Sequential computation and Cartesian Effect Categories

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Categorical semantics for programming languages

- Program / Category

Type	X
Function	Y f(X x);
Arguments	Z f(X x, Y y)
Substitution	f(g(x))

Object	X
Morphism	$f : X \rightarrow Y$
Product	$f : X \times Y \rightarrow Z$
Composition	$f \circ g$

- Effects
  - Non-termination
  - Modification of the state
  - …

⚠ In particular the order of evaluation of arguments has consequences when there are side-effects: $f \times g$

Related work

- Strong Monads *[Moggi 1989]*
- Freyd categories *[Power, Robinson 1997]*
- Haskell's Arrows *[Hughes 2000]*
- Evaluation logic *[Moggi 1995]*

⇨ Quite similar frameworks: *[Heunen, Jacobs 2006], [Atkey 2008]*

☺ Cartesian effect categories: more precise

Contents

- Effect categories
  - Pure morphism
  - Effect of a morphism, same effect relation $\approx$
  - Consistency relations $\lhd$ $\blacktriangleleft$ complements same effect
  - Examples: errors, partiality, state
- Cartesian effect categories
  - Semi-pure product $\rtimes$ $\ltimes$
  - Sequential product ⧕ ⧔
- Comparisons
  - Evaluation logic
  - Haskell's Arrows
  - Freyd categories, strong monads

Effects

- What is an effect ?
  - *Pure morphisms* are effect-free u : X ~~> Y
  - If **v** is pure then **f** and **v**∘**f** have the same effect

- Effect of a morphism f : X → Y
  - everything but the "result"

  Ɛ(f) = ()$_Y$∘f : X → 1

```
void effect ( Y (*f)(X), X x) { Y y = f(x); }
```

- Same effect equivalence relation ≈

$$\forall f : X \to Y \,, \forall f' : X \to Y' \,, f \approx f' \iff \langle\,\rangle_Y \circ f = \langle\,\rangle_{Y'} \circ f'$$

Consistency

- We need also a relation $\lhd$ stating roughly that
  - results are the same,
  - but effects might be different
- We define several relations $\lhd$ $\blacktriangleleft$ $\lhd\rhd$ $\blacktriangleleft\blacktriangleleft$
  - “minimal” requirements or “common” properties
  - Some extensions
- An Effect Category has
  - Same-effect and Consistency relations
  - Such that consistency is complementary to the same-effect:

$$\forall f, f' : X \to Y\ ,\ (f \approx f') \wedge (f \lhd \rhd f') \implies f = f'$$

Consistencies

- A consistency relation f $\lhd$ v
  - Pure reflexivity: $\forall$ v pure : X~~>Y, v $\lhd$ v
  - Compatibility with composition:

$$\begin{tikzcd} X \arrow[r, squiggly, "v"] \arrow[dr, "f"'] \arrow[r, phantom, shift right=3, "\triangledown"] & Y' \arrow[r, squiggly, "w"] \arrow[r, phantom, shift right=3, "\triangledown"] & Z \\ & Y \arrow[u, squiggly, "u"] \arrow[ur, "g"'] & \end{tikzcd} \Longrightarrow \begin{tikzcd} X \arrow[rr, squiggly, "w \circ v"] \arrow[rr, bend right, "g \circ f"'] \arrow[rr, phantom, "\triangledown"] & & Z \end{tikzcd}$$

- f $\lhd\rhd$ g if there exists v *pure* s.t. f $\lhd$ v $\rhd$ g

- Extended consistency $\blacktriangleleft$ between non pure morphisms
  - ✚ Extension: $f \lhd v \Longrightarrow f \blacktriangleleft v$
  - ✚ Substitution: $g \blacktriangleleft g' \Longrightarrow g \circ f \blacktriangleleft g' \circ f$

  - ☝ There exists a smallest extended consistency

$$\begin{tikzcd} X \arrow[r, "f"'] & Y \arrow[r, squiggly, "w"] \arrow[r, bend right=60, "g"'] \arrow[r, phantom, shift right=3, "\triangledown"] & Z \end{tikzcd} \Longleftrightarrow \begin{tikzcd} X \arrow[rr, squiggly, "w \circ f"] \arrow[rr, bend right, "g \circ f"'] \arrow[rr, phantom, "\blacktriangledown"] & & Z \end{tikzcd}$$

Error

- Pure morphisms do not raise errors
- $f \approx g$ iff
  - f and g raise the same errors for the same arguments
- $f \lhd v$ iff
  - f coincides with v on D_f
- $f \lhd \rhd g$ iff
  - f and g coincides on $D_f \cap D_g$
- $f \blacktriangleleft g$ iff
  - f and g coincides on $D_f \subseteq D_g$ and on $\overline{D_g}$

Partial functions

- Pure morphisms are total functions
- Effect is the domain of definition as in *[Curien, Obtulowitz 89]*

- $f \approx g$ iff
  - f and g have the same domain of definition, $()_Y \circ f = ()_{Y'} \circ g$

- $f \lhd v$ iff $f \leq v$

- $f \blacktriangleleft g$ iff $f \leq g$

- Complementarity of $\approx$ and $\lhd$: as axiom

State

- Pure morphisms do not modify the state S
  - We denote $\sigma_X : S \times X \to S$ and $\pi_X : S \times X \to X$ the projections

- $f \approx g$ iff $\sigma_Y \circ f = \sigma_{Y'} \circ g$
  - f and g modify the state in the same manner

- $f \lhd v$ iff $\pi_Y \circ f = v \circ \pi_X$
  - The "result" of f is always v (or that of v)

- $f \blacktriangleleft g$ iff $\pi_Y \circ f = \pi_Y \circ g$
  - f and g "always have the same result"

- $\blacktriangleleft$ is an equivalence relation and $\blacktriangleleft \blacktriangleright$ and $\blacktriangleleft$ are identical

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- Cartesian effect categories
  - Semi-pure product $\rtimes$ $\ltimes$
  - Sequential product $\boldsymbol{\rtimes}$ $\boldsymbol{\ltimes}$
- Related Work
  - Evaluation logic
  - Haskell's Arrows
  - Freyd categories, strong monads

Binary product property

- p, q, r, s, t are projections
- The binary product defines a functor $\times : C^2 \rightarrow C$ s.t. for all v_1 and v_2, the morphism $v_1 \times v_2$ is the unique morphism that satisfies the binary product property:

$$q_1 \circ (v_1 \times v_2) = v_1 \circ p_1$$
$$q_2 \circ (v_1 \times v_2) = v_2 \circ p_2$$

$$\begin{tikzcd}
X_1 \arrow[r, rightsquigarrow, "v_1"] & Y_1 \\
X_1 \times X_2 \arrow[u, rightsquigarrow, "p_1"] \arrow[d, rightsquigarrow, "p_2"'] \arrow[r, rightsquigarrow, "v_1 \times v_2"] \arrow[ur, phantom, "="] \arrow[dr, phantom, "="] & Y_1 \times Y_2 \arrow[u, rightsquigarrow, "q_1"'] \arrow[d, rightsquigarrow, "q_2"] \\
X_2 \arrow[r, rightsquigarrow, "v_2"] & Y_2
\end{tikzcd}$$

Semi-pure product

- $(C \subseteq K, \lhd)$ is an effect category
- A left semi-pure product $\ltimes$ extends $\times$ and satisfies the semi-pure product property:

$$q_1 \circ (v_1 \ltimes f_2) \lhd v_1 \circ p_1$$
$$q_2 \circ (v_1 \ltimes f_2) = f_2 \circ p_2$$

$$\begin{tikzcd}
X_1 \arrow[r, rightsquigarrow, "v_1"] \arrow[d, phantom, ""] & Y_1 \\
X_1 \times X_2 \arrow[u, rightsquigarrow, "p_1"] \arrow[r, "v_1 \ltimes f_2"] \arrow[d, rightsquigarrow, "p_2"'] \arrow[ur, phantom, "\triangledown"] & Y_1 \times Y_2 \arrow[u, rightsquigarrow, "q_1"'] \arrow[d, rightsquigarrow, "q_2"] \\
X_2 \arrow[r, "f_2"] \arrow[ur, phantom, "="] & Y_2
\end{tikzcd}$$

- Complementarity: **$\varepsilon(v_1 \ltimes f_2) = \varepsilon(f_2 \circ p_2) = \varepsilon(q_1 \circ (v_1 \ltimes f_2))$**

Sequential product

- (C $\subseteq$ K, $\lhd$) is an effect category
- A left sequential product $\ltimes$ is composed from semi-pure products as follows:

$$f_1 \ltimes f_2 = (\mathrm{id}_{Y_1} \ltimes f_2) \circ (f_1 \rtimes \mathrm{id}_{X_2})$$

$$\begin{tikzcd}
X_1 \arrow[r, "f_1"] \arrow[dr, phantom, "="] & Y_1 \arrow[r, squiggly, "\mathrm{id}"] \arrow[dr, phantom, "\triangledown"] & Y_1 \\
X_1 \times X_2 \arrow[u, squiggly, "p_1"] \arrow[d, squiggly, "p_2"'] \arrow[r, "f_1 \rtimes \mathrm{id}"] \arrow[dr, phantom, "\triangle"] & Y_1 \times X_2 \arrow[u, squiggly, "r_1"'] \arrow[d, squiggly, "r_2"] \arrow[r, "\mathrm{id} \ltimes f_2"] \arrow[dr, phantom, "="] & Y_1 \times Y_2 \arrow[u, squiggly, "q_1"'] \arrow[d, squiggly, "q_2"] \\
X_2 \arrow[r, squiggly, "\mathrm{id}"] & X_2 \arrow[r, "f_2"] & Y_2
\end{tikzcd}$$

Sequential product properties

- States that two graph homomorphisms $\cdot\!\rightsquigarrow$ and $\leftsquigarrow\!\cdot$ satisfy:

$$
\begin{aligned}
q_1 \circ (f_1 \cdot\!\rightsquigarrow f_2) &\blacktriangleleft f_1 \circ p_1 \\
q_2 \circ (f_1 \cdot\!\rightsquigarrow f_2) &= f_2 \circ r_2 \circ (f_1 \leftsquigarrow\!\cdot \mathrm{id}_{X_2}) \\
q_1 \circ (f_1 \leftsquigarrow\!\cdot f_2) &= f_1 \circ s_1 \circ (\mathrm{id}_{X_2} \cdot\!\rightsquigarrow f_1) \\
q_2 \circ (f_1 \leftsquigarrow\!\cdot f_2) &\blacktriangleleft f_2 \circ p_2
\end{aligned}
$$

$$
\begin{tikzcd}
X_1 \arrow[rr, "f_1"] \arrow[drr, phantom, "\blacktriangledown"] & & Y_1 \\
X_1 \times X_2 \arrow[u, squiggly, "p_1"] \arrow[rr, "f_1 \cdot\!\rightsquigarrow f_2"] \arrow[dr, bend right, "f_1 \leftsquigarrow\!\cdot \mathrm{id}"'] & & Y_1 \times Y_2 \arrow[u, squiggly, "q_1"'] \arrow[dd, squiggly, "q_2"] \\
& Y_1 \times X_2 \arrow[d, squiggly, "r_2"] \arrow[r, phantom, "="] & \\
& X_2 \arrow[r, "f_2"] & Y_2
\end{tikzcd}
$$

$$
\begin{tikzcd}
& X_1 \arrow[r, "f_1"] & Y_1 \\
& X_1 \times Y_2 \arrow[u, squiggly, "s_1"'] \arrow[r, phantom, "="] & \\
X_1 \times X_2 \arrow[ur, bend left, "\mathrm{id} \cdot\!\rightsquigarrow f_2"] \arrow[rr, "f_1 \leftsquigarrow\!\cdot f_2"] \arrow[d, squiggly, "p_2"'] \arrow[drr, phantom, "\blacktriangle"] & & Y_1 \times Y_2 \arrow[uu, squiggly, "q_1"'] \arrow[d, squiggly, "q_2"] \\
X_2 \arrow[rr, "f_2"] & & Y_2
\end{tikzcd}
$$

Theorems and proofs in a Cartesian effect category

- The sequential products $\mathbin{\times\!\!\blacktriangleleft}\ \mathbin{\blacktriangleright\!\!\times}$ satisfy the sequential product property $\cdot\!\leadsto\ \leftsquigarrow\!\cdot$

- Every pure morphism is central
  - $v \ltimes f = v \rtimes f$

- Non ambiguity
  - $v \ltimes f = v \times f$
  - $g \mathbin{\blacktriangleright\!\!\times} f = g \ltimes f$

- $(id \ltimes g) \circ (id \ltimes f) = id \ltimes (g \circ f)$
- $(k \ltimes g) \circ (f_1 \mathbin{\blacktriangleright\!\!\times} f_2) = (k \circ f_1) \mathbin{\blacktriangleright\!\!\times} (g \circ f_2)$

- Associativity, swap, etc.

Error, Partiality, State

- Error

$$(f_1 \ltimes f_2)(x_1, x_2) = \begin{cases} \langle [f_1](x_1), [f_2](x_2) \rangle & \text{if } [f_1](x_1) \in Y_1 \text{ and } [f_2](x_2) \in Y_2 \\ [f_2](x_2) & \text{if } [f_1](x_1) \in Y_1 \text{ and } [f_2](x_2) \in E \\ [f_1](x_1) & \text{if } [f_1](x_1) \in E \end{cases}$$

- Partiality

$$\mathcal{D}_{(f_1 \ltimes f_2)} = \mathcal{D}_{f_1} \times \mathcal{D}_{f_2} \quad \text{and} \quad (f_1 \ltimes f_2)(x_1, x_2) = \langle [f_1](x_1), [f_2](x_2) \rangle$$

- State

$$\forall x_1 \in X_1\,,\ \forall x_2 \in X_2\,,\ \forall s \in S$$

$$[f_1 \ltimes f_2](s, x_1, x_2) = \langle s_2, y_1, y_2 \rangle$$

where $[f_1](s, x_1) = \langle s_1, y_1 \rangle$ and $[f_2](s_1, x_2) = \langle s_2, y_2 \rangle$

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Evaluation logic, Haskell's Arrows

- Evaluation logic

Monad MY	Results [Moggi95] $c \Downarrow a$	Consistency $f \lhd v$
$Y + E$	$c = a$ (thus, c is total)	$c \in Y \implies c = a$
$(Y \times S)^S$	$\exists s \in S\,,\ \exists s' \in S\,,\ c(s) = (a, s')$	$\forall s \in S\,,\ \exists s' \in S\,,\ c(s) = (a, s')$
$\mathcal{L}(Y)$	$a \in c$	$\exists k \in \mathbb{N}\,,\ c = (a)^k$
$\mathcal{P}_{\mathsf{fin}}(Y)$	$a \in c$	$c = \{a\}$ or $c = \emptyset$

- Cartesian effect categories give rise to an Arrow

Cartesian effect categories	Arrows
$K(X,Y)$	$A(X,Y)$
$C(X,Y) \subseteq K(X,Y)$	$\texttt{arr} : C(X,Y) \to A(X,Y)$
$f \mapsto (g \mapsto g \circ f)$	$\ggg : A(X,Y) \to A(Y,Z) \to A(X,Z)$
$f \mapsto f \ltimes \mathsf{id}$	$\texttt{first} : A(X,Y) \to A(X \times Z, Y \times Z)$

Freyd categories, strong monad

- Cartesian effect categories *are* Freyd categories

- Strong monads
  - The strength t can be expressed as a left Kleisli product
    - $]t_{Y1,Y2}[= id_{Y1} \ltimes_{Kl}]id_{MY2}[$
  - C_0 Cartesian category with strong monad (M,t) and a consistency is a weak Cartesian effect category

    if and only if

    the strength is consistent with the identity

Conclusion

- A formalization of computational effects
  - Consistency with effect category
  - Semi-pure and sequential products with Cartesian effect category
  - Universal property of semi-pure products:
    - Powerful tool for definitions and proofs
    - Cartesian effect categories are Freyd categories
    - Conditions for strong monads to be CEC
  - Many kind of effects (error, partiality, state, …) → a CEC

- Clarify evaluation logic / Cartesian effect categories ?
- Combining effects ?