

Deduction and Fractions

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Outline

Introduction

A deduction rule is a fraction

Deduction is the composition of fractions

Conclusion

Motivations

The semantics of **computational effects**

Cf. the talk by **Jean-Guillaume Dumas**:
a framework for dealing with the order of evaluation
of the arguments in a language with effects

Fact 1. Syntax, models, proofs,... : this is **logic**...

Fact 2. **Categories** and **limit sketches** provide tools
for dealing with the semantics *and* with the syntax.

Fact 3. A logic is, essentially, a (bi)category of **fractions**.

What is a logic?

A **logic** should have

- ▶ a **syntax**
which are the sentences of interest?
- ▶ a notion of **models**
what is the meaning of each sentence?
- ▶ a system for **proofs**
how can a sentence be inferred from another one?

In this talk we focus on **proofs**.

In this talk

A deduction rule, written **AS a fraction**

$$\frac{\mathcal{H}}{\mathcal{C}}$$

actually **IS a fraction** (in the categorical sense)

$$\frac{\mathcal{C}}{\mathcal{H}}$$

Propositional logic

Hilbert calculus, restricted to the connector “ \Rightarrow ”.

Syntax. Propositions (formulas) are made of symbols p, q, \dots and a binary operation “ \Rightarrow ”.

Models. Given a set of propositions Σ , a model (interpretation) of Σ associates to each proposition $p \in \Sigma$ a **truth value** $v(p) \in \{0, 1\}$ in accordance with the **truth table** for “ \Rightarrow ”:

A	B	$A \Rightarrow B$
0	0	1
0	1	1
1	0	0
1	1	1

Deduction rules. *modus ponens*

$$\frac{A \quad A \Rightarrow B}{B}$$

and two rules with “empty” premisses

$$A \Rightarrow (B \Rightarrow A) \quad (A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C))$$

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Theories and specifications: two categories

For a given logic:

- ▶ A **theory** Θ is a saturated class of sentences,
called **theorems**:
every sentence derived from Θ with the rules of the logic is in Θ .
- ▶ A **specification** Σ is a class of sentences,
called **axioms**:
new sentences may be derived from Σ with the rules of the logic
(generally).

This provides two **categories**:

- ▶ **T** for theories
- ▶ **S** for specifications

Theories and specifications: two adjoint functors

For a given logic:

- ▶ Every **theory** Θ can be seen as a (huge) specification $\mathbf{R}\Theta$.
- ▶ Every **specification** Σ generates a theory $\mathbf{L}\Sigma$ using the rules of the logic.

This provides two **adjoint functors**:

$$\mathbf{S} \begin{array}{c} \xrightarrow{\mathbf{L}} \\ \xleftarrow{\mathbf{R}} \\ \perp \end{array} \mathbf{T}$$

In addition, every theory Θ is **saturated**:

$$\mathbf{L}\mathbf{R}\Theta \cong \Theta$$

Propositional logic: theories

A propositional theory Θ is:

- ▶ a set $\Theta(F)$ of **formulas**
- ▶ a subset $\Theta(T)$ of **true formulas**
- ▶ a binary operation “ \Rightarrow ”: $\Theta(F)^2 \rightarrow \Theta(F)$
which satisfies the rules: for all p, q, r in $\Theta(F)$:
 - ▶ if $p, p \Rightarrow q \in \Theta(T)$ then $q \in \Theta(T)$
 - ▶ $p \Rightarrow (q \Rightarrow p) \in \Theta(T)$
 - ▶ $(p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r)) \in \Theta(T)$

Example. The theory of booleans \mathcal{B} :

$$\mathcal{B}(F) = \{0, 1\}, \mathcal{B}(T) = \{1\},$$

$$\mathcal{B}(\Rightarrow)(1, 0) = 0, \text{ otherwise } \mathcal{B}(\Rightarrow)(p, q) = 1$$

Propositional logic: specifications

A propositional specification Σ is:

- ▶ a set $\Sigma(F)$ of **formulas**
- ▶ a subset $\Sigma(T)$ of **true formulas**
- ▶ a partial binary operation “ \Rightarrow ”: $\Sigma(F)^2 \rightharpoonup \Sigma(F)$

Example.

$$\Sigma_0(F) = \{p, q\}, \Sigma_0(T) = \emptyset, q = (p \Rightarrow p).$$

Propositional logic is an adjunction

- ▶ $\mathbf{R} : \mathbf{T} \rightarrow \mathbf{S}$ is the inclusion
- ▶ $\mathbf{L} : \mathbf{S} \rightarrow \mathbf{T}$ generates theorems from axioms

A **specification** Σ_0 :

$$\Sigma_0(F) = \{p, q\}, \Sigma_0(T) = \emptyset, q = (p \Rightarrow p).$$

The **models** of Σ_0 :

$$v_0: v_0(p) = 0, v_0(q) = 1.$$

$$v_1: v_1(p) = 1, v_1(q) = 1.$$

The **theory** $\mathbf{L}\Sigma_0$:

$$\mathbf{L}\Sigma_0(F) = \{p, q, p \Rightarrow q, \dots\}, \mathbf{L}\Sigma_0(T) = \{q, \dots\}.$$

$q \in \mathbf{L}\Sigma_0(T)$ because $p \Rightarrow p$ can be deduced, using the propositional rules.

The models of Σ_0 are the **morphisms of theories** $\mathbf{L}\Sigma_0 \rightarrow \mathcal{B}$.

Diagrammatic logic

Definition. (without syntax...)

A **diagrammatic logic** is functor with a full and faithful right adjoint

So, a **logic** is

- ▶ a category of **theories** \mathbf{T}
- ▶ a category of **specifications** \mathbf{S}
- ▶ a **forgetful** functor $\mathbf{R} : \mathbf{T} \rightarrow \mathbf{S}$
- ▶ a **generating** functor $\mathbf{L} : \mathbf{S} \rightarrow \mathbf{T}$

which form an **adjunction**

$$\mathbf{S} \begin{array}{c} \xrightarrow{\mathbf{L}} \\ \xleftarrow{\mathbf{R}} \\ \perp \end{array} \mathbf{T}$$

with \mathbf{R} full and faithful, *i.e.*, $\mathbf{LR}\Theta \cong \Theta$ for every theory Θ

Entailments and fractions

With respect to a logic $\mathbf{L} : \mathbf{S} \rightarrow \mathbf{T}$

- ▶ An **entailment**

$$\Sigma \xleftarrow{\tau} \Sigma'$$

is a morphism $\tau : \Sigma \rightarrow \Sigma'$ in \mathbf{S} such that $\mathbf{L}\tau$ is invertible in \mathbf{T} .

- ▶ A **fraction** $\frac{\sigma}{\tau}$ is a cospan in \mathbf{S} made of a morphism σ (the **numerator**) and an entailment τ (the **denominator**)

$$\Sigma \xrightarrow{\sigma} \Sigma'_1 \xleftarrow{\tau} \Sigma_1$$

Then $\mathbf{L}\left(\frac{\sigma}{\tau}\right) = (\mathbf{L}\tau)^{-1} \circ \mathbf{L}\sigma$ in \mathbf{T}

$$\begin{array}{ccccc} \mathbf{L}\Sigma & \xrightarrow{\mathbf{L}\sigma} & \mathbf{L}\Sigma'_1 & \xleftarrow{\mathbf{L}\tau} & \mathbf{L}\Sigma_1 \\ & & & \xrightarrow{\mathbf{L}\tau^{-1}} & \\ & \searrow & & \nearrow & \\ & & \mathbf{L}\left(\frac{\sigma}{\tau}\right) & & \end{array}$$

“The” theorem

Gabriel and Zisman (1967)

Calculus of Fractions and Homotopy Theory. Ch. 1.

Remark. Every theory Θ is $\Theta = \mathbf{L}\Sigma$ for some specification Σ .

Remark. In general, a morphism of theories $\theta: \mathbf{L}\Sigma \rightarrow \mathbf{L}\Sigma_1$ is **not** $\theta = \mathbf{L}\sigma$ for a morphism of specifications $\sigma: \Sigma \rightarrow \Sigma_1$.
(because Σ_1 is “too small”)

Theorem. Every morphism of theories $\theta: \mathbf{L}\Sigma \rightarrow \mathbf{L}\Sigma_1$ is $\theta = \mathbf{L}(\frac{\sigma}{\tau})$ for some **fraction** $\frac{\sigma}{\tau}: \Sigma \rightarrow \Sigma_1$.

Corollary. (Up to equiv.) \mathbf{T} is the category of fractions of \mathbf{S} with denominators the entailments.

What is a deduction rule?

With respect to a logic $\mathbf{L} : \mathbf{S} \rightarrow \mathbf{T}$

Definition.

A **rule** $\frac{\mathcal{H}}{\mathcal{C}}$ is a fraction $\frac{\mathcal{C}}{\mathcal{H}} : \mathcal{C} \rightarrow \mathcal{H}$

$$\mathcal{H} \xleftarrow{\quad h \quad} \mathcal{H}' \xleftarrow{\quad c \quad} \mathcal{C}$$

This definition includes both **elementary rules** and **derived rules** (or **proofs**)

(the distinction is provided by the syntax of \mathbf{L}).

According to [GZ68], the rules are the morphisms of theories, expressed as fractions.

The modus ponens rule

$$\frac{\mathcal{H} \quad A \quad A \Rightarrow B}{\mathcal{C}} = \frac{A \quad A \Rightarrow B}{B}$$

- ▶ **Static.** A **theory** Θ is a saturated set of **theorems**.
Let Θ be a theory with theorems p and $p \Rightarrow q$.
Then theorem q is also in Θ .
- ▶ **Dynamic.** A **specification** Σ is a set of **axioms**,
which generates a **theory** $\mathbf{L}\Sigma$.
Let Σ be a specification with axioms p and $p \Rightarrow q$.
Then the specification Σ' made of Σ and the axiom q
is **equivalent** to Σ , *i.e.*, $\mathbf{L}\Sigma = \mathbf{L}\Sigma'$.

The modus ponens fraction

Propositional specifications:

$$\mathcal{H} : \mathcal{H}(F) = \{A, B, A \Rightarrow B\}, \quad \mathcal{H}(T) = \{A, A \Rightarrow B\}$$

$$\mathcal{C} : \mathcal{C}(F) = \{B\}, \quad \mathcal{C}(T) = \{B\}$$

$$\mathcal{H}' : \mathcal{H}'(F) = \{A, B, A \Rightarrow B\}, \quad \mathcal{H}'(T) = \{A, B, A \Rightarrow B\}$$

The inclusions of \mathcal{H} and \mathcal{C} in \mathcal{H}' are **morphisms** of specifications and h is an **entailment**

$$\mathcal{H} \xrightarrow{h} \mathcal{H}' \xleftarrow{\mathcal{C}} \mathcal{C}$$

$\mathbf{L}h$ is an **isomorphism** of theories, $\mathbf{L}(\frac{\mathcal{C}}{h}) = \mathbf{L}h^{-1} \circ \mathbf{L}\mathcal{C}$

$$\begin{array}{ccccc} \mathbf{L}\mathcal{H} & \xrightarrow{\mathbf{L}h} & \mathbf{L}\mathcal{H}' & \xleftarrow{\mathbf{L}\mathcal{C}} & \mathbf{L}\mathcal{C} \\ & \swarrow \mathbf{L}h^{-1} & & & \\ & & & \searrow & \\ & & & \mathbf{L}(\frac{\mathcal{C}}{h}) & \end{array}$$

Rules are fractions

RULES	FRACTIONS	numbers
$\mathcal{H}, \mathcal{C} : \text{rules}$	$\mathcal{H}, \mathcal{C} : \text{fractions}$	$2, 3 \in \mathbb{Z}$
$\frac{\mathcal{H}}{\mathcal{C}}$	$\mathcal{H} \begin{array}{c} \xrightarrow{h} \\ \xleftarrow{\quad} \end{array} \mathcal{H}' \begin{array}{c} \xleftarrow{c} \\ \xrightarrow{\quad} \end{array} \mathcal{C}$	$\frac{3}{2} \quad (\frac{3}{2} \neq \frac{6}{4})$ “syntactically”
$\frac{\mathcal{H}}{\mathcal{C}}$	$\mathbf{L}\mathcal{H} \begin{array}{c} \xrightarrow{\mathbf{L}h} \\ \xleftarrow{\quad} \end{array} \mathbf{L}\mathcal{H}' \begin{array}{c} \xleftarrow{\mathbf{L}c} \\ \xrightarrow{\quad} \end{array} \mathbf{L}\mathcal{C}$	$\frac{3}{2} \in \mathbb{Q} \quad (\frac{3}{2} = \frac{6}{4})$

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Deduction

The **deduction process** is
a succession of **deduction steps**.

A **deduction step**:

- ▶ *Input.* A rule $\frac{\mathcal{H}}{\mathcal{C}}$, a specification Σ ,
an **instance** i of \mathcal{H} in Σ .
- ▶ *Output.* The **instance** j of \mathcal{C} in Σ
which corresponds to “applying $\frac{\mathcal{H}}{\mathcal{C}}$ to i ”.

What is a deduction step?

With respect to a logic $\mathbf{L} : \mathbf{S} \rightarrow \mathbf{T}$

Definition.

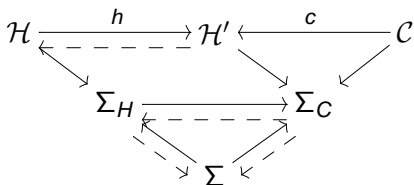
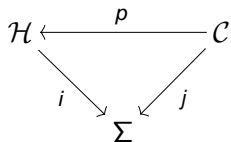
An **instance** of \mathcal{H} in Σ is a fraction $i : \mathcal{H} \rightarrow \Sigma$

Definition.

The **step** applying a rule $p : \mathcal{C} \rightarrow \mathcal{H}$

to an instance $i : \mathcal{H} \rightarrow \Sigma$ of \mathcal{H} in Σ

is the composition of fractions $i \circ p : \mathcal{C} \rightarrow \Sigma$



Deduction process

Since a deduction step is a composition,
the **deduction process** is (essentially)
a succession of compositions....

... combined with **colimits** of specifications
for grouping several hypotheses in a unique one...

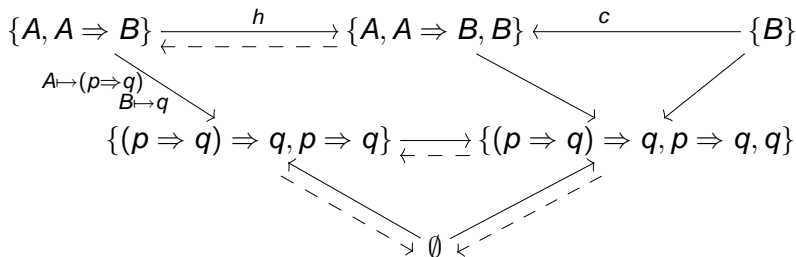
... resulting in the usual **tree-like** representation
of the deduction.

Applying modus ponens

The specification Σ_0 : $\Sigma_0(F) = \{p, q\}$, $\Sigma_0(T) = \emptyset$, $q = (p \Rightarrow p)$ generates the theorem q .

The last step in the proof is an application of *modus ponens*:

$$\frac{(p \Rightarrow q) \Rightarrow q \quad p \Rightarrow q}{q}$$



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- ▶ The category of fractions is the quotient of a **bicategory**, and bicategories are technically difficult...
Cf. the talk by **Pawel Sobocinski**.
- ▶ More about models, syntax, etc...
 - D.D. *How to combine diagrammatic logics*.
- ▶ More examples
 - Jean-Guillaume Dumas , D.D., Jean-Claude Reynaud. *Cartesian effect categories are Freyd-categories*.
 - Cesar Dominguez, D.D. *A parameterization process as a categorical construction*.