "\Rightarrow \kappa_u"] \\
\arrow[r, phantom, "\overset{\Longrightarrow}{\varphi_i}"] & \operatorname{bic}\mathcal{M} \arrow[dl, "h"] \\
X &
\end{tikzcd}
=
\begin{tikzcd}
\mathcal{M}_i \arrow[r, "\mathcal{M}_u"] \arrow[dd, "\lambda_i"'] \arrow[dr, phantom, "\Downarrow \lambda_u"] & \mathcal{M}_j \arrow[ddl, "\lambda_j"] \arrow[r, phantom, "\overset{\varphi_j}{\Longrightarrow}"] & \arrow[d, bend left, "\kappa_j"] \\
& & \operatorname{bic}\mathcal{M} \arrow[dll, "h"] \\
X & &
\end{tikzcd}
$$

UNIVERSAL PROPERTY

(essential uniqueness)

- for any $h, h' : \mathrm{bic}\,\mathcal{M} \to X$, a modification $\psi : \Delta h \circ \kappa \to \Delta h' \circ \kappa$ is $(\Delta\xi) \circ \kappa$ for a unique 2-cell $\xi : h \Rightarrow h'$

- this implies that any mediating morphisms are **essentially unique**, ie any two are isomorphic via a unique isomorphism

PLAN OF TALK

- Categories with structure
  - extensive categories, adhesive categories
  - Van Kampen colimits
- 2-categories, bicategories & bicolimits
- **Van Kampen colimits as a universal property**

MAIN THEOREM

- Let **C** have pullbacks and **J**-colimits. Let Γ : **C**→Span(**C**) be the usual embedding. Then:

$\kappa : \mathcal{F} \to C$ is Van Kampen in **C**

iff

$\Gamma\kappa$ is a bicolimit in Span(**C**)

- *Proof sketch:*
  - lemmas that allow to pass between **C** and Span(**C**)
  - restatement of the universal property of bicolimits so that it matches the VK condition.

SOME COROLLARIES

- **C** a category with pullbacks:
  - **C** has a strict initial object iff it has an initial object and it is preserved by the embedding into Span(**C**)
  - **C** is extensive iff it has binary sums and these are preserved by the embedding into Span(**C**)
  - **C** is adhesive iff it has pushouts along monos and these are preserved by the embedding into Span(**C**)
  - ...

INTUITIONS

- Ordinary universal property of colimits is good enough for **C**
- With Γ: **C**→Span(**C**) we pass into a wilder universe
- VK colimits are "reinforced" colimits that are ready for this shock

EXAMPLE - SYMMETRIES

$$
\begin{tikzcd}[column sep=small]
& 2 \arrow[dl, "!"'] \arrow[dr, "!"] & \\
1 \arrow[dr] & & 1 \arrow[dl] \\
& 1 &
\end{tikzcd}
\quad \overset{\Gamma}{\longmapsto} \quad
\begin{tikzcd}[column sep=small]
& 2 \arrow[dl, "\binom{1}{1}"'] \arrow[dr, "\binom{1}{1}"] & \\
1 \arrow[rr, Rightarrow, "\binom{\mathrm{id}}{\mathrm{id}}"] \arrow[dr, "1"'] & & 1 \arrow[dl, "1"] \\
& 1 &
\end{tikzcd}
$$

Not VK in **Set**

$$
\begin{tikzcd}[column sep=small]
& 2 \arrow[dl, "\binom{1}{1}"'] \arrow[dr, "\binom{1}{1}"] & \\
1 \arrow[rr, Rightarrow, "\binom{\mathrm{id}}{\mathrm{id}}"] \arrow[dr, "2"'] & & 1 \arrow[dl, "2"] \\
& 1 &
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=small]
& 2 \arrow[dl, "\binom{1}{1}"'] \arrow[dr, "\binom{1}{1}"] & \\
1 \arrow[rr, Rightarrow, "\binom{tw}{\mathrm{id}}"] \arrow[dr, "2"'] & & 1 \arrow[dl, "2"] \\
& 1 &
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=small]
& 2 \arrow[dl, "\binom{1}{1}"'] \arrow[dr, "\binom{1}{1}"] & \\
1 \arrow[rr, Rightarrow, "\binom{\mathrm{id}}{tw}"] \arrow[dr, "2"'] & & 1 \arrow[dl, "2"] \\
& 1 &
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=small]
& 2 \arrow[dl, "\binom{1}{1}"'] \arrow[dr, "\binom{1}{1}"] & \\
1 \arrow[rr, Rightarrow, "\binom{tw}{tw}"] \arrow[dr, "2"'] & & 1 \arrow[dl, "2"] \\
& 1 &
\end{tikzcd}
$$

- but only two mediating morphisms, so cannot be a bicolimit
- so VK bicolimits are “stable under symmetries”

FUTURE WORK

- Characterise the VK colimits in **Set**
  - or at least the VK pushouts!

- characterise weakenings of the VK condition by looking at universes between **C** and Span(**C**) (like Par(**C**) or Rel(**C**))
  - obtain (useful?) weakenings of adhesive categories etc