Categorical Quantum Computing: the necessity of Euler decomposition

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Joint work with Ross Duncan (U. Oxford)

Categorical Computer Science'09, Grenoble

Context and Motivations

Categorical axiomatisation of QIP

- †-compact closed categories [Abramsky, Coecke, LiCS'04], categorical axiomatisation of the teleportation underlying the information flow.
- Basis structure [Pavlovic, Coecke,06], categorical semantics of State transfer [Coecke, Paquette, Perdrix, MFPS'08]

- Unbiased basis [Coecke, Duncan, ICALP'08], proof of Shor algorithm.

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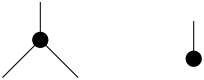

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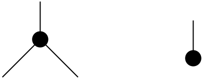

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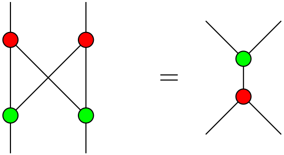
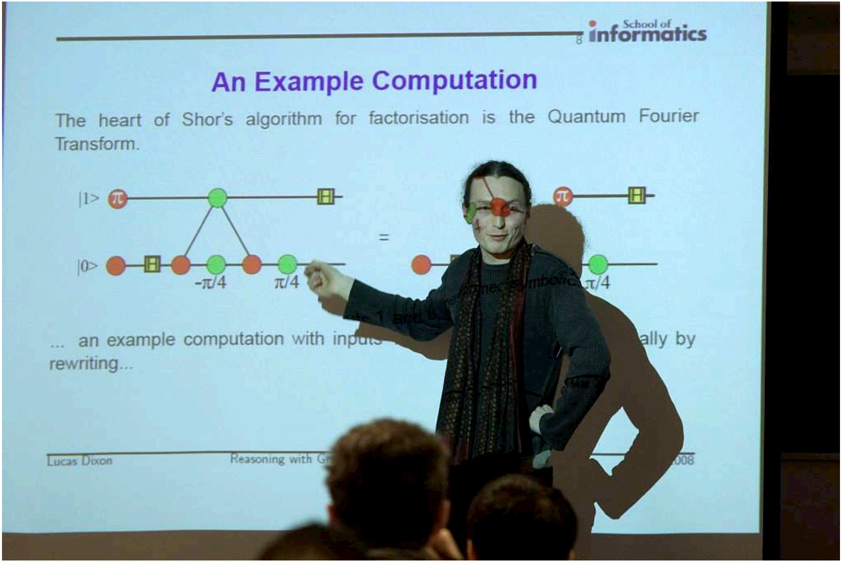
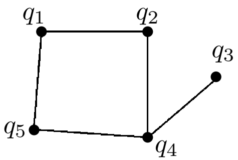




Lucas Dixon (U. Edinburgh) (photo by Dan Oi, U. Strathclyde at QUISCO Inaugural Meeting)

Towards a categorical axiomatisation of entanglement

Graph states:



- Representation of entanglement
- Applications: One-way QC [Raussendorf, Briegel 00], Quantum secret sharing [Markham, Sanders 08].

This talk: Abstract proof of the fundamental properties of graph states.

Objectives:

- Reveal the structures of entanglement.
- Refine the graphical language.

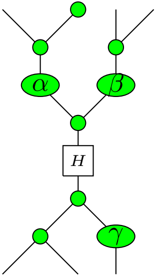
Diagrammatic language

Diagrammatic language

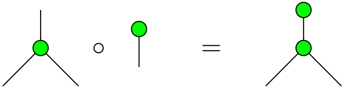
Definition

A *diagram* is a finite undirected open graph generated by the family of vertices:

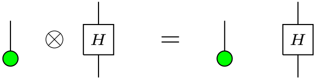
where $\alpha \in [0, 2\pi)$.



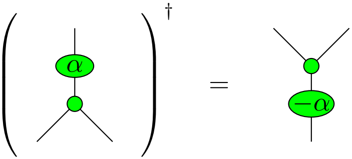
- Composition ($\circ$)



- Tensor ($\otimes$)



- Dagger ($\dagger$)



Spider

Diagrams form a †-compact closed category with basis structures.

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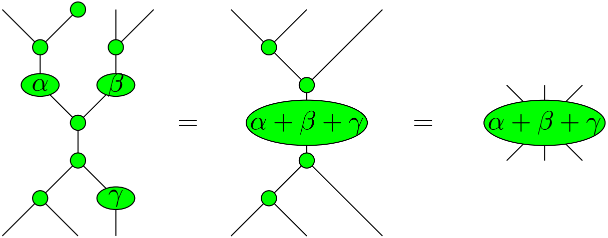
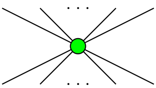
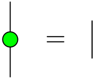
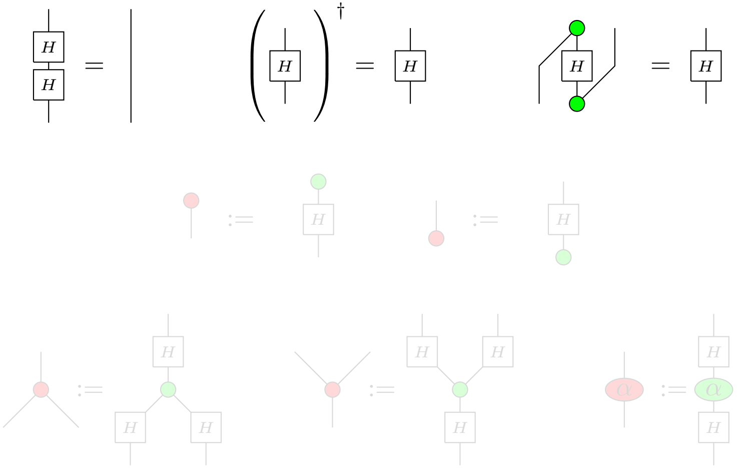






Hadamard



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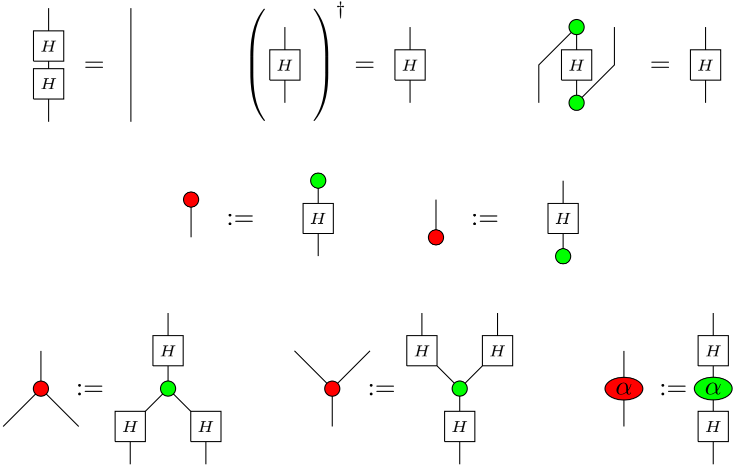
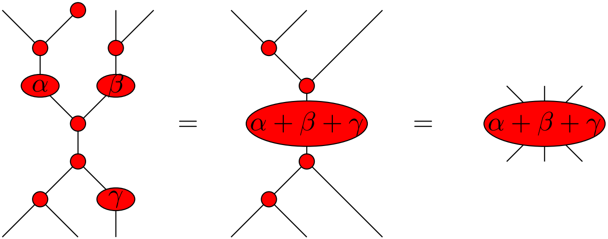
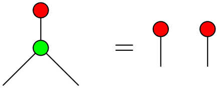


Red Spider

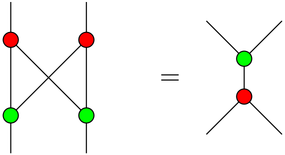


Complementary basis

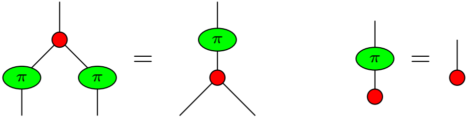
Copying



Bialgebra



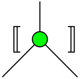
π-Commutation

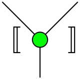


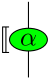
Interpretation in FdHilb

$[\![$  $]\!] = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

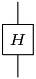
$[\![$  $]\!] = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \end{pmatrix}$

$[\![$  $]\!] = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$

$[\![$  $]\!] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

$[\![$ 

$]\!] = R_z(\alpha) = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{pmatrix}$

$[\![$ 

$]\!] = H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

$[\![$

$]\!] = R_x(\alpha)$

$[\![$

$]\!] = \Lambda Z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

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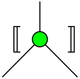
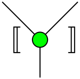
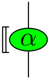
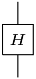
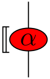
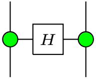
$[\![$  $]\!] = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$

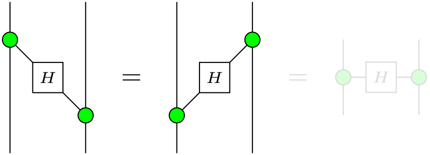
$[\![$  $]\!] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

$[\![$  $]\!] = R_z(\alpha) = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{pmatrix}$

$[\![$  $]\!] = H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

$[\![$  $]\!] = R_x(\alpha)$

$[\![$  $]\!] = \Lambda Z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$


H
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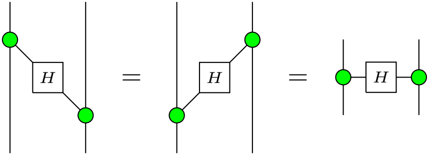
H
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H

Abstract Graph states

Abstract Graph states

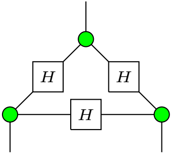
Definition

An **abstract graph state** is a diagram composed of green dots and H only such that:

– every green dot is connected to exactly one input or output

– every H is connecting two green dots

– there is no connection between two green dots



$[\![$ $]\!] = |G_{\text{triangle}}\rangle$:

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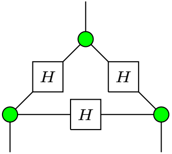
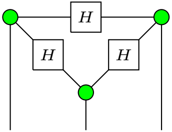
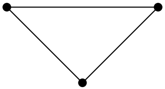


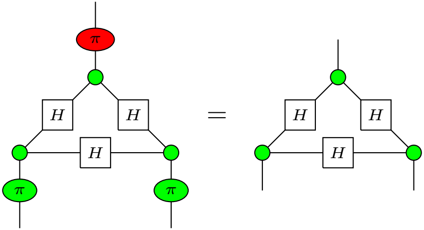
$\llbracket$ 

$\rrbracket = |G_{\mathsf{triangle}}\rangle :$ 

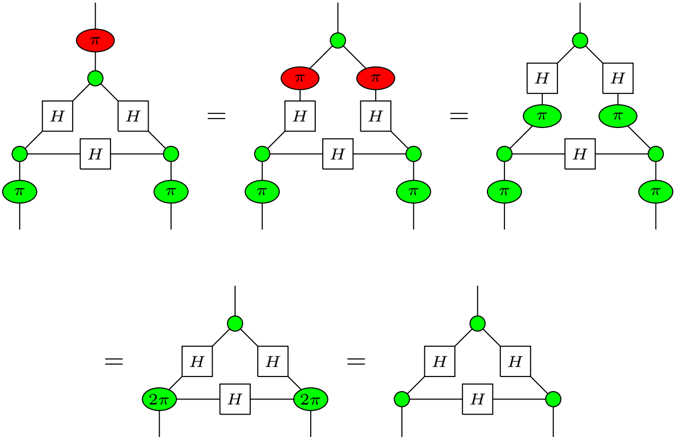
Property (Fixpoint)

Given a graph G and a vertex $u \in V(G)$,

$$R_x(\pi)^{(u)} R_z(\pi)^{(N_G(u))} |G\rangle = |G\rangle$$



Proof fixpoint



Theorem (Van den Nest)

Given a graph G and a vertex $u \in V(G)$,

$$R_x(\pi/2)^{(u)} R_z(-\pi/2)^{(N_G(u))} |G\rangle = |G * u\rangle \ .$$

where $G * u = G \Delta K_{N_G(u)}$ is the graph obtained by applying a local complementation on u in G [Bouchet85].

Two locally equivalent graphs represent the same entanglement

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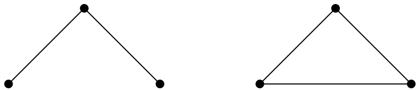

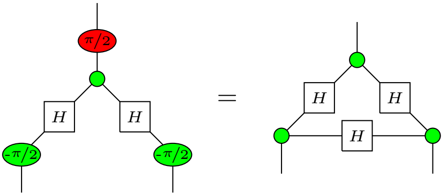

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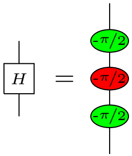
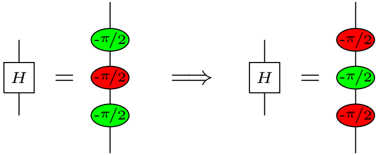
Euler decomposition

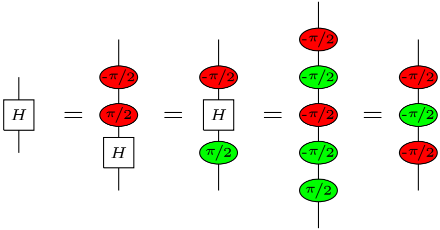


Lemma

The H-decomposition into $\pi/2$ rotations is not unique:

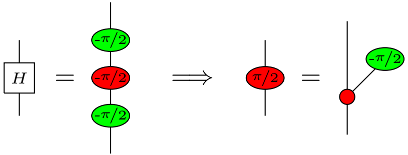


Proof:



Lemma

Each colour of $\pi/2$ rotation may be expressed in terms of the other colour.



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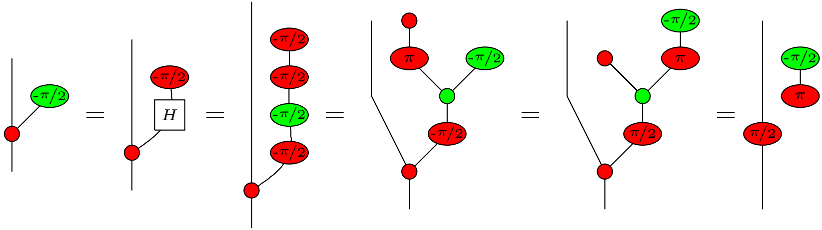
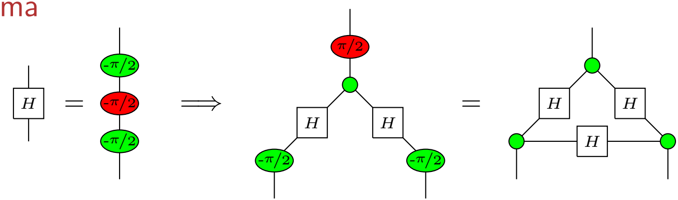


Lemma



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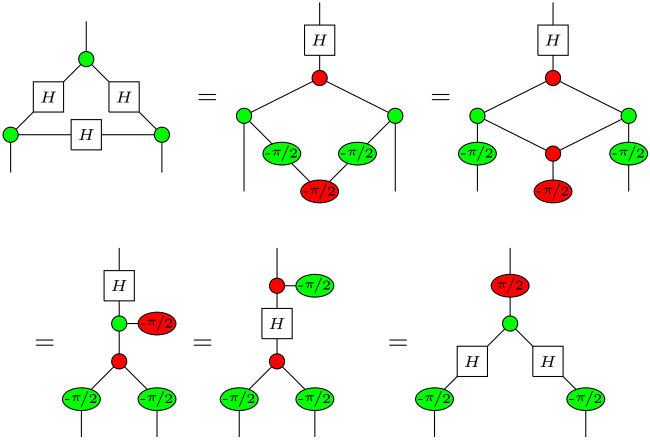
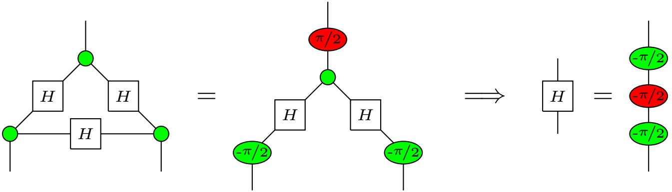
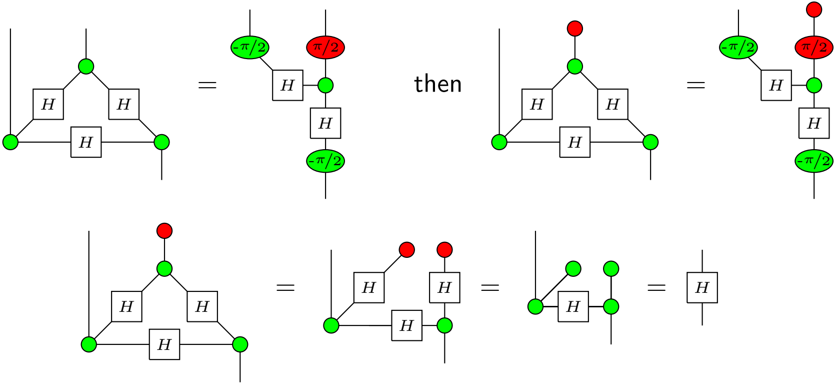


Lemma

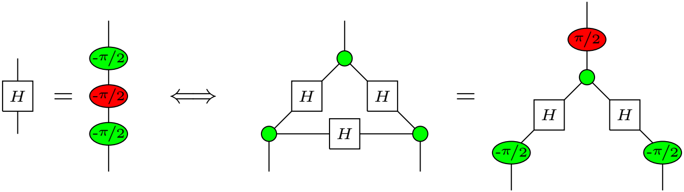
Local complementation implies the H-decomposition:




H
-π/2
π/2
then
-π/2

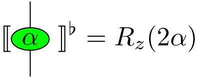
Theorem

Van den Nest's theorem holds if and only if H has a Euler decomposition:

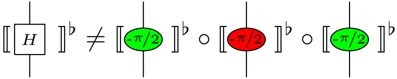


H-decomposition is a new rule

Let $[\![\cdot]\!]^\flat$ be exactly as $[\![\cdot]\!]$ with the following change:



This functor preserves all the axioms of the language, but



hence the Euler decomposition is not derivable from the axioms of the theory.

Conclusion

- Abstract proof of Van den Nest theorem.
- H Euler decomposition as a sufficient and necessary condition for Van den Nest theorem.
- Refine the diagrammatic language and point out a structure of entanglement.

Van den Nest Theorem: Locally equivalent graphs represent the same entanglement.
There exist graphs which are representing the same entanglement but which are not locally equivalent [Ji,Chen,Wei,Ying'08].

- Refine the language for capturing the previous case.
- Apply to states that cannot be represented by graphs.

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Lucas Dixon & Ross Duncan & Aleks Kissinger

http://dream.inf.ed.ac.uk/projects/quantomatic