Homotopical methods in polygraphic rewriting

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References.

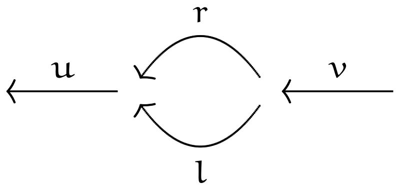
- *Higher-dimensional categories with finite derivation type*, Theory and Applications of Categories, 2009.
- *Identities among relations for higher-dimensional rewriting systems*, arXiv:0910.4538.

Part I. Two-dimensional Homotopy and String Rewriting

String Rewriting

String Rewriting System : X a set , $R \subseteq X^* \times X^*$

$$ulv \to_R urv$$



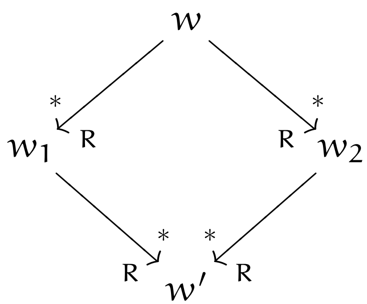
$$(r, l) \in R \quad u, v \in X^*$$

$\to_R^*$: reflexive symetrique closure of $\to_R$

Terminating :

$$w_0 \to_R w_1 \to_R \cdots \to_R w_n \to_R \cdots$$

Confluent



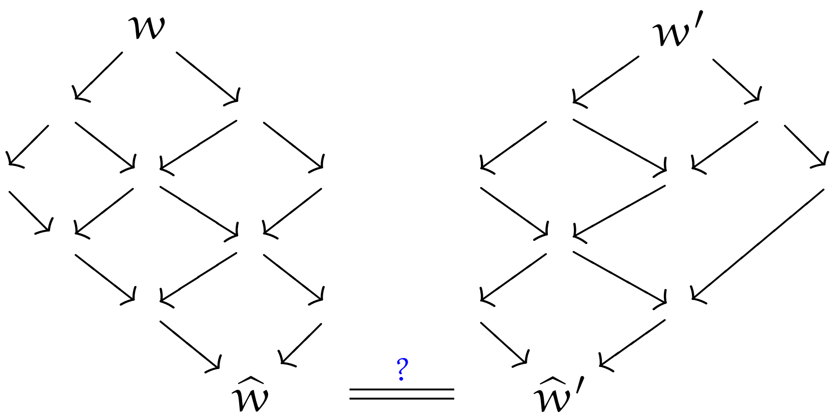
String Rewriting and word problem

Word problem

$$w, w' \in X^*, \text{ is } w = w' \text{ in } X^*/\leftrightarrow^*_R$$

$\leftrightarrow^*_R$: derivation.

Normal form algorithm : (X,R) : finite + convergent (terminating + confluent)



Fact. Monoids having a finite convergent presentation are decidable.

First Squier theorem

Rewriting is not universal to decide the word problem in finite type monoids.

Theorem. (Squier '87) There are finite type decidable monoids which do not have a finite convergent presentation.

Proof :

- A monoid $\mathbf{M}$ having a finite convergent presentation (X, R) is of homological type FP_3.

$$\ker \mathbf{J} \longrightarrow \mathbb{Z}\mathbf{M}[R] \xrightarrow{\mathbf{J}} \mathbb{Z}\mathbf{M}[X] \longrightarrow \mathbb{Z}\mathbf{M} \longrightarrow \mathbb{Z}$$

i.e. module of **homological 3-syzygies** is generated by critical branchings.

- There are finite type decidable monoids which are not of type FP_3.

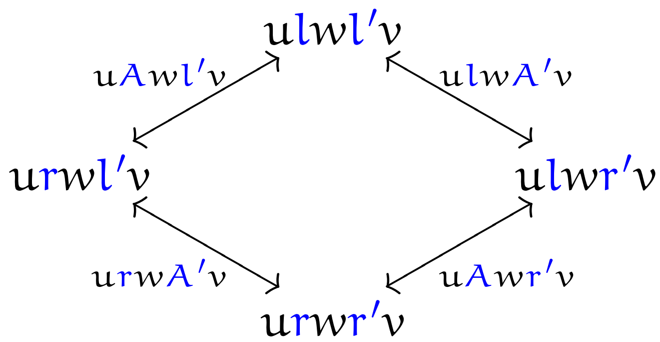
Second Squier Theorem

Theorem. Squier '87 ('94) The homological finiteness condition FP_3 is not sufficient for a finite type decidable monoid to admit a presentation by a finite convergent rewriting system.

Proof : • (X, R) a string rewriting system.

- $S(X, R)$ Squier 2-dimensional combinatorial complex.

0-cells : words on X, 1-cells : derivations $\leftrightarrow_R^*$, 2-cells : Peiffer elements



- (X, R) has **finite derivation type** (FDT) if

X and R are finite and $S(X, R)$ has a finite set of homotopy trivializer.

- Property FDT is Tietze invariant for finite rewriting systems
- A monoid having a finite convergent rewriting system has FDT.
- There are finite type decidable monoids which do not have FDT and which are FP_3.

Part II. Two-dimensional Homotopy for higher-dimensional rewriting systems

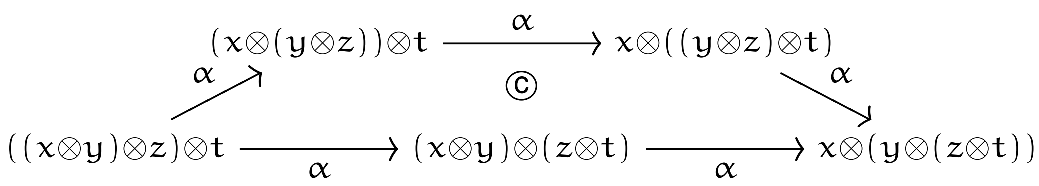
Mac Lane's coherence theorem

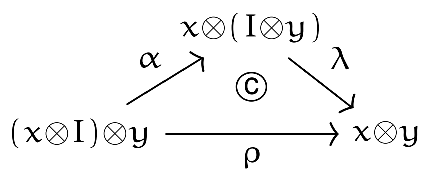
monoidal category is made of:

- a category $\mathcal{C}$,
- functors $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and $I : * \to \mathcal{C}$,
- three natural isomorphisms

$$\alpha_{x,y,z} : (x \otimes y) \otimes z \to x \otimes (y \otimes z) \qquad \lambda_x : I \otimes x \to x \qquad \rho_x : x \otimes I \to x$$

such that the following diagrams commute:





Mac Lane's coherence theorem. "In a monoidal category $(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho)$, all the diagrams built from $\mathcal{C}$, $\otimes$, I, α, λ and ρ are commutative."

Program:

– General setting: *homotopy bases* of *track* n*-categories*.

– Proof method: *rewriting techniques* for presentations of n-categories by *polygraphs*.

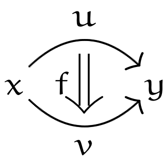
– Algebraic interpretation: *identities among relations*.

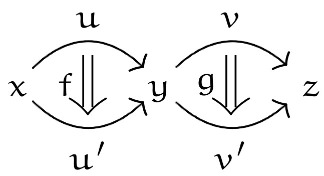
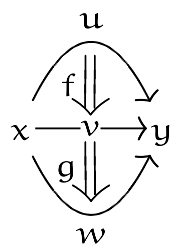
n-categories

An n**-category** $\mathcal{C}$ is made of:

- 0**-cells**
- 1**-cells:** $x \xrightarrow{u} y$ with one composition

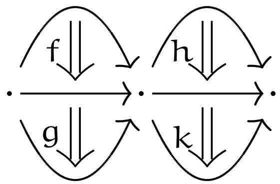
$$u \star_0 v = x \xrightarrow{u} y \xrightarrow{v} z$$

- 2**-cells:**  with two compositions

$f \star_0 g =$  and $f \star_1 g =$ 

Exchange relation:

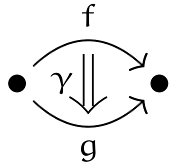
$$(f \star_1 g) \star_0 (h \star_1 k) = (f \star_0 h) \star_1 (g \star_0 k)$$

when 

- 3**-cells** with three compositions $\star_0$, $\star_1$ and $\star_2$, etc.

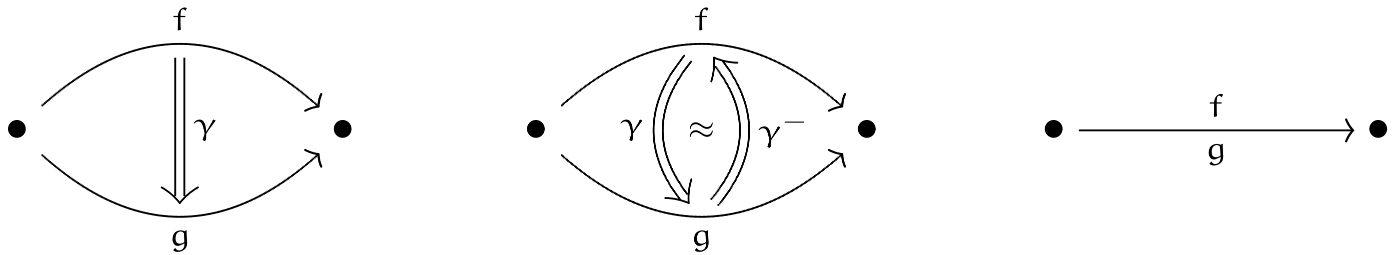
Track n-categories, cellular extensions and polygraphs

A **track n-category** is an n-category whose n-cells are invertible (for $\star_{n-1}$).

A **cellular extension** of $\mathcal{C}$ is a set Γ of $(n+1)$-cells 

with f and g parallel n-cells in $\mathcal{C}$.

$\mathcal{C}[\Gamma]$ $\qquad$ $\mathcal{C}(\Gamma)$ $\qquad$ $\mathcal{C}/\Gamma$



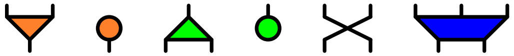
An **n-polygraph** is a family $\Sigma = (\Sigma_0, \ldots, \Sigma_n)$ where each Σ_{k+1} is a cellular extension of $\Sigma_0[\Sigma_1]\cdots[\Sigma_k]$.

Free n-category	Free track n-category	Presented $(n-1)$-category
$\Sigma^* = \Sigma^*_{n-1}[\Sigma_n]$	$\Sigma^\top = \Sigma^*_{n-1}(\Sigma_n)$	$\overline{\Sigma} = \Sigma^*_{n-1}/\Sigma_n$

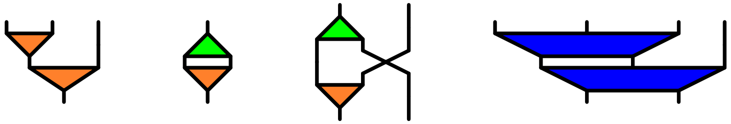
Graphical notations for polygraphs

We draw:

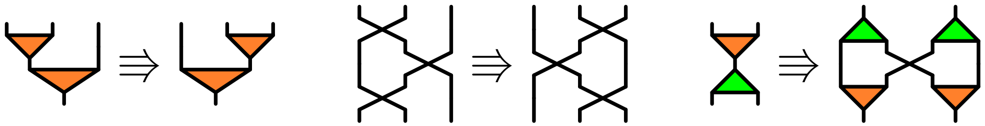
- Generating 2-cells as "circuit components":



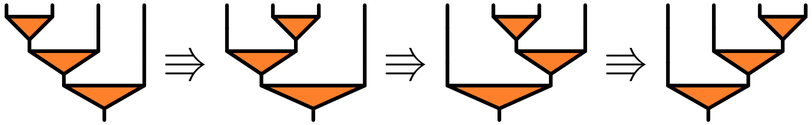
- 2-cells as "circuits":



- Generating 3-cells as "rewriting rules":

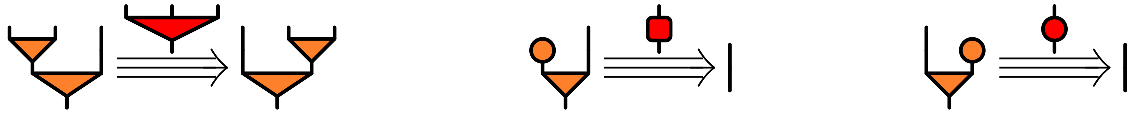


- 3-cells as "rewriting paths":



Example : the 2-category of monoids

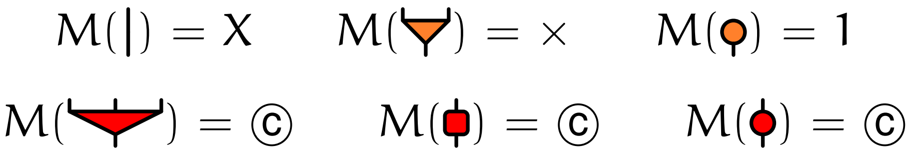
Let Σ be the 3-polygraph with one 0-cell, one 1-cell, two 2-cells and and three 3-cells:



Proposition. The 2-category $\overline{\Sigma}$ is the theory of monoids.

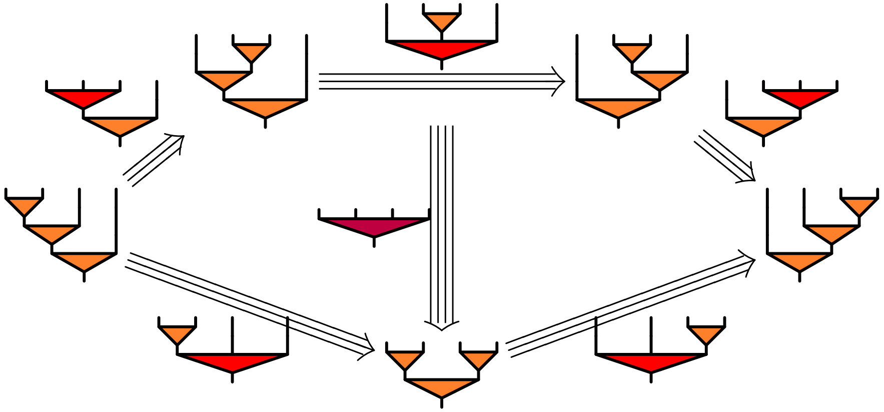
i.e., there is an equivalence:

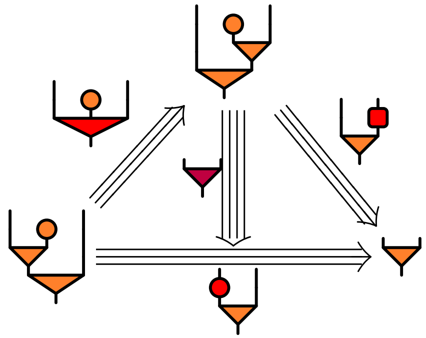
$$\text{Monoids } (X, \times, 1) \text{ in a } 2\text{-category } \mathcal{C} \quad \leftrightarrow \quad 2\text{-functors } M : \overline{\Sigma} \to \mathcal{C}$$



Example : the track 3-category of monoidal categories

Let Γ be the cellular extension of Σ^* with two 4-cells:





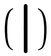
Proposition. The track 3-category $\Sigma^\top/\Gamma$ is the theory of monoidal categories, *i.e.*, there is an equivalence:

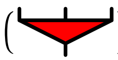
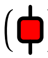
$$\text{Monoidal categories } (\mathcal{C}, \otimes, I, \alpha, \lambda, \rho) \quad \leftrightarrow \quad 3\text{-functors } M : \Sigma^\top/\Gamma \to \mathbf{Cat}$$

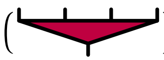
3-category $\mathbf{Cat}$:

- one 0-cell, categories as 1-cells, functors as 2-cells, natural transformations as 3-cells
- $\star_0$ is $\times$, $\star_1$ is the composition of functors, $\star_2$ the vertical composition of natural transformations

The equivalence is given by:

$M($$) = \mathcal{C}$ $M($$) = \otimes$ $M($$) = I$

$M($$) = \alpha$ $M($$) = \lambda$ $M($$) = \rho$

$M($$) = ©$ $M($$) = ©$

Homotopy bases and finite derivation type

A **homotopy basis** of an n-category $\mathcal{C}$ is a cellular extension Γ such that:

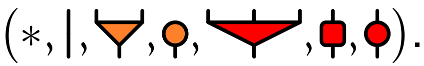
For every n-cells $\cdot \overset{f}{\underset{g}{\rightrightarrows}} \cdot$ in $\mathcal{C}$, there exists an $(n+1)$-cell $\cdot \overset{f}{\underset{g}{\rightrightarrows}} \cdot$ (with $\Downarrow$) in $\mathcal{C}(\Gamma)$, *i.e.*, $\overline{f} = \overline{g}$ in $\mathcal{C}/\Gamma$.

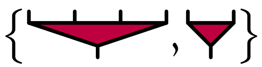
An n-polygraph Σ has **finite derivation type (FDT)** if it is finite and if $\Sigma^{\top}$ admits a finite homotopy basis.

Theorem. Let Σ and Υ be finite and *Tietze-equivalent* n-polygraphs, *i.e.*, $\overline{\Sigma} \simeq \overline{\Upsilon}$. Then:

$$\Sigma \text{ has FDT} \quad \text{iff} \quad \Upsilon \text{ has FDT.}$$

Mac Lane's theorem revisited. Let Σ be the 3-polygraph



Then the cellular extension  of Σ^* is a homotopy basis of $\Sigma^{\top}$.

Part III. Computation of homotopy bases

Rewriting properties of an n-polygraph Σ: termination and confluence

A **reduction** of Σ is a non-identity n-cell $u \xrightarrow{f} v$ of Σ^*.

A **normal form** is an $(n-1)$-cell u of Σ^* such that no reduction $u \xrightarrow{f} v$ exists.

The polygraph Σ **terminates** when it has no infinite sequence of reductions $u_1 \xrightarrow{f_1} u_2 \xrightarrow{f_2} u_3 \xrightarrow{f_3} (\cdots)$

Termination $\Rightarrow$ Existence of normal forms

A **branching** of Σ is a diagram $v \xleftarrow{f} u \xrightarrow{g} w$ in Σ^*.

– It is **local** when f and g contain exactly one generating n-cell of Σ_n.

– It is **confluent** when there exists a diagram $v \xrightarrow{f'} u' \xleftarrow{g'} w$

The polygraph Σ is **(locally) confluent** when every (local) branching is confluent.

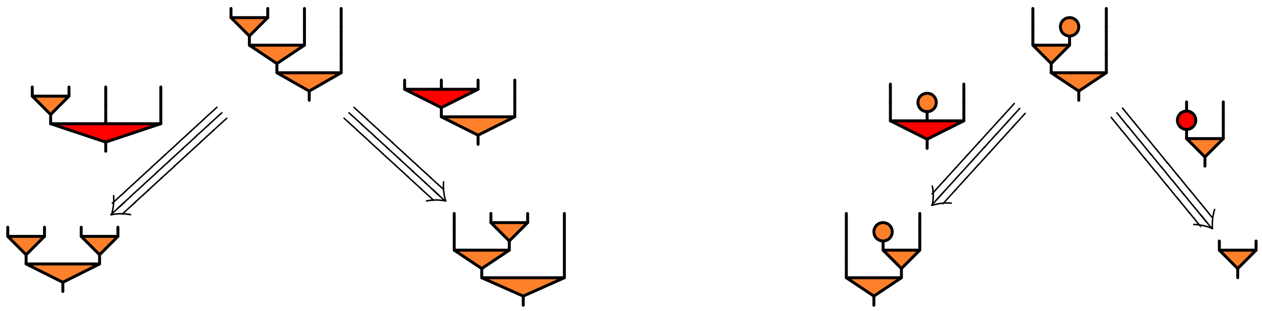
Confluence $\Rightarrow$ Unicity of normal forms

Rewriting properties of an n-polygraph Σ: convergence

The polygraph Σ is **convergent** if it terminates and it is confluent.

Theorem [Newman's lemma]. Termination $+$ local confluence $\Rightarrow$ Convergence.

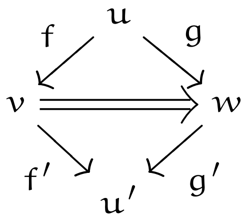
A branching is **critical** when it is "a minimal overlapping" of n-cells, such as:



Theorem. Termination $+$ confluence of critical branchings $\Rightarrow$ Convergence.

The homotopy basis of generating confluences

A **generating confluence** of an n-polygraph Σ is an $(n+1)$-cell



with (f, g) critical.

Theorem. Let Σ be a convergent n-polygraph. Let Γ be a cellular extension of Σ^* made of one generating confluence for each critical branching of Σ. Then Γ is a homotopy basis of $\Sigma^\top$.

Corollary. If Σ is a finite convergent n-polygraph with a finite number of critical branchings, then it has FDT.

Generating confluences : pear necklaces

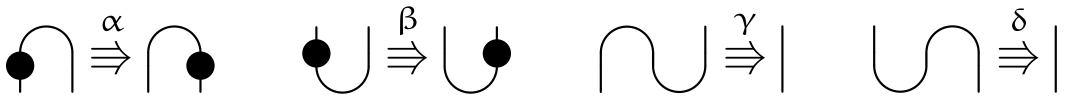
Theorem, Squier '94. If a monoid admits a presentation by a finite convergent word rewriting system, then it has FDT.

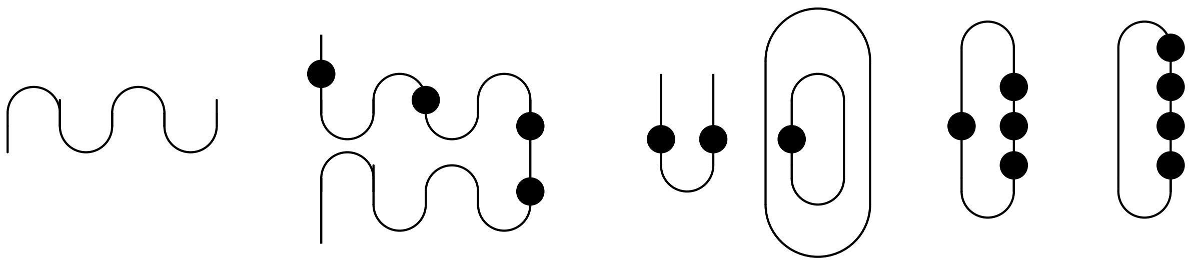
Theorem There exists a 2-category that lacks FDT, even though it admits a presentation by a finite convergent 3-polygraph.

- A 3-polygraph presenting the 2-category of pear necklaces.

one 0-cell, one 1-cell, three 2-cells :

four 3-cells :

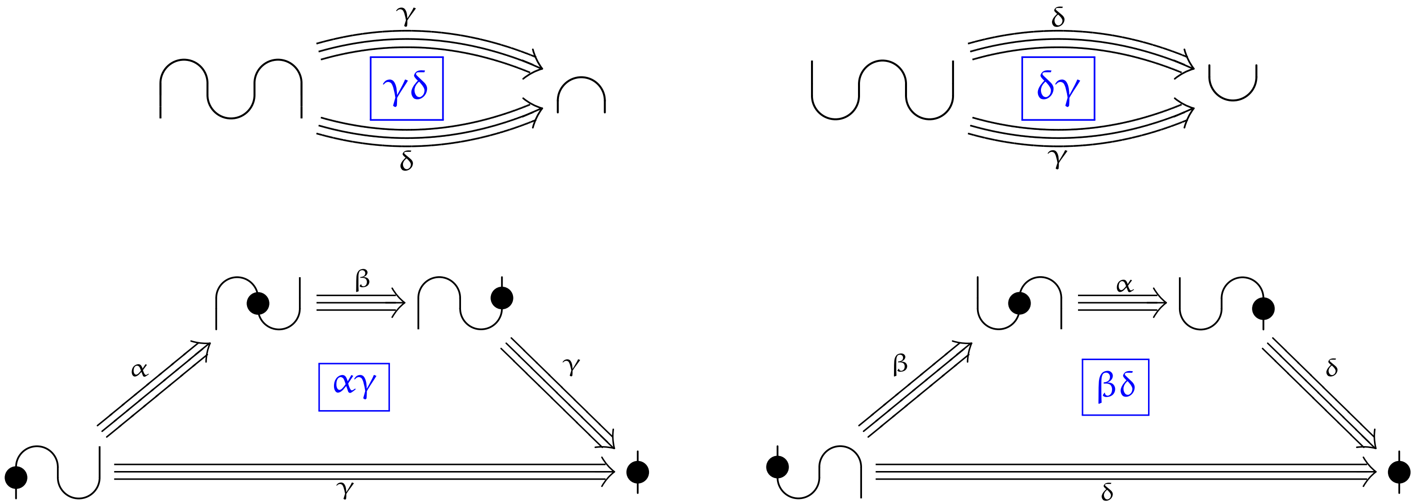




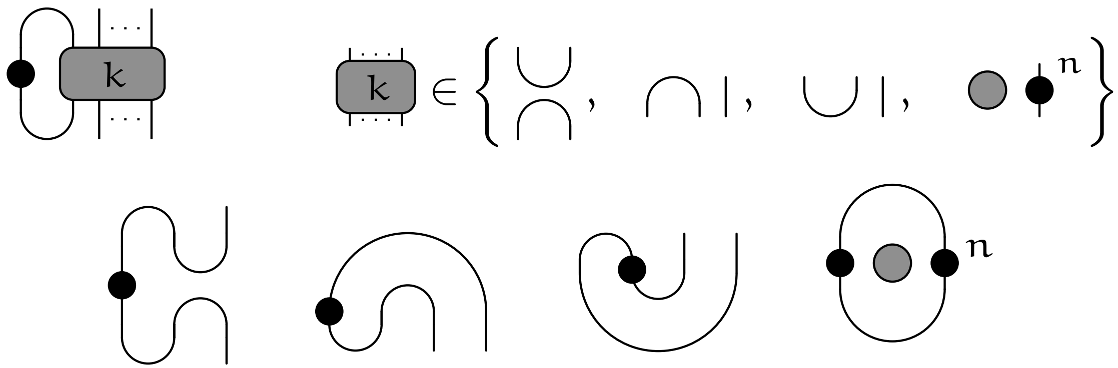
- Σ is finite and convergent but does not have FDT.

Generating confluences : pear necklaces

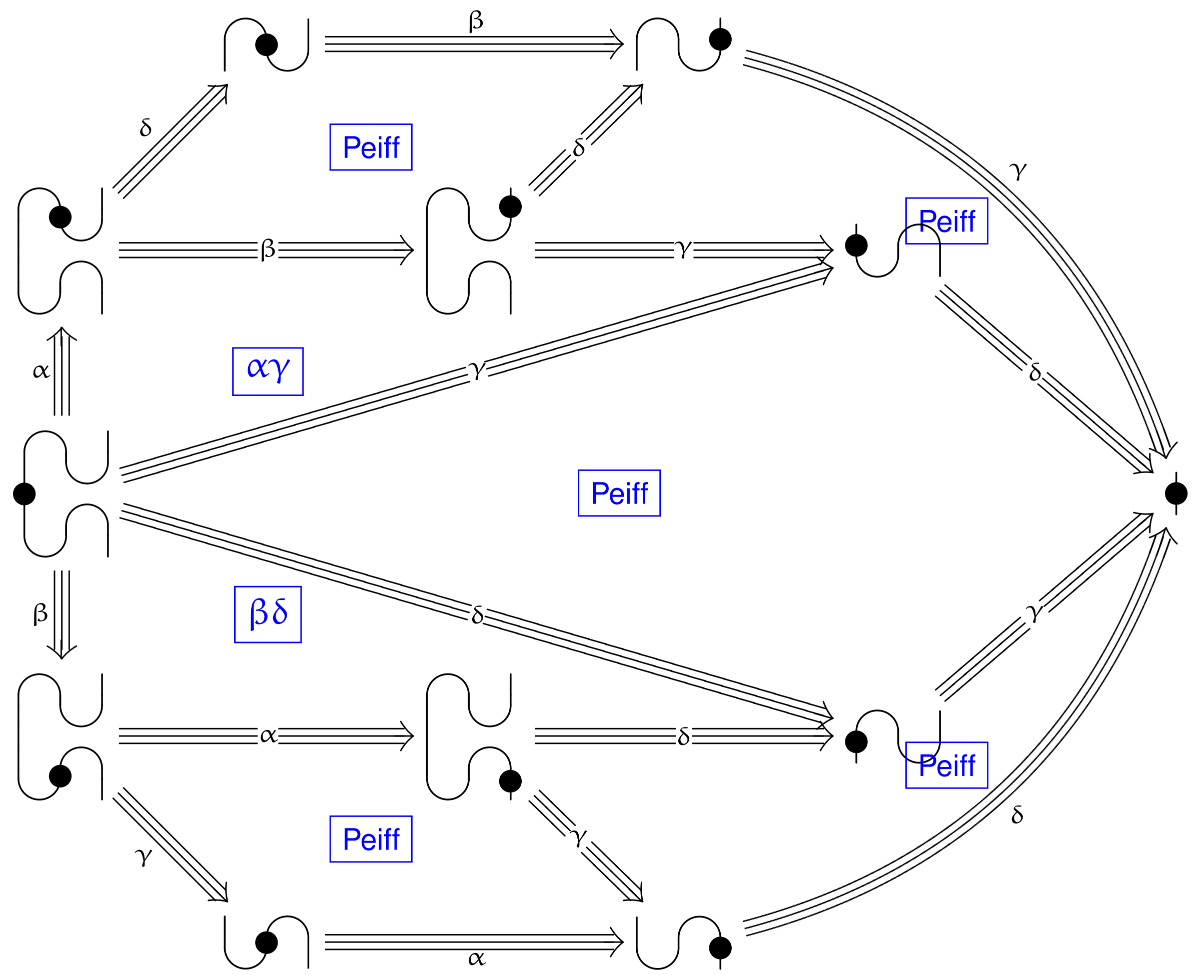
- Four regular critical branching



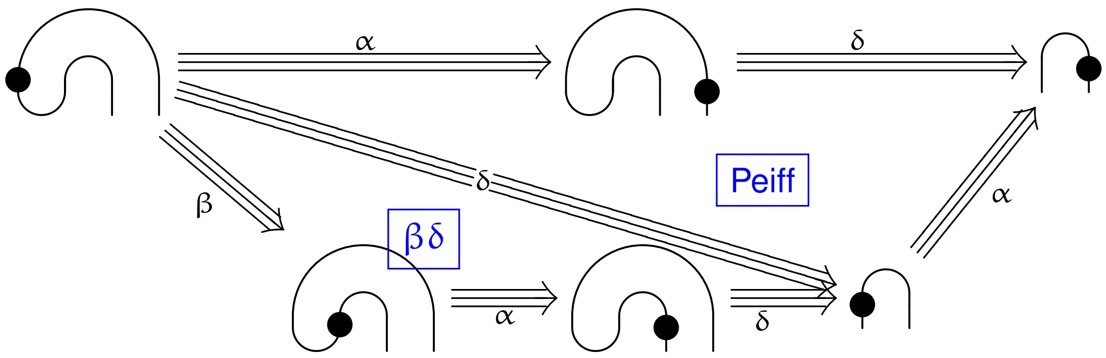
- One right-indexed critical branching

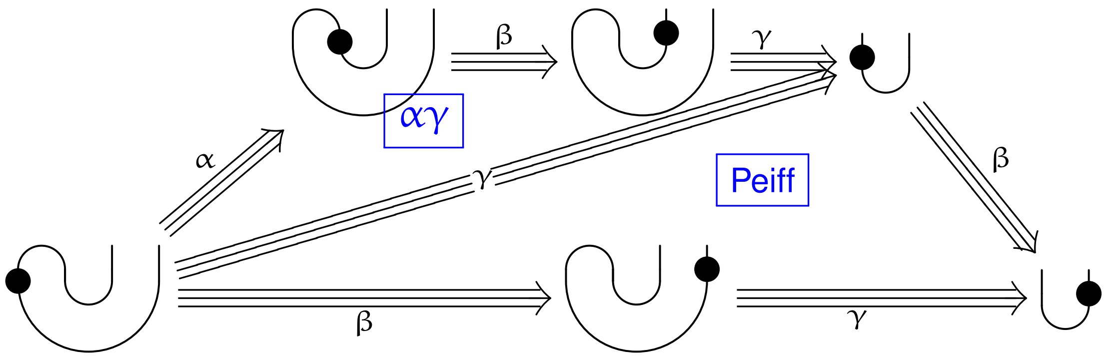


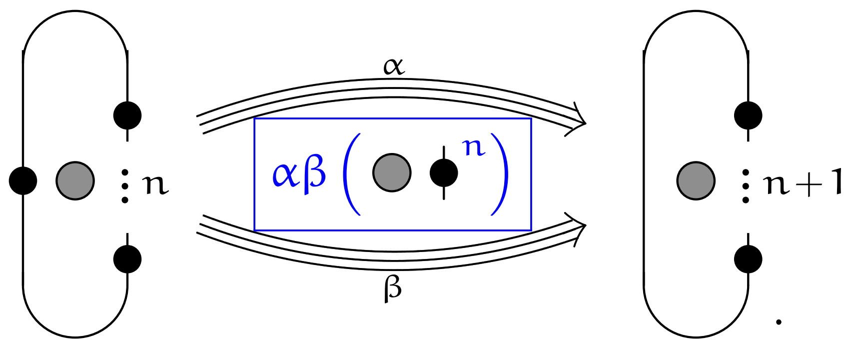
Generating confluences : pear necklaces



Generating confluences : pear necklaces

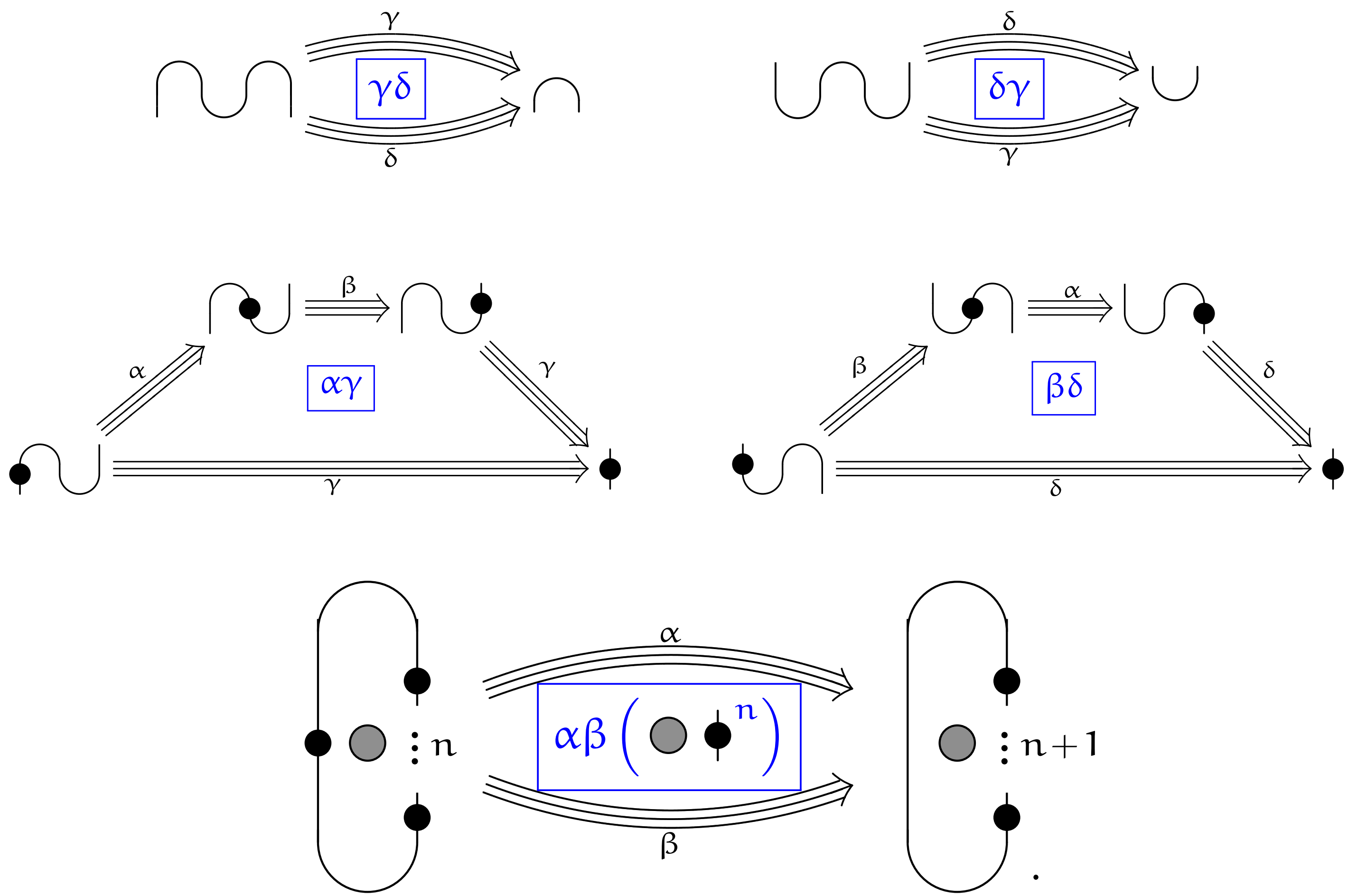






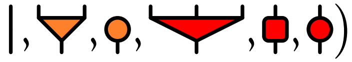
Generating confluences : pear necklaces

- An infinite homotopy base :

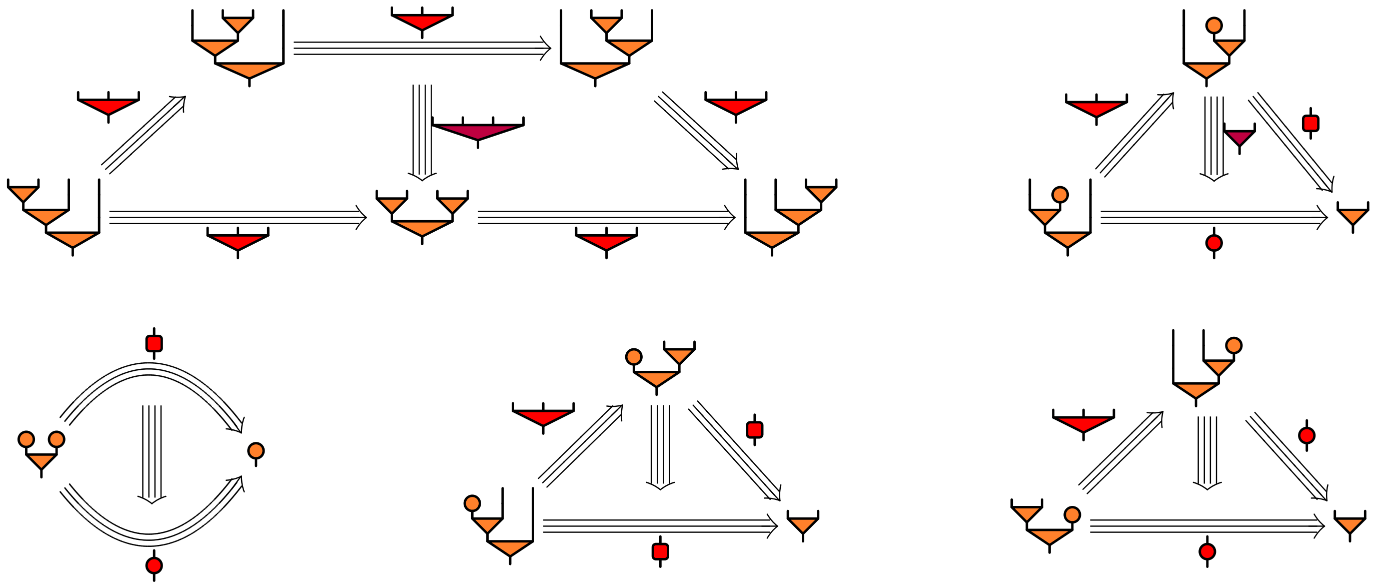


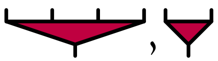
- The 3-polygraph is finite and convergent but does not have finite derivation type

Generating confluences : Mac Lane's coherence theorem

- Let Σ be the finite 3-polygraph $(*, $  $)$.

Lemma. Σ terminates and is locally confluent, with the following five generating confluences:



Theorem. The cellular extension $\{$  $\}$ is a homotopy basis of $\Sigma^{\top}$.

Corollary (Mac Lane's coherence theorem). "In a monoidal category $(\mathcal{C}, \otimes, \mathrm{I}, \alpha, \lambda, \rho)$, all the diagrams built from $\mathcal{C}$, $\otimes$, I, α, λ and ρ are commutative."

Part IV. Identities among relations

Defining identities among relations

The **contexts** of an n-category $\mathcal{C}$ are the partial maps $C : \mathcal{C}_n \to \mathcal{C}_n$ generated by:

$$x \mapsto f \star_i x \qquad \text{and} \qquad x \mapsto x \star_i f$$

The **category of contexts** of $\mathcal{C}$ is the category $\mathbf{C}\mathcal{C}$ with:

– Objects: n-cells of $\mathcal{C}$.

– Morphisms from f to g: contexts C of $\mathcal{C}$ such that $C[f] = g$.

The **natural system of identities among relations** of an n-polygraph Σ is the functor $\Pi(\Sigma) : \mathbf{C}\overline{\Sigma} \to \mathbf{Ab}$ defined as follows:

- If u is an $(n-1)$-cell of $\overline{\Sigma}$, then $\Pi(\Sigma)_u$ is the quotient of

$$\mathbb{Z}\left\{ \lfloor f \rfloor \;\middle|\; v \overset{f}{\circlearrowleft} \text{ in } \Sigma^\top, \overline{v} = u \right\}$$

by (with $\star$ denoting $\star_{n-1}$):

– $\lfloor f \star g \rfloor = \lfloor f \rfloor + \lfloor g \rfloor$ for every $f \circlearrowright v \circlearrowleft g$ with $\overline{v} = u$.

– $\lfloor f \star g \rfloor = \lfloor g \star f \rfloor$ for every $v \underset{g}{\overset{f}{\rightleftarrows}} w$ with $\overline{v} = \overline{w} = u$.

- If C is a context of $\overline{\Sigma}$ from u to v, then $C\lfloor f \rfloor = \lfloor B[f] \rfloor$, with $\overline{B} = C$.

Generating identities among relations

A **generating set** of $\Pi(\Sigma)$ is a part $X \subseteq \Pi(\Sigma)$ such that, for every $\lfloor f \rfloor$:

$$\lfloor f \rfloor = \sum_{i=1}^{k} \pm C_i[x_i], \quad \text{with } x_i \in X,\ C_i \in \mathbf{C}\overline{\Sigma}.$$

Proposition. Let Σ and Υ be finite Tietze-equivalent n-polygraphs. Then

$\Pi(\Sigma)$ is finitely generated *iff* $\Pi(\Upsilon)$ is finitely generated.

Proposition. Let Γ be a homotopy basis of $\Sigma^{\top}$ and $\widetilde{\Gamma} = \left\{ \widetilde{\gamma} = f \star g^{-} \,\middle|\, \gamma : f \to g \text{ in } \Gamma \right\}$.
Then $\left\lfloor \widetilde{\Gamma} \right\rfloor$ is a generating set for $\Pi(\Sigma)$.

3.2. Generating identities among relations

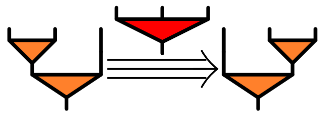
Theorem. If a n-polygraph Σ has FDT, then $\Pi(\Sigma)$ is finitely generated.

Proposition. If Σ is a convergent n-polygraph, then $\Pi(\Sigma)$ is generated by the generating confluences of Σ.

Example. Let Σ be the 2-polygraph $(*, |,$  $)$.

It is a finite convergent presentation of the monoid $\{1, a\}$ with $aa = a$.

It has one generating confluence:



Hence the following element generates $\Pi(\Sigma)$: