

Categorical models of the λ -calculus via linear logic

Categorical Computer Science

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Recall 0: Categorical semantics

λ -calculus

CCC

$\top \times \Rightarrow$

$(A \times B) \Rightarrow C \simeq A \Rightarrow (B \Rightarrow C)$

$A \rightarrow A \times A \quad A \rightarrow \top$

(Intuitionistic) Linear Logic

SMCC

$1 \otimes \multimap$

$(A \otimes B) \multimap C \simeq A \multimap (B \multimap C)$

! co-monad $(!A \rightarrow A \quad !A \rightarrow !!A)$

$!A \rightarrow !A \otimes !A \quad !A \rightarrow 1$

Recall 0: Categorical semantics

λ -calculus

CCC

$\top \times \Rightarrow$

$(A \times B) \Rightarrow C \simeq A \Rightarrow (B \Rightarrow C)$

$A \rightarrow A \times A \quad A \rightarrow \top$

Linear Logic

SMCC

$1 \otimes \multimap$

$(A \otimes B) \multimap C \simeq A \multimap (B \multimap C)$

! co-monad $(!A \rightarrow A \quad !A \rightarrow !!A)$

$!A \rightarrow !A \otimes !A \quad !A \rightarrow 1$

$\perp \quad A^\perp = A \multimap \perp \quad A^{\perp\perp} \simeq A$
 $?A = (!A^\perp)^\perp \dots$

Syntactic translations vs. models

λ -calculus \longrightarrow LL

λ -model \longleftarrow LL-model

4 translations, how many CCCs?

Recall I: Seely

Let \mathcal{C} be a (categorical) model of LL:

- the co-Kleisli category \mathcal{K} for $!$ is a CCC
 $\mathcal{K}(A, B) = \mathcal{C}(!A, B)$

This leads to the adjunction:

$$\begin{array}{ccc} \mathcal{C} & \begin{array}{c} \nearrow \emptyset \\ \searrow ! \\ \vdash \end{array} & \mathcal{K} \end{array}$$

$$\frac{\mathcal{C} : !A \rightarrow B}{\mathcal{K} : A \rightarrow B} \quad \begin{array}{l} !A \multimap B \\ A \Rightarrow B \end{array}$$

Girard's first translation

$$X^{\star} = X$$

$$(A \Rightarrow B)^{\star} = !A^{\star} \multimap B^{\star}$$

$$(\Gamma \vdash A)^{\star} = !\Gamma^{\star} \vdash A^{\star}$$

co-Kleisli category \mathcal{K} \implies λ -calculus

$$\frac{}{f : A \Rightarrow A \Rightarrow B \vdash f : A \Rightarrow A \Rightarrow B} \quad \frac{x : A \vdash x : A}{x : A, f : A \Rightarrow A \Rightarrow B \vdash (f)x : A \Rightarrow B} \quad \frac{}{x : A \vdash x : A}$$
$$\frac{x : A, f : A \Rightarrow A \Rightarrow B \vdash (f)xx : B}{x : A \vdash \lambda f.(f)xx : (A \Rightarrow A \Rightarrow B) \Rightarrow B}$$
$$\frac{}{\vdash \lambda x.\lambda f.(f)xx : A \Rightarrow (A \Rightarrow A \Rightarrow B) \Rightarrow B}$$

$$\frac{}{!A \vdash !A} \quad \frac{\overline{!A \vdash !A} \quad \overline{B \vdash B}}{!A, !A \multimap B \vdash B}$$
$$\frac{}{!A, !A, !A \multimap !A \multimap B \vdash B}$$
$$\frac{}{!A, !A \multimap !A \multimap B \vdash B}$$
$$\frac{}{!A, !(!A \multimap !A \multimap B) \vdash B}$$
$$\frac{}{!A \vdash !(!A \multimap !A \multimap B) \multimap B}$$
$$\frac{}{\vdash !A \multimap !(!A \multimap !A \multimap B) \multimap B}$$

$$\vdash !A \multimap (A \multimap A \multimap B) \multimap B$$

Recall II: Moggi

Let \mathcal{K} be a CCC and T be a **strong monad** on \mathcal{K} ,

$$A \times TB \xrightarrow{s} T(A \times B)$$

the **Kleisli** category for T is a model of the **cbv λ -calculus**.

λ_{ml}

$$\frac{}{x : A \vdash x : A}$$

$$\frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x.t : A \Rightarrow B}$$

$$\frac{\Gamma \vdash t : A \Rightarrow B \quad \Delta \vdash u : A}{\Gamma, \Delta \vdash (t)u : B}$$

$$\frac{\Gamma \vdash t : A}{\Gamma \vdash [t] : TA}$$

$$\frac{\Gamma \vdash M : TA \quad \Delta, x : A \vdash N : TB}{\Gamma, \Delta \vdash \text{let } x = M \text{ in } N : TB}$$

cbv λ -calculus

$$t ::= [V] \mid (t)t \qquad X^v = X \qquad \Gamma^v \vdash t^v : TA^v$$

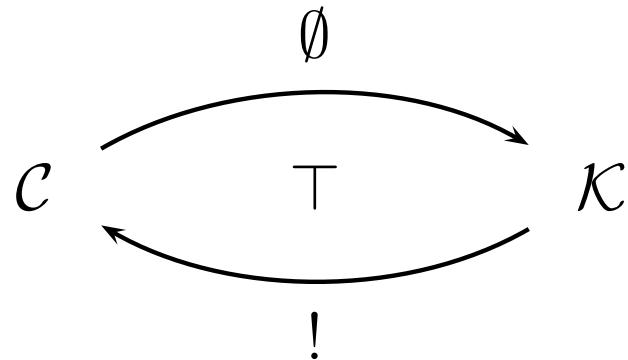
$$V ::= x \mid \lambda x.t \qquad (A \Rightarrow B)^v = A^v \Rightarrow TB^v \qquad \Gamma^v \vdash V^v : A^v$$

$$x^v = x \qquad (\lambda x.t)^v = \lambda x.t^v \qquad [V]^v = [V^v]$$

$$((t)u)^v = \text{let } x = t^v \text{ in let } y = u^v \text{ in } (x)y$$

Recall III: Bierman / Benton

- The adjunction between \mathcal{C} and \mathcal{K} is symmetric monoidal.



- The induced monad on \mathcal{K} is a symmetric monoidal monad (thus **strong!**).

$$\begin{array}{ccccc} & & T(TA \times B) & \xrightarrow{s} & TT(A \times B) \\ & \nearrow s & & & \searrow \mu \\ TA \times TB & & & & T(A \times B) \\ & \searrow s & & & \nearrow \mu \\ & & T(A \times TB) & \xrightarrow{s} & TT(A \times B) \end{array}$$

Girard's second translation

Original presentation

$$X^* = !X$$

$$(A \Rightarrow B)^* = !(A^* \multimap B^*)$$

$$(\Gamma \vdash A)^* = \Gamma^* \vdash A^*$$

Alternative presentation

$$X^* = X$$

$$(A \Rightarrow B)^* = !A^* \multimap !B^*$$

$$(\Gamma \vdash A)^* = !\Gamma^* \vdash !A^*$$

Girard's second translation

$$X^{\star} = X$$

$$(A \Rightarrow B)^{\star} = !A^{\star} \multimap !B^{\star}$$

$$(\Gamma \vdash A)^{\star} = !\Gamma^{\star} \vdash !A^{\star}$$

$!$ is a strong monad on \mathcal{K} \implies cbv λ -calculus

$$\begin{array}{c} \dfrac{\overline{!A \vdash !A} \quad \overline{!B \vdash !B}}{!A, !A \multimap !B \vdash !B} \\ \hline \dfrac{\overline{!A \vdash !A} \quad \overline{!A, !(A \multimap B) \vdash !B}}{!A, !A, !A \multimap !(A \multimap B) \vdash !B} \\ \hline \dfrac{\overline{!A, !A \multimap !(A \multimap B) \vdash !B}}{!A, !A \multimap !(A \multimap B) \vdash !B} \\ \hline \dfrac{\overline{!A, !(A \multimap !(A \multimap B)) \vdash !B}}{!A \vdash !(A \multimap !(A \multimap B)) \multimap !B} \\ \hline \dfrac{\overline{!A \vdash !(A \multimap !(A \multimap B)) \multimap !B}}{!A \vdash !(!(A \multimap !(A \multimap B))) \multimap !B} \\ \hline \dfrac{\overline{!A \vdash !(!(A \multimap !(A \multimap B))) \multimap !B}}{\vdash !A \multimap !(!(A \multimap !(A \multimap B))) \multimap !B} \end{array}$$

Recall IV: Scott

Models of the untyped λ -calculus are based on a **retraction**:

$$D \Rightarrow D \quad \lhd \quad D$$

a **reflexive object** D in a CCC is a model of the $\lambda(\beta)$ -calculus.

(a strict reflexive object $D \simeq D \Rightarrow D$ is a model of the $\lambda(\beta\eta)$ -calculus)

untyped λ -calculus = simply typed λ -calculus /_($\forall A,B \ A \equiv B$)

$$\frac{\frac{x : D \vdash t : D}{\vdash \lambda x.t : D \Rightarrow D} \quad \vdash e(\lambda x.t) : D}{\vdash d \circ e(\lambda x.t) : D \Rightarrow D} \quad \vdash u : D}{\vdash (d \circ e(\lambda x.t))u : D}$$

Preservation:

Retractions are preserved by co-Kleisli

τ -translation

Original presentation

$$X^{\star} = X$$

$$(A \Rightarrow B)^{\star} = !?A^{\star} \multimap ?B^{\star}$$

$$(\Gamma \vdash A)^{\star} = !?\Gamma^{\star} \vdash ?A^{\star}$$

Alternative presentation

$$X^{\star} = ?X$$

$$(A \Rightarrow B)^{\star} = ?(!A^{\star} \multimap B^{\star})$$

$$(\Gamma \vdash A)^{\star} = !\Gamma^{\star} \vdash A^{\star}$$

τ -translation

$$X^* = ?X$$

$$(A \Rightarrow B)^* = ?(!A^* \multimap B^*)$$

$$(\Gamma \vdash A)^* = !\Gamma^* \vdash A^*$$

$$?A^* \multimap A^*$$

$$\frac{\frac{\frac{\frac{?X \vdash ?X}{??X \vdash ?X}}{\frac{!A^* \vdash !A^* \quad B^* \vdash B^*}{!A^* \multimap B^*, !A^* \vdash B^*}}{\frac{!A^* \multimap B^*, !A^* \vdash B^*}{!A^* \multimap B^*, !A^* \vdash ?B^*}}{\frac{?(!A^* \multimap B^*), !A^* \vdash ?B^* \quad ?B^* \vdash B^*}{?(!A^* \multimap B^*), !A^* \vdash B^*}}}{?(!A^* \multimap B^*) \vdash !A^* \multimap B^*}$$

$$\frac{\forall A \quad ?A^* \triangleleft_{\mathcal{C}} A^*}{\forall A, B \quad ?(A \Rightarrow B)^* \triangleleft_{\mathcal{K}} (A \Rightarrow B)^*} \implies \text{ λ -calculus without } \eta$$

τ -translation

$$X^* = ?X$$

$$(A \Rightarrow B)^* = ?(!A^* \multimap B^*)$$

$$(\Gamma \vdash A)^* = !\Gamma^* \vdash A^*$$

$$\frac{\forall A \quad ?A^* \triangleleft_{\mathcal{C}} A^*}{\forall A, B \quad ?(A \Rightarrow B)^* \triangleleft_{\mathcal{K}} (A \Rightarrow B)^*} \implies \text{λ-calculus without } \eta$$

$$\frac{\frac{\frac{!A^* \vdash B^*}{\vdash !A^* \multimap B^*} \quad \vdash ?(!A^* \multimap B^*)}{\vdash !A^* \multimap B^*} \quad \frac{\vdash A^*}{\vdash !A^*} \quad \frac{B^* \vdash B^*}{B^* \vdash B^*}}{!A^* \multimap B^* \vdash B^*}$$

Recall V: Moggi

Let R be an object in a CCC,

$$A \quad \mapsto \quad (A \Rightarrow R) \Rightarrow R = \neg\neg A \quad \text{is a strong monad.}$$

Corresponding λ -terms

$$\eta = \lambda a. \lambda k. (k)a : A \Rightarrow \neg\neg A$$

$$\mu = \lambda m. \lambda k(m) \lambda K(K)k : \neg\neg\neg\neg A \Rightarrow \neg\neg A$$

$$s = \lambda \langle a, B \rangle. \lambda k. (B) \lambda b. (k) \langle a, b \rangle : A \times \neg\neg B \Rightarrow \neg\neg(A \times B)$$

q-translation

Original presentation

$$X^* = X$$

$$(A \Rightarrow B)^* = !A^* \multimap ?!B^*$$

$$(\Gamma \vdash A)^* = !\Gamma^* \vdash ?!A^*$$

Alternative presentation

$$X^* = !X$$

$$(A \Rightarrow B)^* = !(A^* \multimap ?B^*)$$

$$(\Gamma \vdash A)^* = \Gamma^* \vdash ?A^*$$

q-translation

$$X^{\star} = X$$

$$(A \Rightarrow B)^{\star} = !A^{\star} \multimap ?!B^{\star}$$

$$(\Gamma \vdash A)^{\star} = !\Gamma^{\star} \vdash ?!A^{\star}$$

?!A \simeq $!(!A \multimap \perp) \multimap \perp = (A \Rightarrow \perp) \Rightarrow \perp$ is a strong monad on \mathcal{K}
 \implies cbv λ -calculus

$$\begin{array}{c} \frac{}{!A \vdash !A} \quad \frac{}{?!B \vdash ?!B} \\ \hline \frac{}{!A, !A \multimap ?!B \vdash ?!B} \\ \hline \frac{}{!A, !(!A \multimap ?!B) \vdash ?!B} \\ \hline \frac{!A \vdash !A \quad !A, ?!(!A \multimap ?!B) \vdash ?!B}{!A, !A, !A \multimap ?!(!A \multimap ?!B) \vdash ?!B} \\ \hline \frac{}{!A, !A \multimap ?!(!A \multimap ?!B) \vdash ?!B} \\ \hline \frac{}{!A, !(!A \multimap ?!(!A \multimap ?!B)) \vdash ?!B} \\ \hline \frac{}{!A \vdash !(!A \multimap ?!(!A \multimap ?!B)) \multimap ?!B} \\ \hline \frac{}{!A \vdash !(!(!A \multimap ?!(!A \multimap ?!B)) \multimap ?!B)} \\ \hline \frac{}{!A \vdash ?!(!(!A \multimap ?!(!A \multimap ?!B)) \multimap ?!B)} \\ \hline \frac{}{\vdash !A \multimap ?!(!(!A \multimap ?!(!A \multimap ?!B)) \multimap ?!B)} \end{array}$$