## Sequential computation and Cartesian Effect Categories

### Jean-Guillaume Dumas Dominique Duval Jean-Claude Reynaud



Université de Grenoble Laboratoire Jean Kuntzmann Applied Mathematics and Computer Science Department



**Claude Shannon Institute** 

Discrete Mathematics, Coding, Cryptography and Information Security www.shannoninstitute.ie

# Categorical semantics for programming languages

	Program	/
Туре	X	
Function	Y f(X x);	
Arguments	Z f(X x, Y y)	
Substitution	f(g(x))	

Category

Object	X
Morphism	$f: X \to Y$
Product	$f: X \times Y \rightarrow Z$
Composition	f∘g

- Effects
  - Non-termination
  - Modification of the state

- ...

 $\bigtriangleup$  In particular the order of evaluation of arguments has consequences when there are side-effects: f  $\times$  g

#### **Related work**

• Strong Monads

[Moggi 1989]

- Freyd categories [Power, Robinson 1997]
- Haskell's Arrows
   [Hughes 2000]
- Evaluation logic [Moggi 1995]
- ⇒ Quite similar frameworks: [Heunen, Jacobs 2006], [Atkey 2008]
- © Cartesian effect categories: more precise

#### Contents

- Effect categories
  - Pure morphism
  - Effect of a morphism, same effect relation  $\approx$
  - − Consistency relations <> < complements same effect</p>
  - Examples: errors, partiality, state
- Cartesian effect categories
  - Semi-pure product  $\rtimes \ltimes$
  - Sequential product ightarrow 
    ightarrow 
    ightarrow
- Comparisons
  - Evaluation logic
  - Haskell's Arrows
  - Freyd categories, strong monads

#### Effects

- What is an effect ?
  - Pure morphisms are effect-free u : X ~~> Y
  - If **v** is pure then **f** and **v**∘**f** have the same effect
- Effect of a morphism  $f : X \rightarrow Y$ 
  - everything but the "result"

 $\mathcal{E}(f) = ()_{Y} \circ f : X \to 1$ <br/>void effect ( Y (\*f)(X), X x) { Y y = f(x); }

• Same effect equivalence relation  $\approx$ 

 $\forall f: X \to Y, \,\forall f': X \to Y', \, f \approx f' \iff \langle \rangle_Y \circ f = \langle \rangle_{Y'} \circ f'$ 

#### Consistency

- We need also a relation < stating roughly that
  - results are the same,
  - but effects might be different
- We define several relations

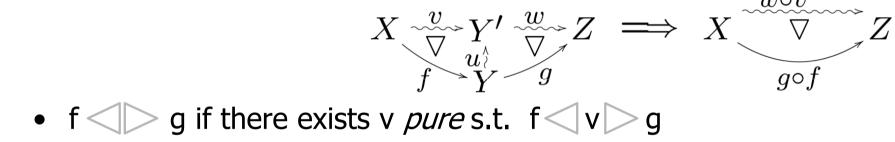


- "minimal" requirements or "common" properties
- Some extensions
- An Effect Category has
  - Same-effect and Consistency relations
  - Such that consistency is complementary to the same-effect:

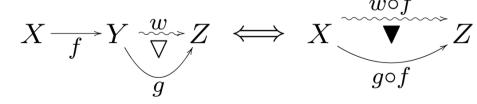
#### $\forall f, f' : X \to Y, \ (f \approx f') \land (f \triangleleft \triangleright f') \implies f = f'$

#### Consistencies

- A consistency relation f < v
  - Pure reflexivity:  $\forall$  v pure : X~~>Y, v  $\lhd$  v
  - Compatibility with composition:



- Extended consistency between non pure morphisms
   ★ Extension: f < v ⇒ f < v</li>
   ★ Substitution: g < g' ⇒ g ∘ f < g' ∘ f</li>
  - There exists a smallest extended consistency



#### Error

- Pure morphisms do not raise errors
- $f \approx g$  iff
  - f and g raise the same errors for the same arguments
- $f \triangleleft v$  iff
  - f coincides with v on  $\rm D_{f}$
- $f \triangleleft \triangleright g$  iff
  - f and g coincides on  $\mathsf{D}_{\mathsf{f}} \cap \mathsf{D}_{\mathsf{g}}$
- $f \blacktriangleleft g$  iff - f and g coincides on  $D_f \subseteq D_q$  and on  $\overline{D_q}$

#### **Partial functions**

- Pure morphisms are total functions
- Effect is the domain of definition as in [Curien, Obtulowitz 89]
- $f \approx g$  iff

– f and g have the same domain of definition, ()<sub>Y</sub> $\circ$ f = ()<sub>Y'</sub> $\circ$ g

- $f \triangleleft v$  iff  $f \leq v$
- $f \blacktriangleleft g$  iff  $f \le g$
- Complementarity of  $\approx$  and  $\triangleleft$ : as axiom

#### State

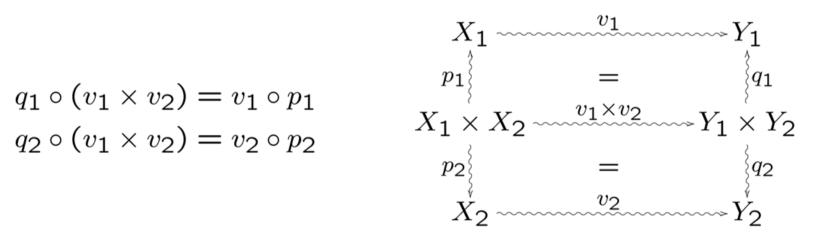
- Pure morphisms do not modify the state S
  - We denote  $\sigma_X:\,S{\times}X\to S$  and  $\pi_X:\,S{\times}X\to X$  the projections
- $f \approx g$  iff  $\sigma_Y \circ f = \sigma_{Y'} \circ g$ - f and g modify the state in the same manner
- $f \triangleleft v$  iff  $\pi_Y \circ f = v \circ \pi_X$ 
  - The "result" of f is always v (or that of v)
- $f \blacktriangleleft g$  iff  $\pi_Y \circ f = \pi_Y \circ g$ 
  - f and g "always have the same result"
- $\blacktriangleleft$  is an equivalence relation and  $\blacktriangleleft \triangleright$  and  $\blacktriangleleft$  are identical

#### Contents

- Effect categories
  - Pure morphism
  - Effect of a morphism, same effect relation  $\approx$
  - − Consistency relations < < complements same effect</p>
  - Examples: errors, partiality, state
- Cartesian effect categories
  - Semi-pure product  $\rtimes \ltimes$
  - Sequential product ightarrow 
    ightarrow 
    ightarrow
- Related Work
  - Evaluation logic
  - Haskell's Arrows
  - Freyd categories, strong monads

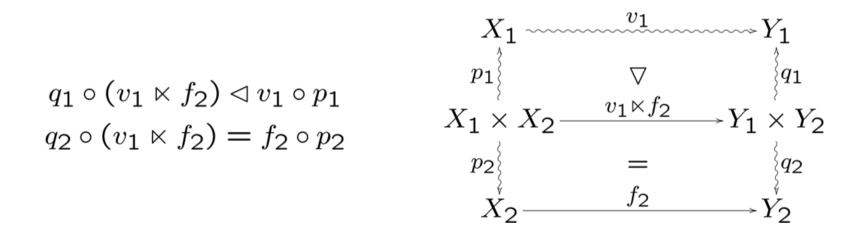
#### **Binary product property**

- p, q, r, s, t are projections
- The binary product defines a functor  $\times$  : C<sup>2</sup>  $\rightarrow$  C s.t. for all  $v_1$  and  $v_2$ , the morphism  $v_1 \times v_2$  is the unique morphism that satisfies the binary product property:



#### **Semi-pure product**

- (C  $\subseteq$  K,  $\triangleleft$ ) is an effect category
- A left semi-pure product ⋉ extends × and satisfies the semi-pure product property:

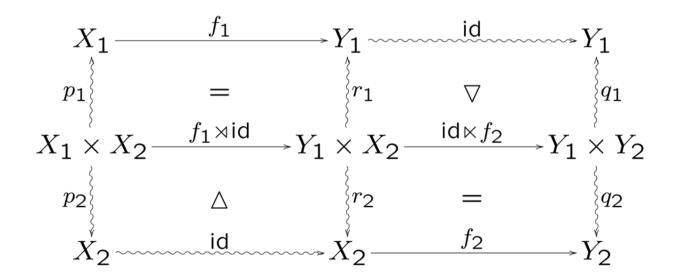


• Complementarity:  $\varepsilon(\mathbf{v_1} \ltimes \mathbf{f_2}) = \varepsilon(\mathbf{f_2} \circ \mathbf{p_2}) = \varepsilon(\mathbf{q_1} \circ (\mathbf{v_1} \ltimes \mathbf{f_2}))$ 

#### **Sequential product**

- (C  $\subseteq$  K,  $\triangleleft$ ) is an effect category
- A left sequential product K is composed from semipure products as follows:

$$f_1 \ltimes f_2 = (\mathrm{id}_{Y_1} \ltimes f_2) \circ (f_1 \rtimes \mathrm{id}_{X_2})$$



#### **Sequential product properties**

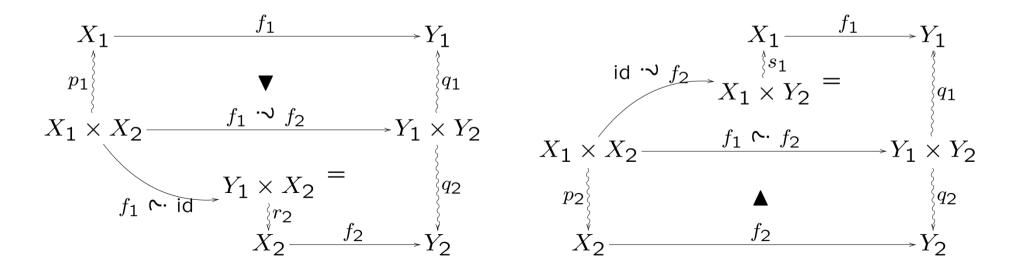
States that two graph homomorphisms -> and -> satisfy:

$$q_1 \circ (f_1 \rightsquigarrow f_2) \blacktriangleleft f_1 \circ p_1$$
  

$$q_2 \circ (f_1 \rightsquigarrow f_2) = f_2 \circ r_2 \circ (f_1 \rightsquigarrow \text{id}_{X_2})$$
  

$$q_1 \circ (f_1 \rightsquigarrow f_2) = f_1 \circ s_1 \circ (\text{id}_{X_2} \rightsquigarrow f_1)$$
  

$$q_2 \circ (f_1 \rightsquigarrow f_2) \blacktriangleleft f_2 \circ p_2$$



#### Theorems and proofs in a Cartesian effect category

- Every pure morphism is central  $- v \ltimes f = v \rtimes f$
- Non ambiguity

   v × f = v × f
   g × f = g × f
- $(id \ltimes g) \circ (id \ltimes f) = id \ltimes (g \circ f)$
- $(\mathbf{k} \ltimes \mathbf{g}) \circ (\mathbf{f}_1 \ltimes \mathbf{f}_2) = (\mathbf{k} \circ \mathbf{f}_1) \ltimes (\mathbf{g} \circ \mathbf{f}_2)$
- Associativity, swap, etc.

#### **Error, Partiality, State**

• Error  

$$(f_1 \ltimes f_2)(x_1, x_2) = \begin{cases} \langle [f_1](x_1), [f_2](x_2) \rangle & \text{if } [f_1](x_1) \in Y_1 \text{ and } [f_2](x_2) \in Y_2 \\ [f_2](x_2) & \text{if } [f_1](x_1) \in Y_1 \text{ and } [f_2](x_2) \in E \\ [f_1](x_1) & \text{if } [f_1](x_1) \in E \end{cases}$$

• Partiality

 $\mathcal{D}_{(f_1 \ltimes f_2)} = \mathcal{D}_{f_1} \times \mathcal{D}_{f_2} \text{ and } (f_1 \ltimes f_2)(x_1, x_2) = \langle [f_1](x_1), [f_2](x_2) \rangle$ 

• State

 $\forall x_1 \in X_1, \forall x_2 \in X_2, \forall s \in S$  $[f_1 \ltimes f_2](s, x_1, x_2) = \langle s_2, y_1, y_2 \rangle$ where  $[f_1](s, x_1) = \langle s_1, y_1 \rangle$  and  $[f_2](s_1, x_2) = \langle s_2, y_2 \rangle$ 

#### Contents

- Effect categories
  - Pure morphism
  - Effect of a morphism, same effect relation  $\approx$
  - − Consistency relations <> < complements same effect</p>
  - Examples: errors, partiality, state
- Cartesian effect categories
  - Semi-pure product  $\rtimes \ltimes$
  - Sequential product ightarrow 
    ightarrow 
    ightarrow
- Comparisons
  - Evaluation logic
  - Haskell's Arrows
  - Freyd categories, strong monads

#### **Evaluation logic, Haskell's Arrows**

• Evaluation logic

Monad	Results [Moggi95]	Consistency
MY	$c\Downarrow a$	$f \lhd v$
Y + E	c = a (thus, c is total)	$c \in Y \implies c = a$
$(Y \times S)^S$	$\exists s \in S, \exists s' \in S, c(s) = (a, s')$	$\forall s \in S, \exists s' \in S, c(s) = (a, s')$
$\mathcal{L}(Y)$	$a \in c$	$\exists k \in \mathbb{N} , c = (a)^k$
$\mathcal{P}_{fin}(Y)$	$a \in c$	$c = \{a\}$ or $c = \emptyset$

• Cartesian effect categories give rise to an Arrow

Cartesian effect categories	Arrows
K(X,Y)	A(X,Y)
$C(X,Y) \subseteq K(X,Y)$	$\operatorname{arr}: C(X,Y) \to A(X,Y)$
$f\mapsto (g\mapsto g\circ f)$	$\implies: A(X,Y) \to A(Y,Z) \to A(X,Z)$
$f\mapsto f\ltimes id$	$\texttt{first}: A(X,Y) \to A(X \times Z, Y \times Z)$

#### Freyd categories, strong monad

- Cartesian effect categories *are* Freyd categories
- Strong monads
  - The strength t can be expressed as a left Kleisli product

•  $]t_{Y1,Y2}[ = id_{Y1} \ltimes_{KI} ]id_{MY2}[$ 

- C<sub>0</sub> Cartesian category with strong monad (M,t) and a consistency is a weak Cartesian effect category if and only if *the strength is consistent with the identity*

#### Conclusion

- A formalization of computational effects
  - Consistency with effect category
  - Semi-pure and sequential products with Cartesian effect category
  - Universal property of semi-pure products:
    - Powerful tool for definitions and proofs
    - Cartesian effect categories are Freyd categories
    - Conditions for strong monads to be CEC
  - Many kind of effects (error, partiality, state, ...)  $\rightarrow$  a CEC
- Clarify evaluation logic / Cartesian effect categories ?
- Combining effects ?