Fast matrix arithmetic for rank structured matrices using non hierarchical representations

Clément Pernet,
joint work with C-P. Jeannerod, T. Mary, D. Roche and A. Storjohann

Univ. Grenoble Alpes,
Laboratoire Jean Kuntzmann.

ILAS 2019, Rio de Janeiro, Brazil. July 8, 2019

Supported by OpenDreamKit Horizon 2020 European RI project (#676541)

Quasiseparable matrices

Definition

$\mathsf{M} \in \mathrm{K}^{n\times n}$ is (r_L, r_U)-quasiseparable if $\begin{cases} \mathsf{rank}(\mathsf{M}_{k+1..n,1..k}) & \leq & r_L \\ \mathsf{rank}(\mathsf{M}_{1..k,k+1..n}) & \leq & r_U \end{cases}$

Properties

- (s,t)-*QS* $\times$ (u,v)-*QS* $=$ $(s+u, t+v)$-*QS*
- $((s,t)\text{-}QS)^{-1} = (s,t)$-*QS*

$$\begin{bmatrix} 4 & 2 & 3 & & & & & & & \\ 4 & 0 & 1 & 4 & & & & & & \\ 4 & 3 & 5 & 6 & 3 & & & & & \\ & 5 & 1 & 1 & 4 & 5 & & & & \\ & & 4 & 3 & 1 & 1 & 4 & & & \\ & & & 4 & 0 & 5 & 1 & 2 & & \\ & & & & 4 & 3 & 1 & 2 & 6 & \\ & & & & & 2 & 1 & 3 & 1 & 1 \\ & & & & & & 1 & 5 & 4 & 3 \\ & & & & & & & 3 & 0 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} 2 & 6 & 1 & 0 & 1 & 4 & 6 & 5 & 2 & 6 \\ 6 & 6 & 2 & 5 & 6 & 1 & 6 & 1 & 3 & 1 \\ 3 & 2 & 2 & 6 & 4 & 1 & 2 & 2 & 0 & 3 \\ 6 & 6 & 2 & 2 & 5 & 1 & 4 & 5 & 5 & 2 \\ 0 & 1 & 6 & 2 & 4 & 4 & 5 & 0 & 3 & 3 \\ 2 & 0 & 5 & 6 & 3 & 1 & 0 & 6 & 2 & 4 \\ 6 & 2 & 6 & 1 & 3 & 4 & 6 & 5 & 2 & 6 \\ 1 & 1 & 5 & 5 & 2 & 1 & 3 & 4 & 5 & 4 \\ 0 & 1 & 6 & 2 & 4 & 4 & 3 & 3 & 2 & 3 \\ 1 & 1 & 5 & 5 & 2 & 1 & 3 & 4 & 5 & 6 \end{bmatrix}$$

Quasiseparable matrices

Definition

$\mathsf{M} \in \mathrm{K}^{n\times n}$ is (r_L, r_U)-quasiseparable if $\begin{cases} \operatorname{rank}(\mathsf{M}_{k+1..n,1..k}) & \leq & r_L \\ \operatorname{rank}(\mathsf{M}_{1..k,k+1..n}) & \leq & r_U \end{cases}$

Properties

- (s,t)-*QS* $\times$ (u,v)-*QS* $=$ $(s+u, t+v)$-*QS*
- $((s,t)\text{-}QS)^{-1} = (s,t)$-*QS*

$$
\begin{bmatrix}
4 & 2 & 3 & & & & & & & \\
4 & 0 & 1 & 4 & & & & & & \\
4 & 3 & 5 & 6 & 3 & & & & & \\
 & 5 & 1 & 1 & 4 & 5 & & & & \\
 & & 4 & 3 & 1 & 1 & 4 & & & \\
 & & & 4 & 0 & 5 & 1 & 2 & & \\
 & & & & 4 & 3 & 1 & 2 & 6 & \\
 & & & & & 2 & 1 & 3 & 1 & 1 \\
 & & & & & & 1 & 5 & 4 & 3 \\
 & & & & & & & 3 & 0 & 4
\end{bmatrix}^{-1}
=
\begin{bmatrix}
2 & 6 & 1 & 0 & 1 & 4 & 6 & 5 & 2 & 6 \\
6 & 6 & 2 & 5 & 6 & 1 & 6 & 1 & 3 & 1 \\
3 & 2 & 2 & 6 & 4 & 1 & 2 & 2 & 0 & 3 \\
6 & 6 & 2 & 2 & 5 & 1 & 4 & 5 & 5 & 2 \\
0 & 1 & 6 & 2 & 4 & 4 & 5 & 0 & 3 & 3 \\
2 & 0 & 5 & 6 & 3 & 1 & 0 & 6 & 2 & 4 \\
6 & 2 & 6 & 1 & 3 & 4 & 6 & 5 & 2 & 6 \\
1 & 1 & 5 & 5 & 2 & 1 & 3 & 4 & 5 & 4 \\
0 & 1 & 6 & 2 & 4 & 4 & 3 & 3 & 2 & 3 \\
1 & 1 & 5 & 5 & 2 & 1 & 3 & 4 & 5 & 6
\end{bmatrix}
$$

Quasiseparable matrices

Definition

$\mathsf{M} \in \mathrm{K}^{n\times n}$ is (r_L, r_U)-quasiseparable if $\begin{cases} \mathrm{rank}(\mathsf{M}_{k+1..n,1..k}) & \leq & r_L \\ \mathrm{rank}(\mathsf{M}_{1..k,k+1..n}) & \leq & r_U \end{cases}$

Properties

- (s,t)-*QS* $\times$ (u,v)-*QS* $= (s+u, t+v)$-*QS*
- $((s,t)\text{-}QS)^{-1} = (s,t)$-*QS*

$$
\begin{bmatrix}
4 & 2 & 3 & & & & & & & \\
4 & 0 & 1 & 4 & & & & & & \\
4 & 3 & 5 & 6 & 3 & & & & & \\
 & 5 & 1 & 1 & 4 & 5 & & & & \\
 & & 4 & 3 & 1 & 1 & 4 & & & \\
 & & & 4 & 0 & 5 & 1 & 2 & & \\
 & & & & 4 & 3 & 1 & 2 & 6 & \\
 & & & & & 2 & 1 & 3 & 1 & 1 \\
 & & & & & & 1 & 5 & 4 & 3 \\
 & & & & & & & 3 & 0 & 4
\end{bmatrix}^{-1}
=
\begin{bmatrix}
2 & 6 & 1 & 0 & 1 & 4 & 6 & 5 & 2 & 6 \\
6 & 6 & 2 & 5 & 6 & 1 & 6 & 1 & 3 & 1 \\
3 & 2 & 2 & 6 & 4 & 1 & 2 & 2 & 0 & 3 \\
6 & 6 & 2 & 2 & 5 & 1 & 4 & 5 & 5 & 2 \\
0 & 1 & 6 & 2 & 4 & 4 & 5 & 0 & 3 & 3 \\
2 & 0 & 5 & 6 & 3 & 1 & 0 & 6 & 2 & 4 \\
6 & 2 & 6 & 1 & 3 & 4 & 6 & 5 & 2 & 6 \\
1 & 1 & 5 & 5 & 2 & 1 & 3 & 4 & 5 & 4 \\
0 & 1 & 6 & 2 & 4 & 4 & 3 & 3 & 2 & 3 \\
1 & 1 & 5 & 5 & 2 & 1 & 3 & 4 & 5 & 6
\end{bmatrix}
$$

Quasiseparable matrices

Definition

$\mathsf{M} \in \mathrm{K}^{n\times n}$ is (r_L, r_U)-quasiseparable if $\begin{cases} \mathrm{rank}(\mathsf{M}_{k+1..n,1..k}) & \leq & r_L \\ \mathrm{rank}(\mathsf{M}_{1..k,k+1..n}) & \leq & r_U \end{cases}$

Properties

- (s,t)-*QS* $\times$ (u,v)-*QS* $=$ $(s+u, t+v)$-*QS*
- $((s,t)$-*QS*$)^{-1}$ $=$ (s,t)-*QS*

$$\begin{bmatrix} 4 & 2 & 3 & & & & & & & \\ 4 & 0 & 1 & 4 & & & & & & \\ 4 & 3 & 5 & 6 & 3 & & & & & \\ & 5 & 1 & 1 & 4 & 5 & & & & \\ & & 4 & 3 & 1 & 1 & 4 & & & \\ & & & 4 & 0 & 5 & 1 & 2 & & \\ & & & & 4 & 3 & 1 & 2 & 6 & \\ & & & & & 2 & 1 & 3 & 1 & 1 \\ & & & & & & 1 & 5 & 4 & 3 \\ & & & & & & & 3 & 0 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} 2 & 6 & 1 & 0 & 1 & 4 & 6 & 5 & 2 & 6 \\ 6 & 6 & 2 & 5 & 6 & 1 & 6 & 1 & 3 & 1 \\ 3 & 2 & 2 & 6 & 4 & 1 & 2 & 2 & 0 & 3 \\ 6 & 6 & 2 & 2 & 5 & 1 & 4 & 5 & 5 & 2 \\ 0 & 1 & 6 & 2 & 4 & 4 & 5 & 0 & 3 & 3 \\ 2 & 0 & 5 & 6 & 3 & 1 & 0 & 6 & 2 & 4 \\ 6 & 2 & 6 & 1 & 3 & 4 & 6 & 5 & 2 & 6 \\ 1 & 1 & 5 & 5 & 2 & 1 & 3 & 4 & 5 & 4 \\ 0 & 1 & 6 & 2 & 4 & 4 & 3 & 3 & 2 & 3 \\ 1 & 1 & 5 & 5 & 2 & 1 & 3 & 4 & 5 & 6 \end{bmatrix}$$

Quasiseparable matrices

Definition

$\mathsf{M} \in \mathrm{K}^{n\times n}$ is (r_L, r_U)-quasiseparable if $\begin{cases} \mathrm{rank}(\mathsf{M}_{k+1..n,1..k}) & \leq & r_L \\ \mathrm{rank}(\mathsf{M}_{1..k,k+1..n}) & \leq & r_U \end{cases}$

Properties

- (s,t)-*QS* $\times$ (u,v)-*QS* $= (s+u, t+v)$-*QS*
- $((s,t)$-*QS*$)^{-1} = (s,t)$-*QS*

$$\begin{bmatrix} 4 & 2 & 3 & & & & & & & \\ 4 & 0 & 1 & 4 & & & & & & \\ 4 & 3 & 5 & 6 & 3 & & & & & \\ & 5 & 1 & 1 & 4 & 5 & & & & \\ & & 4 & 3 & 1 & 1 & 4 & & & \\ & & & 4 & 0 & 5 & 1 & 2 & & \\ & & & & 4 & 3 & 1 & 2 & 6 & \\ & & & & & 2 & 1 & 3 & 1 & 1 \\ & & & & & & 1 & 5 & 4 & 3 \\ & & & & & & & 3 & 0 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} 2 & 6 & 1 & 0 & 1 & 4 & 6 & 5 & 2 & 6 \\ 6 & 6 & 2 & 5 & 6 & 1 & 6 & 1 & 3 & 1 \\ 3 & 2 & 2 & 6 & 4 & 1 & 2 & 2 & 0 & 3 \\ 6 & 6 & 2 & 2 & 5 & 1 & 4 & 5 & 5 & 2 \\ 0 & 1 & 6 & 2 & 4 & 4 & 5 & 0 & 3 & 3 \\ 2 & 0 & 5 & 6 & 3 & 1 & 0 & 6 & 2 & 4 \\ 6 & 2 & 6 & 1 & 3 & 4 & 6 & 5 & 2 & 6 \\ 1 & 1 & 5 & 5 & 2 & 1 & 3 & 4 & 5 & 4 \\ 0 & 1 & 6 & 2 & 4 & 4 & 3 & 3 & 2 & 3 \\ 1 & 1 & 5 & 5 & 2 & 1 & 3 & 4 & 5 & 6 \end{bmatrix}$$

Quasiseparable matrices

Definition

M $\in \mathrm{K}^{n\times n}$ is (r_L, r_U)-quasiseparable if $\begin{cases} \mathrm{rank}(\mathsf{M}_{k+1..n,1..k}) & \leq & r_L \\ \mathrm{rank}(\mathsf{M}_{1..k,k+1..n}) & \leq & r_U \end{cases}$

Properties

- (s,t)-*QS* $\times$ (u,v)-*QS* $= (s+u, t+v)$-*QS*
- $((s,t)\text{-}QS)^{-1} = (s,t)$-*QS*

$$\begin{bmatrix} 4 & 2 & 3 & & & & & & & \\ 4 & 0 & 1 & 4 & & & & & & \\ 4 & 3 & 5 & 6 & 3 & & & & & \\ & 5 & 1 & 1 & 4 & 5 & & & & \\ & & 4 & 3 & 1 & 1 & 4 & & & \\ & & & 4 & 0 & 5 & 1 & 2 & & \\ & & & & 4 & 3 & 1 & 2 & 6 & \\ & & & & & 2 & 1 & 3 & 1 & 1 \\ & & & & & & 1 & 5 & 4 & 3 \\ & & & & & & & 3 & 0 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} 2 & 6 & 1 & 0 & 1 & 4 & 6 & 5 & 2 & 6 \\ 6 & 6 & 2 & 5 & 6 & 1 & 6 & 1 & 3 & 1 \\ 3 & 2 & 2 & 6 & 4 & 1 & 2 & 2 & 0 & 3 \\ 6 & 6 & 2 & 2 & 5 & 1 & 4 & 5 & 5 & 2 \\ 0 & 1 & 6 & 2 & 4 & 4 & 5 & 0 & 3 & 3 \\ 2 & 0 & 5 & 6 & 3 & 1 & 0 & 6 & 2 & 4 \\ 6 & 2 & 6 & 1 & 3 & 4 & 6 & 5 & 2 & 6 \\ 1 & 1 & 5 & 5 & 2 & 1 & 3 & 4 & 5 & 4 \\ 0 & 1 & 6 & 2 & 4 & 4 & 3 & 3 & 2 & 3 \\ 1 & 1 & 5 & 5 & 2 & 1 & 3 & 4 & 5 & 6 \end{bmatrix}$$

Quasiseparable matrices

Definition

$\mathsf{M} \in \mathrm{K}^{n\times n}$ is (r_L, r_U)-quasiseparable if $\begin{cases} \mathrm{rank}(\mathsf{M}_{k+1..n,1..k}) & \leq & r_L \\ \mathrm{rank}(\mathsf{M}_{1..k,k+1..n}) & \leq & r_U \end{cases}$

Properties

- (s,t)-*QS* $\times$ (u,v)-*QS* $= (s+u, t+v)$-*QS*
- $((s,t)\text{-}QS)^{-1} = (s,t)$-*QS*

$$\begin{bmatrix} 4&2&3&&&&&&& \\ 4&0&1&4&&&&&& \\ 4&3&5&6&3&&&&& \\ &5&1&1&4&5&&&& \\ &&4&3&1&1&4&&& \\ &&&4&0&5&1&2&& \\ &&&&4&3&1&2&6& \\ &&&&&2&1&3&1&1 \\ &&&&&&1&5&4&3 \\ &&&&&&&3&0&4 \end{bmatrix}^{-1} = \begin{bmatrix} 2&6&1&0&1&4&6&5&2&6 \\ 6&6&2&5&6&1&6&1&3&1 \\ 3&2&2&6&4&1&2&2&0&3 \\ 6&6&2&2&5&1&4&5&5&2 \\ 0&1&6&2&4&4&5&0&3&3 \\ 2&0&5&6&3&1&0&6&2&4 \\ 6&2&6&1&3&4&6&5&2&6 \\ 1&1&5&5&2&1&3&4&5&4 \\ 0&1&6&2&4&4&3&3&2&3 \\ 1&1&5&5&2&1&3&4&5&6 \end{bmatrix}$$

Quasiseparable matrices

Definition

$\mathsf{M} \in \mathrm{K}^{n\times n}$ is (r_L, r_U)-quasiseparable if $\begin{cases} \mathrm{rank}(\mathsf{M}_{k+1..n,1..k}) & \leq & r_L \\ \mathrm{rank}(\mathsf{M}_{1..k,k+1..n}) & \leq & r_U \end{cases}$

Properties

- (s,t)-*QS* $\times$ (u,v)-*QS* $=$ $(s+u, t+v)$-*QS*
- $((s,t)\text{-}QS)^{-1} = (s,t)$-*QS*

$$\begin{bmatrix} 4 & 2 & 3 & & & & & & & \\ 4 & 0 & 1 & 4 & & & & & & \\ 4 & 3 & 5 & 6 & 3 & & & & & \\ & 5 & 1 & 1 & 4 & 5 & & & & \\ & & 4 & 3 & 1 & 1 & 4 & & & \\ & & & 4 & 0 & 5 & 1 & 2 & & \\ & & & & 4 & 3 & 1 & 2 & 6 & \\ & & & & & 2 & 1 & 3 & 1 & 1 \\ & & & & & & 1 & 5 & 4 & 3 \\ & & & & & & & 3 & 0 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} 2 & 6 & 1 & 0 & 1 & 4 & 6 & 5 & 2 & 6 \\ 6 & 6 & 2 & 5 & 6 & 1 & 6 & 1 & 3 & 1 \\ 3 & 2 & 2 & 6 & 4 & 1 & 2 & 2 & 0 & 3 \\ 6 & 6 & 2 & 2 & 5 & 1 & 4 & 5 & 5 & 2 \\ 0 & 1 & 6 & 2 & 4 & 4 & 5 & 0 & 3 & 3 \\ 2 & 0 & 5 & 6 & 3 & 1 & 0 & 6 & 2 & 4 \\ 6 & 2 & 6 & 1 & 3 & 4 & 6 & 5 & 2 & 6 \\ 1 & 1 & 5 & 5 & 2 & 1 & 3 & 4 & 5 & 4 \\ 0 & 1 & 6 & 2 & 4 & 4 & 3 & 3 & 2 & 3 \\ 1 & 1 & 5 & 5 & 2 & 1 & 3 & 4 & 5 & 6 \end{bmatrix}$$

Quasiseparable matrices

Definition

$\mathsf{M} \in \mathrm{K}^{n\times n}$ is (r_L, r_U)-quasiseparable if $\begin{cases} \operatorname{rank}(\mathsf{M}_{k+1..n,1..k}) & \leq & r_L \\ \operatorname{rank}(\mathsf{M}_{1..k,k+1..n}) & \leq & r_U \end{cases}$

Properties

- (s,t)-*QS* $\times$ (u,v)-*QS* $= (s+u, t+v)$-*QS*
- $((s,t)\text{-}QS)^{-1} = (s,t)$-*QS*

$$\begin{bmatrix}
4 & 2 & 3 & & & & & & & \\
4 & 0 & 1 & 4 & & & & & & \\
4 & 3 & 5 & 6 & 3 & & & & & \\
 & 5 & 1 & 1 & 4 & 5 & & & & \\
 & & 4 & 3 & 1 & 1 & 4 & & & \\
 & & & 4 & 0 & 5 & 1 & 2 & & \\
 & & & & 4 & 3 & 1 & 2 & 6 & \\
 & & & & & 2 & 1 & 3 & 1 & 1 \\
 & & & & & & 1 & 5 & 4 & 3 \\
 & & & & & & & 3 & 0 & 4
\end{bmatrix}^{-1} = \begin{bmatrix}
2 & 6 & 1 & 0 & 1 & 4 & 6 & 5 & 2 & 6 \\
6 & 6 & 2 & 5 & 6 & 1 & 6 & 1 & 3 & 1 \\
3 & 2 & 2 & 6 & 4 & 1 & 2 & 2 & 0 & 3 \\
6 & 6 & 2 & 2 & 5 & 1 & 4 & 5 & 5 & 2 \\
0 & 1 & 6 & 2 & 4 & 4 & 5 & 0 & 3 & 3 \\
2 & 0 & 5 & 6 & 3 & 1 & 0 & 6 & 2 & 4 \\
6 & 2 & 6 & 1 & 3 & 4 & 6 & 5 & 2 & 6 \\
1 & 1 & 5 & 5 & 2 & 1 & 3 & 4 & 5 & 4 \\
0 & 1 & 6 & 2 & 4 & 4 & 3 & 3 & 2 & 3 \\
1 & 1 & 5 & 5 & 2 & 1 & 3 & 4 & 5 & 6
\end{bmatrix}$$

Quasiseparable matrices

Definition

M $\in \mathrm{K}^{n\times n}$ is (r_L, r_U)-quasiseparable if $\begin{cases} \text{rank}(M_{k+1..n,1..k}) & \leq & r_L \\ \text{rank}(M_{1..k,k+1..n}) & \leq & r_U \end{cases}$

Properties

- (s,t)-*QS* $\times$ (u,v)-*QS* $= (s+u, t+v)$-*QS*
- $((s,t)\text{-}QS)^{-1} = (s,t)$-*QS*

$$
\begin{bmatrix}
4 & 2 & 3 & & & & & & & \\
4 & 0 & 1 & 4 & & & & & & \\
4 & 3 & 5 & 6 & 3 & & & & & \\
 & 5 & 1 & 1 & 4 & 5 & & & & \\
 & & 4 & 3 & 1 & 1 & 4 & & & \\
 & & & 4 & 0 & 5 & 1 & 2 & & \\
 & & & & 4 & 3 & 1 & 2 & 6 & \\
 & & & & & 2 & 1 & 3 & 1 & 1 \\
 & & & & & & 1 & 5 & 4 & 3 \\
 & & & & & & & 3 & 0 & 4
\end{bmatrix}^{-1}
=
\begin{bmatrix}
2 & 6 & 1 & 0 & 1 & 4 & 6 & 5 & 2 & 6 \\
6 & 6 & 2 & 5 & 6 & 1 & 6 & 1 & 3 & 1 \\
3 & 2 & 2 & 6 & 4 & 1 & 2 & 2 & 0 & 3 \\
6 & 6 & 2 & 2 & 5 & 1 & 4 & 5 & 5 & 2 \\
0 & 1 & 6 & 2 & 4 & 4 & 5 & 0 & 3 & 3 \\
2 & 0 & 5 & 6 & 3 & 1 & 0 & 6 & 2 & 4 \\
6 & 2 & 6 & 1 & 3 & 4 & 6 & 5 & 2 & 6 \\
1 & 1 & 5 & 5 & 2 & 1 & 3 & 4 & 5 & 4 \\
0 & 1 & 6 & 2 & 4 & 4 & 3 & 3 & 2 & 3 \\
1 & 1 & 5 & 5 & 2 & 1 & 3 & 4 & 5 & 6
\end{bmatrix}
$$

Applications

Numerical linear algebra: naturally occur in solving

- generalized eigenvalue problems,
- Fast Multipole Method (n-body simulation)

Applications

Numerical linear algebra: naturally occur in solving

- generalized eigenvalue problems,
- Fast Multipole Method (n-body simulation)

Computer algebra: over $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{Z}/p\mathbb{Z}$

- normal forms of polynomial matrices over $\mathbb{Z}/p\mathbb{Z}$
- `LinBox`: black-box exact linear algebra library,

Applications

Numerical linear algebra: naturally occur in solving

- generalized eigenvalue problems,
- Fast Multipole Method (n-body simulation)

Computer algebra: over $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{Z}/p\mathbb{Z}$

- normal forms of polynomial matrices over $\mathbb{Z}/p\mathbb{Z}$
- `LinBox`: black-box exact linear algebra library,

Efficient representation: take advantage of the structure

Space: ideally, storage linear in n and s

Structure: recursive (hierarchical) or flat

Arithmetic cost for most common ops:

- QS $\times$ Vect.
- QS $\times$ Tall-Skinny matrix (s Vect).
- QS $\times$ QS.
- LU, QR factorization, LinSys

Structured representation of a quasiseparable matrix

Sequentially SemiSeparable (SSS) [Eidelman Gohberg 99] [5]

$$\mathsf{M}_{i,j} = \begin{cases} p(i)^T \mathsf{a}(i-1)\dots\mathsf{a}(j+1)q(j), & 1 \le j < i \le n \\ d(i), & 1 \le i = j \le n \\ g(i)^T \mathsf{b}(i+1)\dots\mathsf{b}(i-1)h(j), & 1 \le i < j \le n \end{cases}$$

where $p(i), q(i), g(i), h(i) \in \mathrm{K}^s$, $\mathsf{a}(i), \mathsf{b}(i) \in \mathrm{K}^{s\times s}$ for $1 \le i \le n-1$

Structured representation of a quasiseparable matrix

Sequentially SemiSeparable (SSS) [Eidelman Gohberg 99] [5]

$$\mathsf{M}_{i,j} = \begin{cases} p(i)^T \mathsf{a}(i-1)\dots\mathsf{a}(j+1)q(j), & 1 \leq j < i \leq n \\ d(i), & 1 \leq i = j \leq n \\ g(i)^T \mathsf{b}(i+1)\dots\mathsf{b}(i-1)h(j), & 1 \leq i < j \leq n \end{cases}$$

where $p(i), q(i), g(i), h(i) \in \mathrm{K}^s$, $\mathsf{a}(i), \mathsf{b}(i) \in \mathrm{K}^{s\times s}$ for $1 \leq i \leq n-1$

	SSS [5]
Size	$O(s^2 n)$
Construction	$O(s^2 n^2)$

Structured representation of a quasiseparable matrix

Sequentially SemiSeparable (SSS) [Eidelman Gohberg 99] [5]

$$\mathsf{M}_{i,j} = \begin{cases} p(i)^T \mathsf{a}(i-1)\dots\mathsf{a}(j+1)q(j), & 1 \leq j < i \leq n \\ d(i), & 1 \leq i = j \leq n \\ g(i)^T \mathsf{b}(i+1)\dots\mathsf{b}(i-1)h(j), & 1 \leq i < j \leq n \end{cases}$$

where $p(i), q(i), g(i), h(i) \in \mathrm{K}^s$, $\mathsf{a}(i), \mathsf{b}(i) \in \mathrm{K}^{s\times s}$ for $1 \leq i \leq n-1$

	SSS [5]
Size	$O(s^2 n)$
Construction	$O(s^2 n^2)$
QS $\times$ Vec	$O(s^2 n)$

Structured representation of a quasiseparable matrix

Sequentially SemiSeparable (SSS) [Eidelman Gohberg 99] [5]

$$\mathsf{M}_{i,j} = \begin{cases} p(i)^T \mathsf{a}(i-1)\dots\mathsf{a}(j+1)q(j), & 1 \leq j < i \leq n \\ d(i), & 1 \leq i = j \leq n \\ g(i)^T \mathsf{b}(i+1)\dots\mathsf{b}(i-1)h(j), & 1 \leq i < j \leq n \end{cases}$$

where $p(i), q(i), g(i), h(i) \in \mathrm{K}^s$, $\mathsf{a}(i), \mathsf{b}(i) \in \mathrm{K}^{s\times s}$ for $1 \leq i \leq n-1$

	SSS [5]
Size	$O(s^2 n)$
Construction	$O(s^2 n^2)$
QS $\times$ Vec QS $\times$ s Vec	$O(s^2 n)$ $O(s^3 n)$

Structured representation of a quasiseparable matrix

Sequentially SemiSeparable (SSS) [Eidelman Gohberg 99] [5]

$$\mathsf{M}_{i,j} = \begin{cases} p(i)^T \mathsf{a}(i-1)\dots\mathsf{a}(j+1)q(j), & 1 \le j < i \le n \\ d(i), & 1 \le i = j \le n \\ g(i)^T \mathsf{b}(i+1)\dots\mathsf{b}(i-1)h(j), & 1 \le i < j \le n \end{cases}$$

where $p(i), q(i), g(i), h(i) \in \mathrm{K}^s$, $\mathsf{a}(i), \mathsf{b}(i) \in \mathrm{K}^{s \times s}$ for $1 \le i \le n-1$

	SSS [5]
Size	$O(s^2 n)$
Construction	$O(s^2 n^2)$
QS $\times$ Vec	$O(s^2 n)$
QS $\times$ s Vec	$O(s^3 n)$
QS $\times$ QS	$O(s^3 n)$

Structured representation of a quasiseparable matrix

Sequentially SemiSeparable (SSS) [Eidelman Gohberg 99] [5]

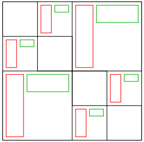
$$\mathsf{M}_{i,j} = \begin{cases} p(i)^T \mathsf{a}(i-1)\dots\mathsf{a}(j+1)q(j), & 1 \leq j < i \leq n \\ d(i), & 1 \leq i = j \leq n \\ g(i)^T \mathsf{b}(i+1)\dots\mathsf{b}(i-1)h(j), & 1 \leq i < j \leq n \end{cases}$$

where $p(i), q(i), g(i), h(i) \in \mathrm{K}^s$, $\mathsf{a}(i), \mathsf{b}(i) \in \mathrm{K}^{s\times s}$ for $1 \leq i \leq n-1$

	SSS [5]
Size	$O(s^2 n)$
Construction	$O(s^2 n^2)$
QS $\times$ Vec	$O(s^2 n)$
QS $\times$ s Vec	$O(s^3 n)$
QS $\times$ QS	$O(s^3 n)$
LU/QR/LinSys	$O(s^3 n)$

Structured representation of a quasiseparable matrix

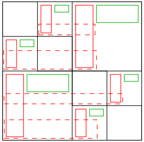
Hierarchical ($\mathcal{H}$) [6]



	SSS [5]	$\mathcal{H}$ [6]
Size	$O(s^2n)$	$O(sn\lg\frac{n}{s})$
Construction	$O(s^2n^2)$	$O(sn^2)$
QS $\times$ Vec	$O(s^2n)$	$O(sn\lg\frac{n}{s})$
QS $\times$ s Vec	$O(s^3n)$	$O(s^2n\lg\frac{n}{s})$
QS $\times$ QS	$O(s^3n)$	$O(s^2n\lg^2\frac{n}{s})$
LU/QR/LinSys	$O(s^3n)$	$O(s^2n\lg^2\frac{n}{s})$

Structured representation of a quasiseparable matrix

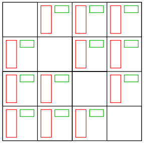
Hierarchically SemiSeparable (HSS) [2]: cf Fast Multipole Method



	SSS [5]	$\mathcal{H}$ [6]	HSS [2]
Size	$O(s^2n)$	$O(sn\lg\frac{n}{s})$	$O(sn)$
Construction	$O(s^2n^2)$	$O(sn^2)$	$O(sn^2)$
QS $\times$ Vec	$O(s^2n)$	$O(sn\lg\frac{n}{s})$	$O(sn)$
QS $\times$ s Vec	$O(s^3n)$	$O(s^2n\lg\frac{n}{s})$	$O(s^2n)$
QS $\times$ QS	$O(s^3n)$	$O(s^2n\lg^2\frac{n}{s})$	$O(s^2n)$
LU/QR/LinSys	$O(s^3n)$	$O(s^2n\lg^2\frac{n}{s})$	$O(s^2n)$

Structured representation of a quasiseparable matrix

Block Low Rank (BLR) [1]



	SSS [5]	$\mathcal{H}$ [6]	HSS [2]	BLR [1]
Size	$O(s^2n)$	$O(sn\lg\frac{n}{s})$	$O(sn)$	$O(s^{0.5}n^{1.5})$
Construction	$O(s^2n^2)$	$O(sn^2)$	$O(sn^2)$	$O(sn^2)$
QS $\times$ Vec	$O(s^2n)$	$O(sn\lg\frac{n}{s})$	$O(sn)$	$O(s^{0.5}n^{1.5})$
QS $\times$ s Vec	$O(s^3n)$	$O(s^2n\lg\frac{n}{s})$	$O(s^2n)$	$O(s^{1.5}n^{1.5})$
QS $\times$ QS	$O(s^3n)$	$O(s^2n\lg^2\frac{n}{s})$	$O(s^2n)$	$O(sn^2)$
LU/QR/LinSys	$O(s^3n)$	$O(s^2n\lg^2\frac{n}{s})$	$O(s^2n)$	$O(sn^2)$

Focus

Reduction to dense matrix multiplication

- Improving theoretical complexities complexities: $O(n^\omega)$

  $\omega = 3$ schoolbook multiplication

  $\omega \approx 2.8074$ [Strassen 69]

  $\omega \approx 2.3729$ [Le Gall 14]

- Efficiency in practice: gather arithmetic in compute intensive kernels

Focus

Reduction to dense matrix multiplication

- Improving theoretical complexities complexities: $O(n^\omega)$

  $\omega = 3$ schoolbook multiplication

  $\omega \approx 2.8074$ [Strassen 69]

  $\omega \approx 2.3729$ [Le Gall 14]

- Efficiency in practice: gather arithmetic in compute intensive kernels

Flat representations

- much better suited for large scale parallelization

Focus

Reduction to dense matrix multiplication

- Improving theoretical complexities complexities: $O(n^\omega)$

$\omega = 3$ schoolbook multiplication
$\omega \approx 2.8074$ [Strassen 69]
$\omega \approx 2.3729$ [Le Gall 14]

- Efficiency in practice: gather arithmetic in compute intensive kernels

Flat representations

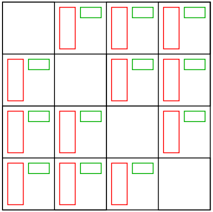
- much better suited for large scale parallelization

Contributions

- speed-up BLR with fast matrix multiplication;
- Connecting notions of **quasiseparability** and **rank profile matrix** [3];
- A new flat represention matching the HSS complexities.

Contributions

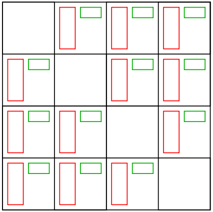
Block Low Rank (BLR) using fast matrix multiplication



	...	HSS [2]	BLR [1]
Size	...	$O(sn)$	$O(s^{0.5}n^{1.5})$
Construction	...	$O(sn^2)$	$O(sn^2)$
QS $\times$ Vec	...	$O(sn)$	$O(s^{0.5}n^{1.5})$
QS $\times$ s Vec	...	$O(s^2n)$	$O(sn^2)$
QS $\times$ QS	...	$O(s^2n)$	$O(sn^2)$
LU/QR/LinSys	...	$O(s^2n)$	$O(sn^2)$

Contributions

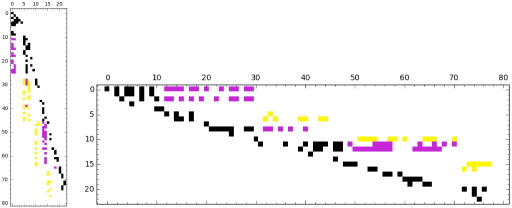
Block Low Rank (BLR) using fast matrix multiplication



	...	HSS [2]	BLR [1]	fast BLR
Size	...	$O(sn)$	$O(s^{0.5}n^{1.5})$	$O(s^{0.5}n^{1.5})$
Construction	...	$O(sn^2)$	$O(sn^2)$	$O(sn^2)$
QS $\times$ Vec	...	$O(sn)$	$O(s^{0.5}n^{1.5})$	$O(s^{0.5}n^{1.5})$
QS $\times$ s Vec	...	$O(s^2n)$	$O(sn^2)$	$O(s^{\omega-1.5}n^{1.5})$
QS $\times$ QS	...	$O(s^2n)$	$O(sn^2)$	$O(s^{\frac{\omega-1}{2}}n^{\frac{\omega+1}{2}})$
LU/QR/LinSys	...	$O(s^2n)$	$O(sn^2)$	$O(s^{\frac{\omega-1}{2}}n^{\frac{\omega+1}{2}})$

Contributions

Compact Bruhat generator



	…	HSS [2]	BLR [1]	fast BLR	CB
Size	…	$O(sn)$	$O(s^{0.5}n^{1.5})$	$O(s^{0.5}n^{1.5})$	$O(sn)$
Construction	…	$O(sn^2)$	$O(sn^2)$	$O(sn^2)$	$O(s^{\omega-2}n^2)$
QS × Vec	…	$O(sn)$	$O(s^{0.5}n^{1.5})$	$O(s^{0.5}n^{1.5})$	$O(sn)$
QS × s Vec	…	$O(s^2n)$	$O(sn^2)$	$O(s^{\omega-1.5}n^{1.5})$	$O(s^{\omega-1}n)$
QS × QS	…	$O(s^2n)$	$O(sn^2)$	$O(s^{\frac{\omega-1}{2}}n^{\frac{\omega+1}{2}})$	$O(s^{\omega-1}n)$
LU/QR/LinSys	…	$O(s^2n)$	$O(sn^2)$	$O(s^{\frac{\omega-1}{2}}n^{\frac{\omega+1}{2}})$	$O(s^{\omega-1}n)$

Outline

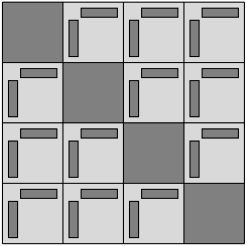
1. Block Low Rank representation
   - First attempt
   - New approach
2. The compact Bruhat representation
   - The rank profile matrix
   - Warm-up : computing the orders of quasiseparability
   - The Bruhat generator
   - The Compact Bruhat generator

Outline

1. Block Low Rank representation
   - First attempt
   - New approach

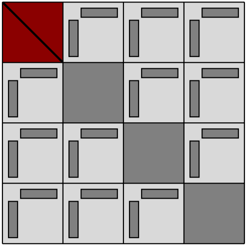
2. The compact Bruhat representation
   - The rank profile matrix
   - Warm-up : computing the orders of quasiseparability
   - The Bruhat generator
   - The Compact Bruhat generator

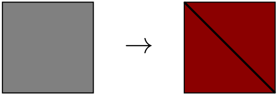
BLR matrix LU factorization: classical algorithm



Kernel costs:

BLR matrix LU factorization: classical algorithm

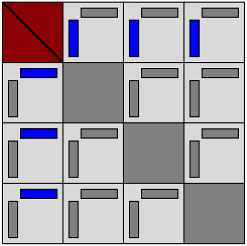


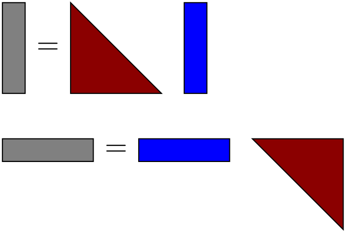


Kernel costs:

- **Factor** kernel: $O(b^3)$

BLR matrix LU factorization: classical algorithm

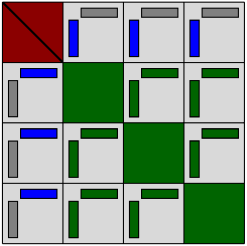


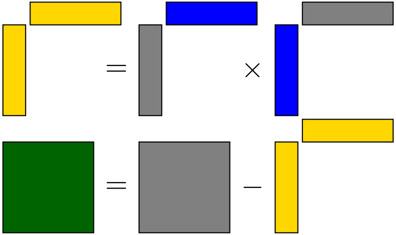


Kernel costs:

- **Factor** kernel: $O(b^3)$
- **Solve** kernel: $O(b^2 r)$

BLR matrix LU factorization: classical algorithm

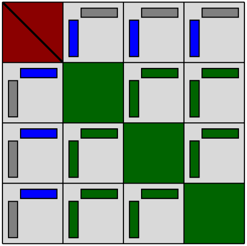


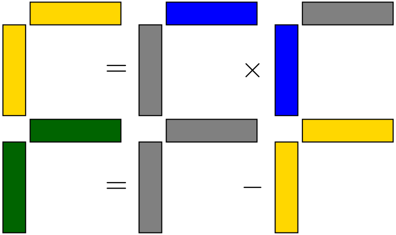


Kernel costs:

- **Factor** kernel: $O(b^3)$
- **Solve** kernel: $O(b^2 r)$
- **Update** kernel:
  - FR target: $O(b^2 r)$

BLR matrix LU factorization: classical algorithm

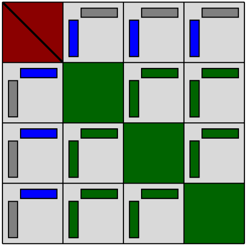




Kernel costs:

- **Factor** kernel: $O(b^3)$
- **Solve** kernel: $O(b^2 r)$
- **Update** kernel:
  - FR target: $O(b^2 r)$
  - LR target: $O(br^2)$

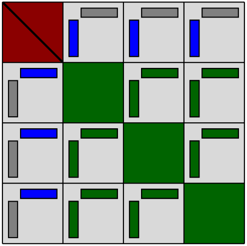
BLR matrix LU factorization: classical algorithm



Kernel costs:

- **Factor** kernel: $O(b^3) \to O(b^\omega)$
- **Solve** kernel: $O(b^2 r) \to O(b^2 r^{\omega-2})$
- **Update** kernel:
  - FR target: $O(b^2 r) \to O(b^2 r^{\omega-2})$
  - LR target: $O(br^2) \to O(br^{\omega-1})$

BLR matrix LU factorization: classical algorithm



Total:

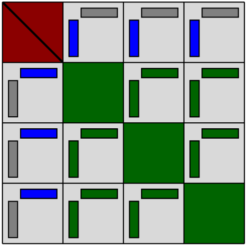
$$O(pb^\omega + p^2b^2r^{\omega-2} + p^3br^{\omega-1})$$
$$\subset O(n^2r^{\omega-2}) \qquad \text{for } b = \Theta(\sqrt{nr})$$

- deceptive speed-up ($r^{3-\omega}$)
- granularity of LR is too low

Kernel costs:

- **Factor** kernel: $O(b^3) \to O(b^\omega)$
- **Solve** kernel: $O(b^2r) \to O(b^2r^{\omega-2})$
- **Update** kernel:
  - FR target: $O(b^2r) \to O(b^2r^{\omega-2})$
  - LR target: $O(br^2) \to O(br^{\omega-1})$

BLR matrix LU factorization: classical algorithm



Total:

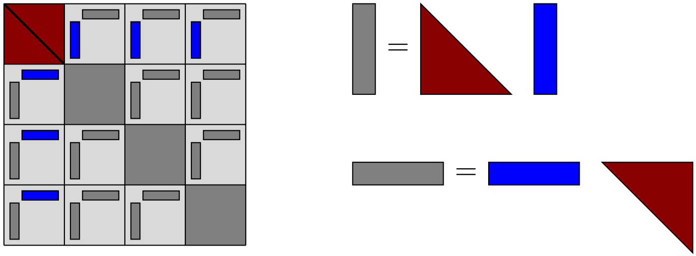
$$O(pb^\omega + p^2b^2r^{\omega-2} + p^3br^{\omega-1})$$
$$\subset O(n^2r^{\omega-2}) \qquad \text{for } b = \Theta(\sqrt{nr})$$

- deceptive speed-up $(r^{3-\omega})$
- granularity of LR is too low
- How to reduce the exponent in n?

Kernel costs:

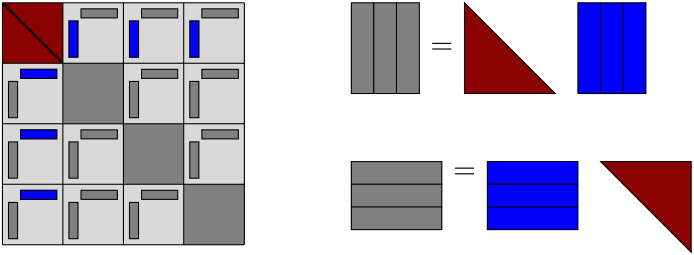
- **Factor** kernel: $O(b^3) \to O(b^\omega)$
- **Solve** kernel: $O(b^2r) \to O(b^2r^{\omega-2})$
- **Update** kernel:
  - FR target: $O(b^2r) \to O(b^2r^{\omega-2})$
  - LR target: $O(br^2) \to O(br^{\omega-1})$

New Solve kernel



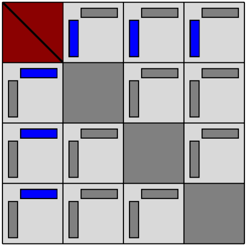
- Classical: $O(p^2)$ calls of cost $O(b^2r^{\omega-2}) \Rightarrow O(p^2b^2r^{\omega-2})$

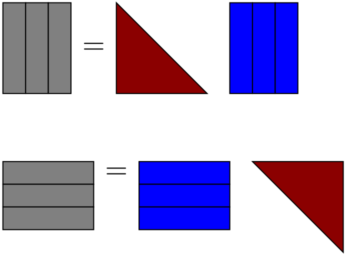
New Solve kernel



- Classical: $O(p^2)$ calls of cost $O(b^2 r^{\omega-2}) \Rightarrow O(p^2 b^2 r^{\omega-2})$
- New: $O(p)$ calls of cost $O(b^{\omega-1} pr) \Rightarrow O(p^2 b^{\omega-1} r)$

New Solve kernel



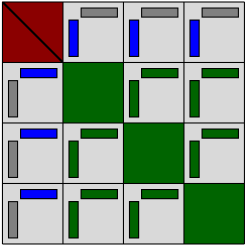


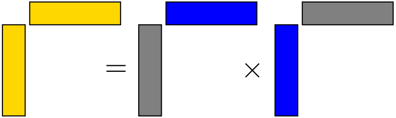
- Classical: $O(p^2)$ calls of cost $O(b^2r^{\omega-2}) \Rightarrow O(p^2b^2r^{\omega-2})$
- New: $O(p)$ calls of cost $O(b^{\omega-1}pr) \Rightarrow O(p^2b^{\omega-1}r)$

$\Rightarrow$ Reduction by a factor $O((b/r)^{3-\omega})$:

- no asymptotic gain if $\omega = 3$ (we only rearranged computations)
- no gain if $r \sim b$ (good enough granularity... but $O(n^\omega)$ complexity)
- possibly large asymptotic gain in general!

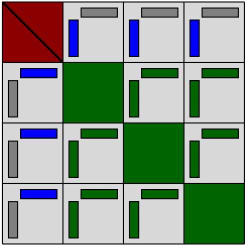
New Update kernel

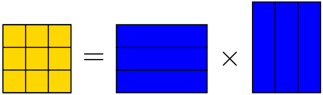




- Classical product: $O(p^3)$ calls of cost $O(br^{\omega-1}) \Rightarrow O(p^3br^{\omega-1})$

New Update kernel

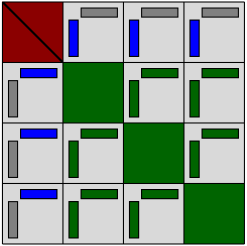


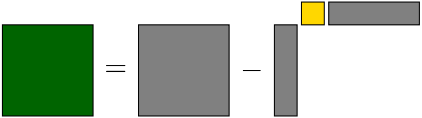


- Classical product: $O(p^3)$ calls of cost $O(br^{\omega-1}) \Rightarrow O(p^3br^{\omega-1})$
- New product: $O(p)$ calls of cost $O(p^2b^{\omega-2}r^2) \Rightarrow O(p^3b^{\omega-2}r^2)$

$\Rightarrow$ Reduction by a factor $O((b/r)^{3-\omega})$

New Update kernel



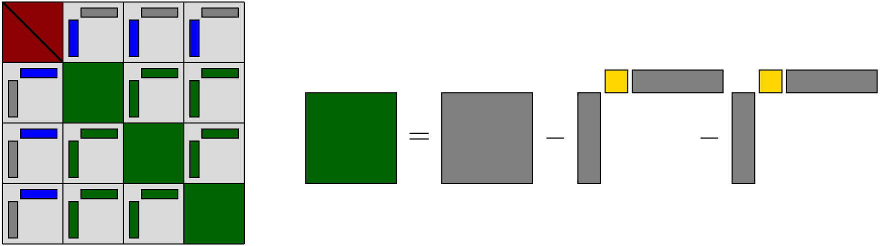


- Classical product: $O(p^3)$ calls of cost $O(br^{\omega-1}) \Rightarrow O(p^3br^{\omega-1})$
- New product: $O(p)$ calls of cost $O(p^2b^{\omega-2}r^2) \Rightarrow O(p^3b^{\omega-2}r^2)$

$\Rightarrow$ Reduction by a factor $O((b/r)^{3-\omega})$

- FR target subtractions: $O(p^2b^2r^{\omega-2})$

New Update kernel

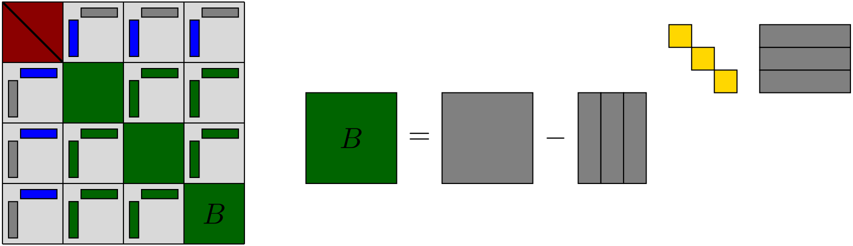


- Classical product: $O(p^3)$ calls of cost $O(br^{\omega-1}) \Rightarrow O(p^3 br^{\omega-1})$
- New product: $O(p)$ calls of cost $O(p^2 b^{\omega-2} r^2) \Rightarrow O(p^3 b^{\omega-2} r^2)$

$\Rightarrow$ Reduction by a factor $O((b/r)^{3-\omega})$

- FR target subtractions: $O(p^2 b^2 r^{\omega-2})$

New Update kernel



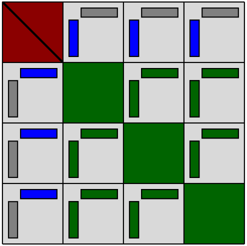
- Classical product: $O(p^3)$ calls of cost $O(br^{\omega-1}) \Rightarrow O(p^3br^{\omega-1})$
- New product: $O(p)$ calls of cost $O(p^2b^{\omega-2}r^2) \Rightarrow O(p^3b^{\omega-2}r^2)$

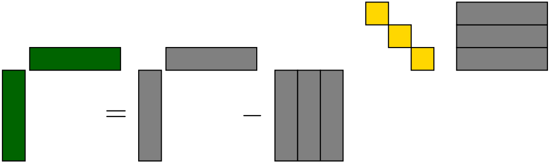
$\Rightarrow$ Reduction by a factor $O((b/r)^{3-\omega})$

- FR target subtractions: $O(p^2b^2r^{\omega-2}) \rightarrow O(p^2b^{\omega-1}r)$

$\Rightarrow$ Reduction by a factor $O((b/r)^{3-\omega})$

New Update kernel





- Classical product: $O(p^3)$ calls of cost $O(br^{\omega-1}) \Rightarrow O(p^3br^{\omega-1})$
- New product: $O(p)$ calls of cost $O(p^2b^{\omega-2}r^2) \Rightarrow O(p^3b^{\omega-2}r^2)$

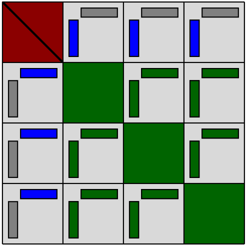
$\Rightarrow$ Reduction by a factor $O((b/r)^{3-\omega})$

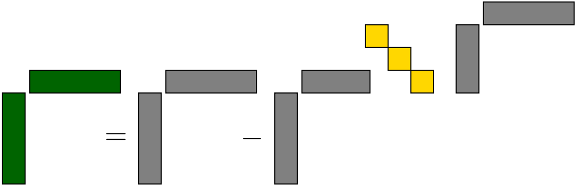
- FR target subtractions: $O(p^2b^2r^{\omega-2}) \rightarrow O(p^2b^{\omega-1}r)$

$\Rightarrow$ Reduction by a factor $O((b/r)^{3-\omega})$

- LR target subtractions: $O(p^3br^{\omega-1})$

New Update kernel





- Classical product: $O(p^3)$ calls of cost $O(br^{\omega-1}) \Rightarrow O(p^3br^{\omega-1})$
- New product: $O(p)$ calls of cost $O(p^2b^{\omega-2}r^2) \Rightarrow O(p^3b^{\omega-2}r^2)$

$\Rightarrow$ Reduction by a factor $O((b/r)^{3-\omega})$

- FR target subtractions: $O(p^2b^2r^{\omega-2}) \to O(p^2b^{\omega-1}r)$

$\Rightarrow$ Reduction by a factor $O((b/r)^{3-\omega})$

- LR target subtractions: $O(p^3br^{\omega-1}) \to O(p^2br^{\omega-1} + p^3r^{\omega})$

$\Rightarrow$ Reduction by a factor $O(b/r) \Rightarrow$ gain even for $\omega = 3$!

Complexity of the new algorithm

- Putting everything together, the complexity of the new algorithm is:

$$O(pb^\omega + p^2 b^{\omega-1} r + p^3 b^{\omega-2} r^2)$$

$$\subset O(n^{(\omega+1)/2} r^{(\omega-1)/2}) \qquad \text{for } b = \Theta((nr)^{1/2})$$

Complexity of the new algorithm

- Putting everything together, the complexity of the new algorithm is:

$$O(pb^{\omega} + p^2 b^{\omega-1} r + p^3 b^{\omega-2} r^2)$$

$$\subset O(n^{(\omega+1)/2} r^{(\omega-1)/2}) \qquad \text{for } b = \Theta((nr)^{1/2})$$

- $\omega = 3 \Rightarrow O(n^2 r)$
- $\omega = 2 \Rightarrow O(n^{3/2} r^{1/2}) \equiv$ BLR storage complexity $\Rightarrow$ nice!
- $\omega = \log_2 7 \Rightarrow O(n^{1.9} r^{0.9})$

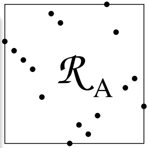
Outline

1. Block Low Rank representation
   - First attempt
   - New approach
2. The compact Bruhat representation
   - The rank profile matrix
   - Warm-up : computing the orders of quasiseparability
   - The Bruhat generator
   - The Compact Bruhat generator

The rank profile matrix [Dumas, P. and Sultan'15]

Definition (Rank Profile matrix)

The unique $\mathcal{R}_A \in \{0,1\}^{m\times n}$ such that any pair of (i,j)-leading sub-matrix of $\mathcal{R}_A$ and of A have the same rank.



A

1	2	3	4
2	4	5	8
1	2	3	4
3	5	9	12

→

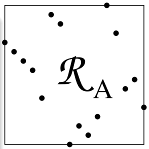
$\mathcal{R}$

1	0	0	0
0	0	1	0
0	0	0	0
0	1	0	0

The rank profile matrix [Dumas, P. and Sultan'15]

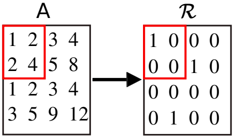
Definition (Rank Profile matrix)

The unique $\mathcal{R}_A \in \{0,1\}^{m \times n}$ such that any pair of (i,j)-leading sub-matrix of $\mathcal{R}_A$ and of A have the same rank.



Theorem

- *RowRP and ColRP read directly on* $\mathcal{R}_A$
- *Same holds for any* (i,j)*-leading submatrix.*

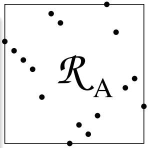


RowRP $= \{1\}$
ColRP $= \{1\}$

The rank profile matrix [Dumas, P. and Sultan'15]

Definition (Rank Profile matrix)

The unique $\mathcal{R}_A \in \{0,1\}^{m\times n}$ such that any pair of (i,j)-leading sub-matrix of $\mathcal{R}_A$ and of A have the same rank.



Theorem

- *RowRP and ColRP read directly on* $\mathcal{R}_A$
- *Same holds for any* (i,j)*-leading submatrix.*

A

1	2	3	4
2	4	5	8
1	2	3	4
3	5	9	12

→ $\mathcal{R}$

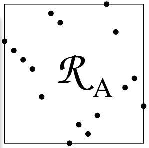
1	0	0	0
0	0	1	0
0	0	0	0
0	1	0	0

RowRP = {1,2}
ColRP = {1,3}

The rank profile matrix [Dumas, P. and Sultan'15]

Definition (Rank Profile matrix)

The unique $\mathcal{R}_A \in \{0,1\}^{m\times n}$ such that any pair of (i,j)-leading sub-matrix of $\mathcal{R}_A$ and of A have the same rank.



Theorem

- *RowRP and ColRP read directly on $\mathcal{R}_A$*
- *Same holds for any (i,j)-leading submatrix.*

A

1	2	3	4
2	4	5	8
1	2	3	4
3	5	9	12

→

$\mathcal{R}$

1	0	0	0
0	0	1	0
0	0	0	0
0	1	0	0

RowRP = {1,4}
ColRP = {1,2}

Computing the rank profile matrix

From a PLUQ decomposition

$$A = PLUQ = P \begin{bmatrix} L & 0 \\ M & I_{m-r} \end{bmatrix} \qquad \begin{bmatrix} I_r & \\ & 0 \end{bmatrix} \qquad \begin{bmatrix} U & V \\ & I_{n-r} \end{bmatrix} Q$$

Computing the rank profile matrix

From a PLUQ decomposition

$$A = PLUQ = P \begin{bmatrix} L & 0 \\ M & I_{m-r} \end{bmatrix} P^T P \begin{bmatrix} I_r & \\ & 0 \end{bmatrix} Q Q^T \begin{bmatrix} U & V \\ & I_{n-r} \end{bmatrix} Q$$

Computing the rank profile matrix

From a PLUQ decomposition

$$A = PLUQ = \underbrace{P \begin{bmatrix} L & 0 \\ M & I_{m-r} \end{bmatrix} P^T}_{\overline{L}} \underbrace{P \begin{bmatrix} I_r & \\ & 0 \end{bmatrix} Q}_{\mathcal{R}^A} \underbrace{Q^T \begin{bmatrix} U & V \\ & I_{n-r} \end{bmatrix} Q}_{\overline{U}}$$

Computing the rank profile matrix

From a PLUQ decomposition

$$A = PLUQ = \underbrace{P \begin{bmatrix} L & 0 \\ M & I_{m-r} \end{bmatrix} P^T}_{\overline{L}} \underbrace{P \begin{bmatrix} I_r & \\ & 0 \end{bmatrix} Q}_{\mathcal{R}^A} \underbrace{Q^T \begin{bmatrix} U & V \\ & I_{n-r} \end{bmatrix} Q}_{\overline{U}}$$

Theorem (Dumas, P. and Sultan 15)

- *The RPM can be recovered from a PLUQ decomposition computed by any Gaussian elimination algorithm under some conditions on its pivoting strategy.*
- *in time* $O(mnr^{\omega-2})$
- *in-place*

Finding the quasiseparability orders

Left triangular matrices

Of the form: $\begin{bmatrix} * & * & * & * & 0 \\ * & * & * & 0 & \\ * & * & 0 & & \\ * & 0 & & & \\ 0 & & & & \end{bmatrix}$

Examples: $\begin{bmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{bmatrix} \times$ Lower or Upper $\times \begin{bmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{bmatrix}$

$$\begin{bmatrix} 2 & 6 & 1 & 0 & 1 & 4 & 6 & 5 & 2 & 6 \\ 6 & 6 & 2 & 5 & 6 & 1 & 6 & 1 & 3 & 1 \\ 3 & 2 & 2 & 6 & 4 & 1 & 2 & 2 & 0 & 3 \\ 6 & 6 & 2 & 2 & 5 & 1 & 4 & 5 & 5 & 2 \\ 0 & 1 & 6 & 2 & 4 & 4 & 5 & 0 & 3 & 3 \\ 2 & 0 & 5 & 6 & 3 & 1 & 0 & 6 & 2 & 4 \\ 6 & 2 & 6 & 1 & 3 & 4 & 6 & 5 & 2 & 6 \\ 1 & 1 & 5 & 5 & 2 & 1 & 3 & 4 & 5 & 4 \\ 0 & 1 & 6 & 2 & 4 & 4 & 3 & 3 & 2 & 3 \\ 1 & 1 & 5 & 5 & 2 & 1 & 3 & 4 & 5 & 6 \end{bmatrix}$$

Finding the quasiseparability orders

Left triangular matrices

Of the form: $\begin{bmatrix} * & * & * & * & 0 \\ * & * & * & 0 & \\ * & * & 0 & & \\ * & 0 & & & \\ 0 & & & & \end{bmatrix}$

Examples: $\begin{bmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{bmatrix} \times$ Lower or Upper $\times \begin{bmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{bmatrix}$

$$\begin{bmatrix}
0 & & & & & & & & & \\
6 & 0 & & & & & & & & \\
3 & 2 & 0 & & & & & & & \\
6 & 6 & 2 & 0 & & & & & & \\
0 & 1 & 6 & 2 & 0 & & & & & \\
2 & 0 & 5 & 6 & 3 & 0 & & & & \\
6 & 2 & 6 & 1 & 3 & 4 & 0 & & & \\
1 & 1 & 5 & 5 & 2 & 1 & 3 & 0 & & \\
0 & 1 & 6 & 2 & 4 & 4 & 3 & 3 & 0 & \\
1 & 1 & 5 & 5 & 2 & 1 & 3 & 4 & 5 & 0
\end{bmatrix},
\begin{bmatrix}
0 & 6 & 1 & 0 & 1 & 4 & 6 & 5 & 2 & 6 \\
 & 0 & 2 & 5 & 6 & 1 & 6 & 1 & 3 & 1 \\
 & & 0 & 6 & 4 & 1 & 2 & 2 & 0 & 3 \\
 & & & 0 & 5 & 1 & 4 & 5 & 5 & 2 \\
 & & & & 0 & 4 & 5 & 0 & 3 & 3 \\
 & & & & & 0 & 0 & 6 & 2 & 4 \\
 & & & & & & 0 & 5 & 2 & 6 \\
 & & & & & & & 0 & 5 & 4 \\
 & & & & & & & & 0 & 3 \\
 & & & & & & & & & 0
\end{bmatrix}$$

Finding the quasiseparability orders

Left triangular matrices

Of the form: $\begin{bmatrix} * & * & * & * & 0 \\ * & * & * & 0 & \\ * & * & 0 & & \\ * & 0 & & & \\ 0 & & & & \end{bmatrix}$

Examples: $\begin{bmatrix} & & 1 \\ & \cdot^{\cdot} & \\ 1 & & \end{bmatrix} \times$ Lower or Upper $\times \begin{bmatrix} & & 1 \\ & \cdot^{\cdot} & \\ 1 & & \end{bmatrix}$

$$\begin{bmatrix} 1 & 1 & 5 & 5 & 2 & 1 & 3 & 4 & 5 & 0 \\ 0 & 1 & 6 & 2 & 4 & 4 & 3 & 3 & 0 & \\ 1 & 1 & 5 & 5 & 2 & 1 & 3 & 0 & & \\ 6 & 2 & 6 & 1 & 3 & 4 & 0 & & & \\ 2 & 0 & 5 & 6 & 3 & 0 & & & & \\ 0 & 1 & 6 & 2 & 0 & & & & & \\ 6 & 6 & 2 & 0 & & & & & & \\ 3 & 2 & 0 & & & & & & & \\ 6 & 0 & & & & & & & & \\ 0 & & & & & & & & & \end{bmatrix}, \begin{bmatrix} 6 & 2 & 5 & 6 & 4 & 1 & 0 & 1 & 6 & 0 \\ 1 & 3 & 1 & 6 & 1 & 6 & 5 & 2 & 0 & \\ 3 & 0 & 2 & 2 & 1 & 4 & 6 & 0 & & \\ 2 & 5 & 5 & 4 & 1 & 5 & 0 & & & \\ 3 & 3 & 0 & 5 & 4 & 0 & & & & \\ 4 & 2 & 6 & 0 & 0 & & & & & \\ 6 & 2 & 5 & 0 & & & & & & \\ 4 & 5 & 0 & & & & & & & \\ 3 & 0 & & & & & & & & \\ 0 & & & & & & & & & \end{bmatrix}$$

Finding the quasiseparability orders

Left triangular matrices

Of the form: $\begin{bmatrix} * & * & * & * & 0 \\ * & * & * & 0 & \\ * & * & 0 & & \\ * & 0 & & & \\ 0 & & & & \end{bmatrix}$

Examples: $\begin{bmatrix} & & 1 \\ & \cdot^{\cdot} & \\ 1 & & \end{bmatrix} \times$ Lower or Upper $\times \begin{bmatrix} & & 1 \\ & \cdot^{\cdot} & \\ 1 & & \end{bmatrix}$

$$\begin{bmatrix} 1 & 1 & 5 & 5 & 2 & 1 & 3 & 4 & 5 & 0 \\ 0 & 1 & 6 & 2 & 4 & 4 & 3 & 3 & 0 & \\ 1 & 1 & 5 & 5 & 2 & 1 & 3 & 0 & & \\ 6 & 2 & 6 & 1 & 3 & 4 & 0 & & & \\ 2 & 0 & 5 & 6 & 3 & 0 & & & & \\ 0 & 1 & 6 & 2 & 0 & & & & & \\ 6 & 6 & 2 & 0 & & & & & & \\ 3 & 2 & 0 & & & & & & & \\ 6 & 0 & & & & & & & & \\ 0 & & & & & & & & & \end{bmatrix}, \begin{bmatrix} 6 & 2 & 5 & 6 & 4 & 1 & 0 & 1 & 6 & 0 \\ 1 & 3 & 1 & 6 & 1 & 6 & 5 & 2 & 0 & \\ 3 & 0 & 2 & 2 & 1 & 4 & 6 & 0 & & \\ 2 & 5 & 5 & 4 & 1 & 5 & 0 & & & \\ 3 & 3 & 0 & 5 & 4 & 0 & & & & \\ 4 & 2 & 6 & 0 & 0 & & & & & \\ 6 & 2 & 5 & 0 & & & & & & \\ 4 & 5 & 0 & & & & & & & \\ 3 & 0 & & & & & & & & \\ 0 & & & & & & & & & \end{bmatrix}$$

Finding the quasiseparability orders

Left triangular matrices

Of the form: $\begin{bmatrix} * & * & * & * & 0 \\ * & * & * & 0 & \\ * & * & 0 & & \\ * & 0 & & & \\ 0 & & & & \end{bmatrix}$

Examples: $\begin{bmatrix} & & 1 \\ & \cdot^{\cdot} & \\ 1 & & \end{bmatrix} \times$ Lower or Upper $\times \begin{bmatrix} & & 1 \\ & \cdot^{\cdot} & \\ 1 & & \end{bmatrix}$

$$\begin{bmatrix} 1 & 1 & 5 & 5 & 2 & 1 & 3 & 4 & 5 & 0 \\ 0 & 1 & 6 & 2 & 4 & 4 & 3 & 3 & 0 & \\ 1 & 1 & 5 & 5 & 2 & 1 & 3 & 0 & & \\ 6 & 2 & 6 & 1 & 3 & 4 & 0 & & & \\ 2 & 0 & 5 & 6 & 3 & 0 & & & & \\ 0 & 1 & 6 & 2 & 0 & & & & & \\ 6 & 6 & 2 & 0 & & & & & & \\ 3 & 2 & 0 & & & & & & & \\ 6 & 0 & & & & & & & & \\ 0 & & & & & & & & & \end{bmatrix}, \begin{bmatrix} 6 & 2 & 5 & 6 & 4 & 1 & 0 & 1 & 6 & 0 \\ 1 & 3 & 1 & 6 & 1 & 6 & 5 & 2 & 0 & \\ 3 & 0 & 2 & 2 & 1 & 4 & 6 & 0 & & \\ 2 & 5 & 5 & 4 & 1 & 5 & 0 & & & \\ 3 & 3 & 0 & 5 & 4 & 0 & & & & \\ 4 & 2 & 6 & 0 & 0 & & & & & \\ 6 & 2 & 5 & 0 & & & & & & \\ 4 & 5 & 0 & & & & & & & \\ 3 & 0 & & & & & & & & \\ 0 & & & & & & & & & \end{bmatrix}$$

Finding the quasiseparability orders

Left triangular matrices

Of the form: $\begin{bmatrix} * & * & * & * & 0 \\ * & * & * & 0 & \\ * & * & 0 & & \\ * & 0 & & & \\ 0 & & & & \end{bmatrix}$

Examples: $\begin{bmatrix} & & 1 \\ & \cdot^{\cdot} & \\ 1 & & \end{bmatrix} \times$ Lower or Upper $\times \begin{bmatrix} & & 1 \\ & \cdot^{\cdot} & \\ 1 & & \end{bmatrix}$

$$\begin{bmatrix}
1 & 1 & 5 & 5 & 2 & 1 & 3 & 4 & 5 & 0 \\
0 & 1 & 6 & 2 & 4 & 4 & 3 & 3 & 0 & \\
1 & 1 & 5 & 5 & 2 & 1 & 3 & 0 & & \\
6 & 2 & 6 & 1 & 3 & 4 & 0 & & & \\
2 & 0 & 5 & 6 & 3 & 0 & & & & \\
0 & 1 & 6 & 2 & 0 & & & & & \\
6 & 6 & 2 & 0 & & & & & & \\
3 & 2 & 0 & & & & & & & \\
6 & 0 & & & & & & & & \\
0 & & & & & & & & &
\end{bmatrix}, \begin{bmatrix}
6 & 2 & 5 & 6 & 4 & 1 & 0 & 1 & 6 & 0 \\
1 & 3 & 1 & 6 & 1 & 6 & 5 & 2 & 0 & \\
3 & 0 & 2 & 2 & 1 & 4 & 6 & 0 & & \\
2 & 5 & 5 & 4 & 1 & 5 & 0 & & & \\
3 & 3 & 0 & 5 & 4 & 0 & & & & \\
4 & 2 & 6 & 0 & 0 & & & & & \\
6 & 2 & 5 & 0 & & & & & & \\
4 & 5 & 0 & & & & & & & \\
3 & 0 & & & & & & & & \\
0 & & & & & & & & &
\end{bmatrix}$$

Quasiseparability orders on the rank profile matrix

$$
\begin{bmatrix}
2 & 2 & 1 & 0 & 2 & 1 & 1 & 1 & 0 & 0\\
2 & 2 & 1 & 0 & 2 & 0 & 1 & 2 & 0 & 0\\
2 & 2 & 1 & 0 & 2 & 2 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
2 & 2 & 1 & 0 & 2 & 0 & 0 & 0 & 0 & 0\\
2 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix},
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & & & \\
0 & 0 & 0 & 0 & 1 & & & & & \\
0 & 1 & 0 & 0 & & & & & & \\
0 & 0 & 0 & & & & & & & \\
0 & 0 & 1 & & & & & & & \\
0 & & & & & & & & &
\end{bmatrix}
$$

Rank $= 1$ Quasiseparability order $= 1$

Quasiseparability orders on the rank profile matrix

$$
\begin{bmatrix}
2 & 2 & 1 & 0 & 2 & 1 & 1 & 1 & 0 & 0\\
2 & 2 & 1 & 0 & 2 & 0 & 1 & 2 & 0 & 0\\
2 & 2 & 1 & 0 & 2 & 2 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
2 & 2 & 1 & 0 & 2 & 0 & 0 & 0 & 0 & 0\\
2 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix},
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & & & \\
0 & 0 & 0 & 0 & 1 & & & & & \\
0 & 1 & 0 & 0 & & & & & & \\
0 & 0 & 0 & & & & & & & \\
0 & 0 & 1 & & & & & & & \\
0 & & & & & & & & &
\end{bmatrix}
$$

Rank $= 2$ Quasiseparability order $= 2$

Quasiseparability orders on the rank profile matrix

$$
\begin{bmatrix}
2 & 2 & 1 & 0 & 2 & 1 & 1 & 1 & 0 & 0 \\
2 & 2 & 1 & 0 & 2 & 0 & 1 & 2 & 0 & 0 \\
2 & 2 & 1 & 0 & 2 & 2 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 2 & 1 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\
2 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix},
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & & & \\
0 & 0 & 0 & 0 & 1 & & & & & \\
0 & 1 & 0 & 0 & & & & & & \\
0 & 0 & 0 & & & & & & & \\
0 & 0 & 1 & & & & & & & \\
0 & & & & & & & & &
\end{bmatrix}
$$

Rank $= 2$ Quasiseparability order $= 2$

Quasiseparability orders on the rank profile matrix

$$\begin{bmatrix}
2&2&1&0&2&1&1&1&0&0\\
2&2&1&0&2&0&1&2&0&0\\
2&2&1&0&2&2&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
2&2&1&0&2&0&0&0&0&0\\
2&2&1&0&0&0&0&0&0&0\\
2&1&0&0&0&0&0&0&0&0\\
1&2&0&0&0&0&0&0&0&0\\
2&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0
\end{bmatrix},
\begin{bmatrix}
1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&\\
0&0&0&0&0&0&0&0&&\\
0&0&0&0&0&0&0&&&\\
0&0&0&0&0&0&1&&&\\
0&0&0&0&1&&&&&\\
0&1&0&0&&&&&&\\
0&0&0&&&&&&&\\
0&0&1&&&&&&&\\
0&&&&&&&&&
\end{bmatrix}$$

Rank $=1$ Quasiseparability order $=2$

Quasiseparability orders on the rank profile matrix

$$
\begin{bmatrix}
2 & 2 & 1 & 0 & 2 & 1 & 1 & 1 & 0 & 0\\
2 & 2 & 1 & 0 & 2 & 0 & 1 & 2 & 0 & 0\\
2 & 2 & 1 & 0 & 2 & 2 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
2 & 2 & 1 & 0 & 2 & 0 & 0 & 0 & 0 & 0\\
2 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix},
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & & & \\
0 & 0 & 0 & 0 & 1 & & & & & \\
0 & 1 & 0 & 0 & & & & & & \\
0 & 0 & 0 & & & & & & & \\
0 & 0 & 1 & & & & & & & \\
0 & & & & & & & & &
\end{bmatrix}
$$

Rank $= 1$ Quasiseparability order $= 2$

Major difficulty

Complexity in $O(n^2 r^{\omega-2})$ where $r = \text{rank}(A) \gg r_L, r_U$. But

Quasiseparability orders on the rank profile matrix

$$\begin{bmatrix} 2 & 2 & 1 & 0 & 2 & 1 & 1 & 1 & 0 & 0 \\ 2 & 2 & 1 & 0 & 2 & 0 & 1 & 2 & 0 & 0 \\ 2 & 2 & 1 & 0 & 2 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 1 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & & & \\ 0 & 0 & 0 & 0 & 1 & & & & & \\ 0 & 1 & 0 & 0 & & & & & & \\ 0 & 0 & 0 & & & & & & & \\ 0 & 0 & 1 & & & & & & & \\ 0 & & & & & & & & & \end{bmatrix}$$

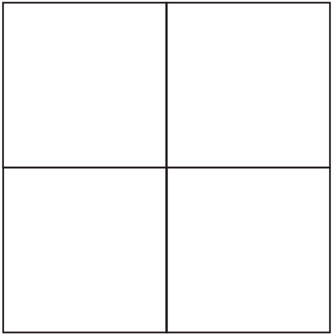
Rank $= 1$ Quasiseparability order $= 2$

Major difficulty

Complexity in $O(n^2 r^{\omega-2})$ where $r = \text{rank}(A) \gg r_L, r_U$. But

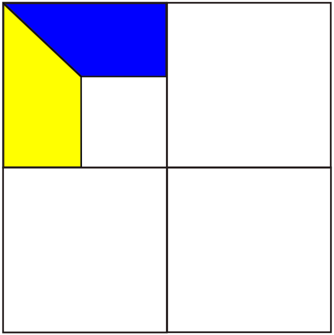
- only a few pivots ($O(r_L, r_U)$) near the top left corner
- when numerous, most pivots must be near the anti-diagonal
  $\rightsquigarrow$ cheaper elimination

Computing the left-triangular part of a rank profile matrix



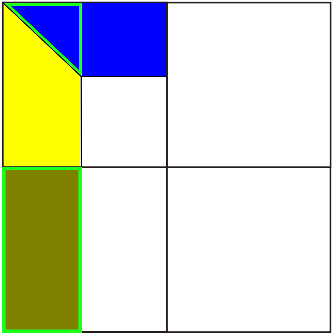
2×2 block splitting

Computing the left-triangular part of a rank profile matrix



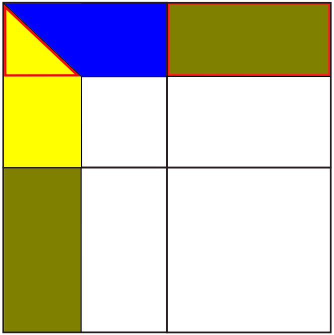
Recursive call

Computing the left-triangular part of a rank profile matrix



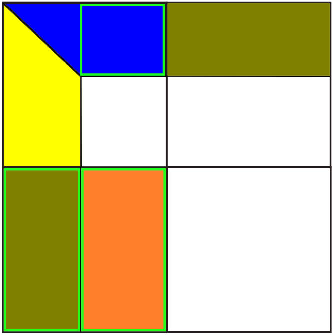
TRSM: $B \leftarrow BU^{-1}$

Computing the left-triangular part of a rank profile matrix



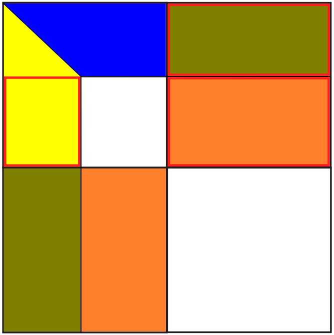
TRSM: $B \leftarrow L^{-1}B$

Computing the left-triangular part of a rank profile matrix



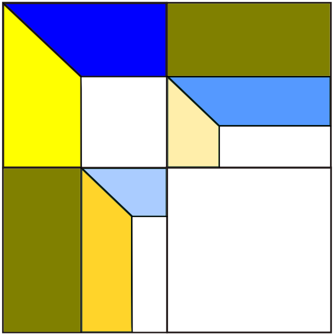
`MatMul`: $C \leftarrow C - A \times B$

Computing the left-triangular part of a rank profile matrix



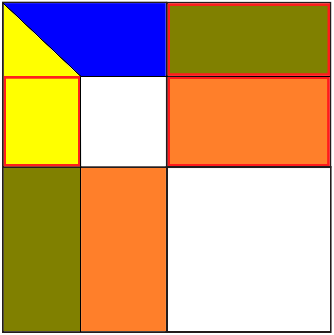
MatMul: $C \leftarrow C - A \times B$

Computing the left-triangular part of a rank profile matrix



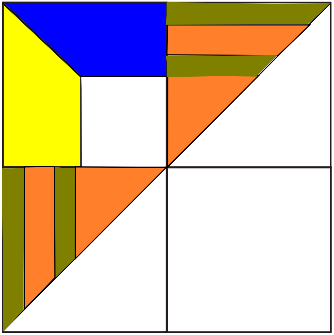
2 recursive calls : not possible as no longer left-triangular

Computing the left-triangular part of a rank profile matrix



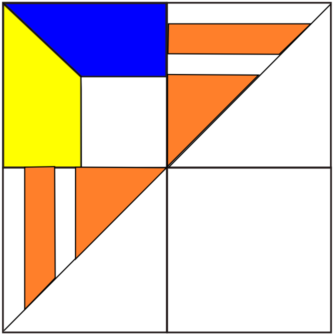
2 recursive calls : not possible as no longer left-triangular

Computing the left-triangular part of a rank profile matrix



2 recursive calls : not possible as no longer left-triangular
$\rightsquigarrow$ permute back to original position

Computing the left-triangular part of a rank profile matrix

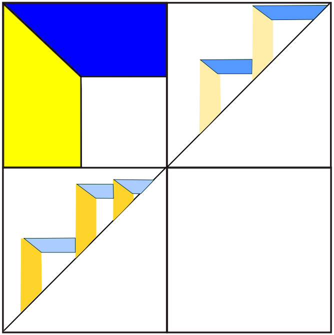


2 recursive calls : not possible as no longer left-triangular

$\rightsquigarrow$ permute back to original position

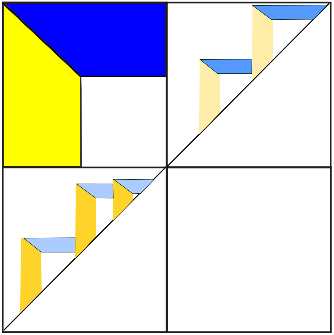
$\rightsquigarrow$ clear out cols/rows already processed

Computing the left-triangular part of a rank profile matrix



2 recursive calls : not possible as no longer left-triangular
$\rightsquigarrow$ permute back to original position
$\rightsquigarrow$ clear out cols/rows already processed
recursive call

Computing the left-triangular part of a rank profile matrix



2 recursive calls : not possible as no longer left-triangular

⇝ permute back to original position

⇝ clear out cols/rows already processed

recursive call

- $O(s^{\omega-2}n^2)$
- in place

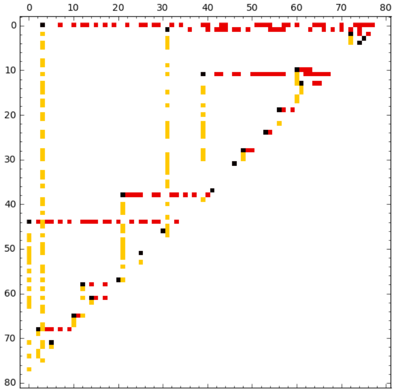
The Bruhat generator

Idea: *When numerous, most pivots must lie near the anti-diagonal.*

Definition (Bruhat Generator)

From a PLUQ decomposition revealing the rank profile matrix E : the triple $(\mathcal{L}, \mathcal{E}, \mathcal{U})$

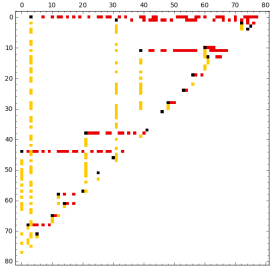
$$\begin{aligned}\mathcal{L} &= \text{Left}(P \begin{bmatrix} L & 0 \end{bmatrix} Q),\\ \mathcal{E} &= \text{Left}(E),\\ \mathcal{U} &= \text{Left}(P \begin{bmatrix} U \\ 0 \end{bmatrix} Q).\end{aligned}$$



The Bruhat Generator

For a quasiseparability order s:

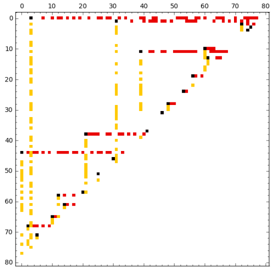
- ✓Generator: $A = \mathsf{Left}(\mathcal{L}\mathcal{E}^T\mathcal{U})$



The Bruhat Generator

For a quasiseparability order s:

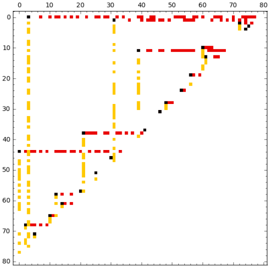
- ✓Generator: $A = \text{Left}(\mathcal{L}\mathcal{E}^T\mathcal{U})$
- ✓Size: $O(ns)$



The Bruhat Generator

For a quasiseparability order s:

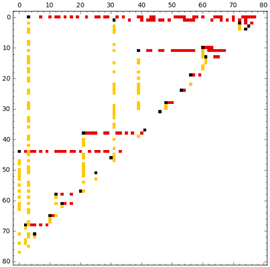
- ✓Generator: $A = \mathsf{Left}(\mathcal{L}\mathcal{E}^T\mathcal{U})$
- ✓Size: $O(ns)$
- ✓Computed in $O(n^2 s^{\omega-2})$
  $\rightsquigarrow$ Adapted from the tile rec. PLUQ algo (same as for the QS orders)



The Bruhat Generator

For a quasiseparability order s:

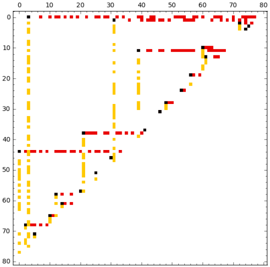
- ✓Generator: $A = \text{Left}(\mathcal{L}\mathcal{E}^T\mathcal{U})$
- ✓Size: $O(ns)$
- ✓Computed in $O(n^2 s^{\omega-2})$
  ⇝ Adapted from the tile rec. PLUQ algo (same as for the QS orders)
- ✓QuasiSep × Vector in $O(\text{Size}) = O(ns)$



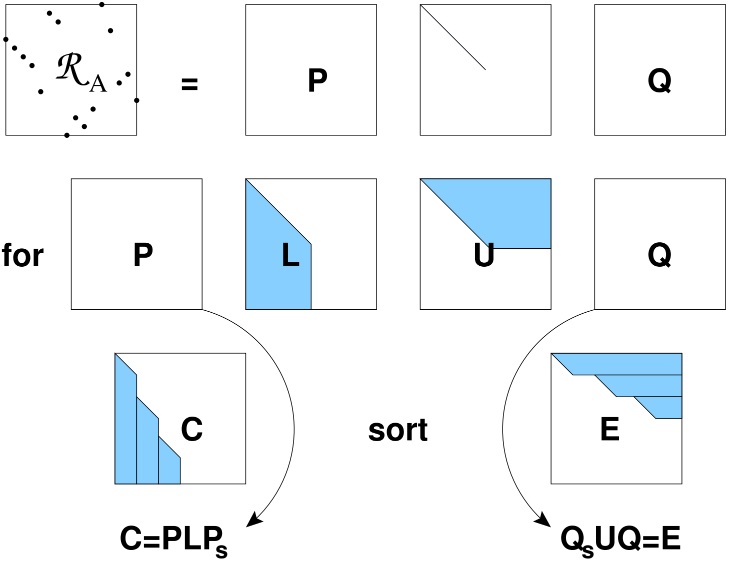
The Bruhat Generator

For a quasiseparability order s:

- ✓Generator: $A = \text{Left}(\mathcal{L}\mathcal{E}^T\mathcal{U})$
- ✓Size: $O(ns)$
- ✓Computed in $O(n^2 s^{\omega-2})$
  $\rightsquigarrow$ Adapted from the tile rec. PLUQ algo (same as for the QS orders)
- ✓QuasiSep $\times$ Vector in $O(\text{Size}) = O(ns)$
- ✗Scattered$\rightsquigarrow$ no fast matrix multiplication



Row and column echelon forms from PLUQ

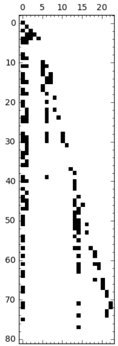


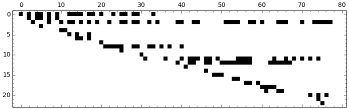
The compact Bruhat Generator

- $C = \mathcal{L}\mathcal{Q}$: Col. permutation to column echelon form
- $E = \mathcal{P}\mathcal{U}$: Row permutation to row echelon form

$$A = \text{Left}(\text{CRE})$$

for an $r \times r$ permutation R.
(generalized Bruhat decomposition [Manthey Helmke 07][7]).

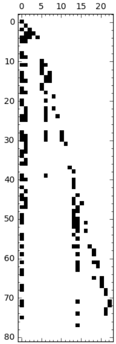


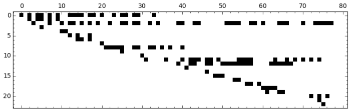


The compact Bruhat Generator

Major difficulty

Again $r = \text{rank}(A) \gg s$



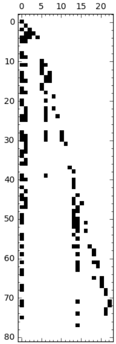


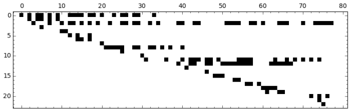
The compact Bruhat Generator

Major difficulty

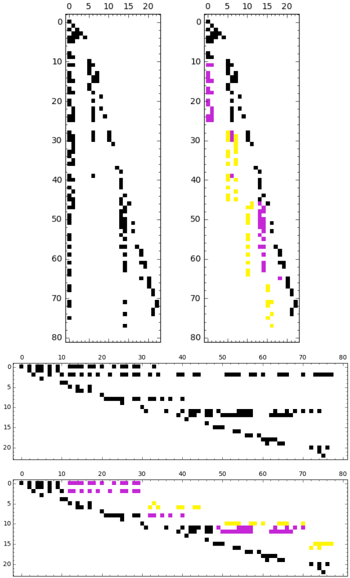
Again $r = \mathrm{rank}(A) \gg s$

- But double structure: either echelon or Left-triangular





The compact Bruhat Generator



Major difficulty

Again $r = \text{rank}(A) \gg s$

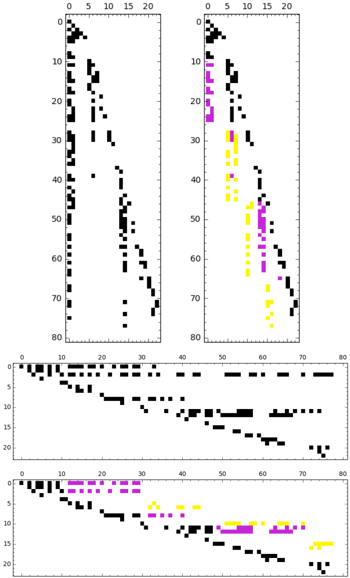
- But double structure: either echelon or Left-triangular
- Fold them in blocks of size s

The compact Bruhat Generator

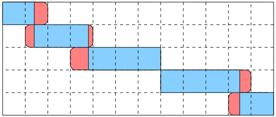
Major difficulty

Again $r = \text{rank}(A) \gg s$

- But double structure: either echelon or Left-triangular
- Fold them in blocks of size s
- forming a **dense** $O(ns)$ block bi-diagonal structure

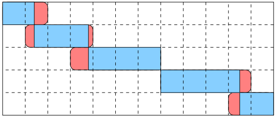


Complexities



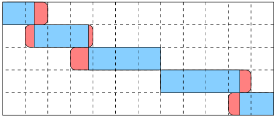
- QS $\times$ Vector: $O(ns)$

Complexities



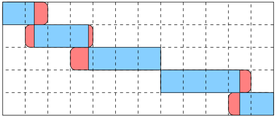
- QS $\times$ Vector: $O(ns)$
- QS $\times$ TallSkinny $(n \times s)$: $O(ns^{\omega-1})$ [P. Storjohann 16]

Complexities



- QS $\times$ Vector: $O(ns)$
- QS $\times$ TallSkinny ($n \times s$): $O(ns^{\omega-1})$ [P. Storjohann 16]
- QS $\times$ QS : $O(ns^{\omega-1})$ using randomize recompression [in preparation]

Complexities



- QS × Vector: $O(ns)$
- QS × TallSkinny ($n \times s$): $O(ns^{\omega-1})$ [P. Storjohann 16]
- QS × QS : $O(ns^{\omega-1})$ using randomize recompression [in preparation]
- LU(QS) : $O(ns^{\omega-1})$ using randomize recompression [in preparation]
- Solve(QS) : $O(ns^{\omega-1})$ using randomize recompression [in preparation]

Perspectives

Faster construction

- [Storjohann Yang 15][9] computes RowRP in Las-Vegas $\tilde{O}(|A| + r^\omega)$
- [Dumas P. Sultan 16][4] extends it to RPM

Perspectives

Faster construction

- [Storjohann Yang 15][9] computes RowRP in Las-Vegas $\tilde{O}(|A| + r^\omega)$
- [Dumas P. Sultan 16][4] extends it to RPM
- Does it applies to computing
  - the QS orders ?
  - a CB generator ?

Perspectives

Faster construction

- [Storjohann Yang 15][9] computes RowRP in Las-Vegas $\tilde{O}(|A| + r^\omega)$
- [Dumas P. Sultan 16][4] extends it to RPM
- Does it applies to computing
  - the QS orders ?
  - a CB generator ?

Numerical stability

- related to the use of subcubic matrix multiplication
- related to the pivoting strategy releaving the RPM

References

P. Amestoy et al. "Improving Multifrontal Methods by Means of Block Low-Rank Representations". In: *SIAM Journal on Scientific Computing* 37.3 (Jan. 2015), A1451–A1474. DOI: 10.1137/120903476.

S. Chandrasekaran, M. Gu, and T. Pals. "A Fast ULV Decomposition Solver for Hierarchically Semiseparable Representations". In: *SIAM Journal on Matrix Analysis and Applications* 28.3 (Jan. 2006), pp. 603–622. DOI: 10.1137/S0895479803436652.

J.-G. Dumas, C. Pernet, and Z. Sultan. "Computing the Rank Profile Matrix". In: *Proc. ISSAC'15*. Bath, United Kingdom: ACM, 2015, pp. 149–156. DOI: 10.1145/2755996.2756682.

J.-G. Dumas, C. Pernet, and Z. Sultan. "Fast Computation of the Rank Profile Matrix and the Generalized Bruhat Decomposition". In: *Journal of Symbolic Computation* (2016). arXiv:1601.01798 [cs.SC].

Y. Eidelman and I. Gohberg. "On a new class of structured matrices". en. In: *Integral Equations and Operator Theory* 34.3 (Sept. 1999), pp. 293–324. DOI: 10.1007/BF01300581.

W. Hackbusch. *Hierarchical Matrices: Algorithms and Analysis*. Vol. 49. Dec. 2015. DOI: 10.1007/978-3-662-47324-5.

W. Manthey and U. Helmke. "Bruhat canonical form for linear systems". In: *Linear Algebra and its Applications* 425.2–3 (2007). Special Issue in honor of Paul Fuhrmann, pp. 261–282. DOI: 10.1016/j.laa.2007.01.022.

Z. Sheng, P. Dewilde, and S. Chandrasekaran. "Algorithms to Solve Hierarchically Semi-separable Systems". en. In: *System Theory, the Schur Algorithm and Multidimensional Analysis*. Ed. by D. Alpay and V. Vinnikov. Operator Theory: Advances and Applications 176. DOI: 10.1007/978-3-7643-8137-0_5. Birkhäuser Basel, 2007, pp. 255–294.