Deterministic computation of the characteristic polynomial in the time of matrix multiplication

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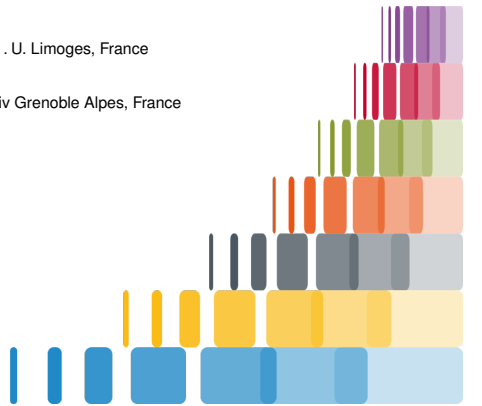

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Outline

- Context, problem, state of the art
- Obstacles, overview of the approach
- Complexity, Spin-off results

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- field $\mathbb{K}$, algebraic complexity (counting operations in $\mathbb{K}$)
- ω: exponent of MatMul over $\mathbb{K}$: $m \times m$ by $m \times m$ in $O(m^\omega)$

Reductions: of most problems to matrix multiplication

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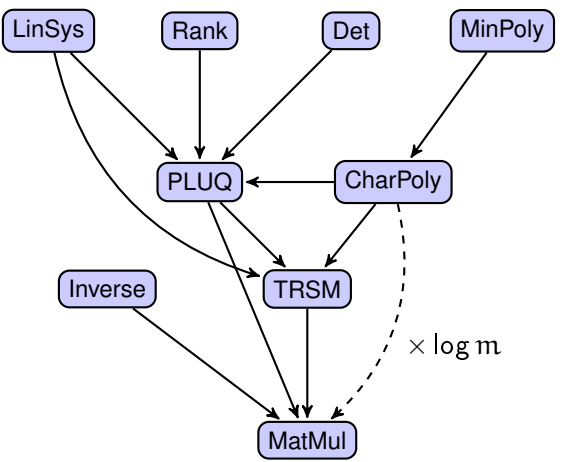

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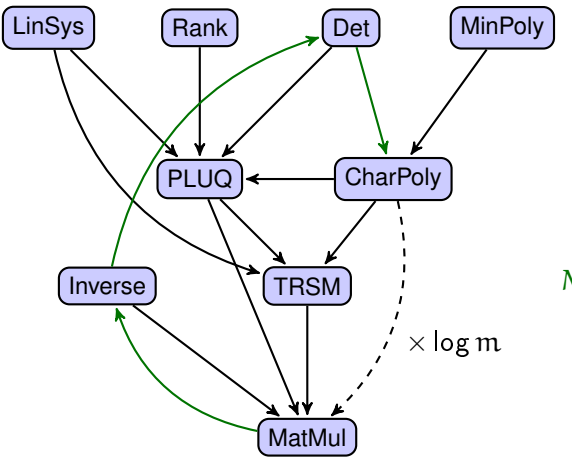

$$\left.\begin{matrix} \mathrm{LinSys} \\ \mathrm{Det} \\ \mathrm{Rank} \\ \mathrm{PLUQ} \\ \mathrm{TRSM} \\ \mathrm{Inverse} \end{matrix}\right\} = O(\mathrm{MatMul})$$

$$\mathrm{MatMul} = O\left(\begin{matrix} \mathrm{Det}, \\ \mathrm{PLUQ}, \\ \mathrm{CharPoly}, \\ \mathrm{Inverse} \end{matrix}\right)$$

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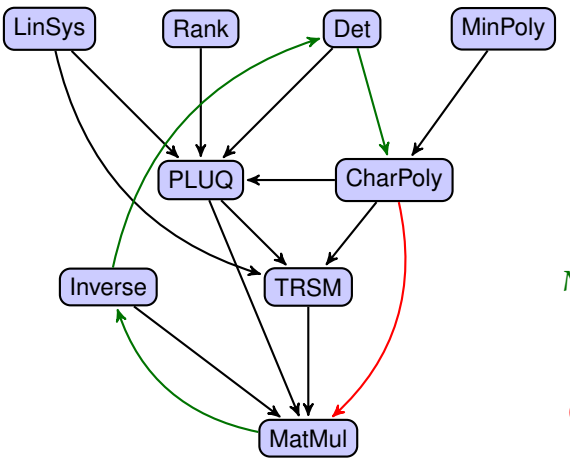

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$$\text{MatMul} = O(\begin{matrix}\text{Det},\\ \text{PLUQ},\\ \text{CharPoly},\\ \text{Inverse}\end{matrix})$$

$$\text{CharPoly} = O(\text{MatMul})\ ?$$

Problem and result

Characteristic polynomial...

given $\mathbf{M} \in \mathbb{K}^{m \times m}$, compute $\det(x\mathbf{I}_m - \mathbf{M}) \in \mathbb{K}[x]$

- deterministic, general: $O(m^\omega \log(m))$ [Keller-Gehrig 1985]
- deterministic, generic input: $O(m^\omega)$ [Giorgi & Jeannerod & Villard 2003]
- randomized, general: $O(m^\omega)$ [P. & Storjohann 2007]

...in the time of matrix multiplication

Deterministic charpoly algorithm in $O(m^\omega)$

using any MatMul algorithm in $O(m^\omega)$ with $2 < \omega \leqslant 3$

(i.e. not relying on a $\tilde{O}(m^{\omega-\varepsilon})$ MatMul algorithm...)

arXiv: 2010.04662 / HAL: hal-02963147

Motivation

Remark

In theory, for all ω, $\mathrm{CharPoly} = \mathrm{O}(\mathrm{m}^\omega)$ *is achieved by running*

- *any* $\mathrm{O}(\mathrm{m}^\omega \log \mathrm{m})$ *algorithm (like Keller-Gehrig's)*
- *using an* $\mathrm{O}(\mathrm{m}^{\omega-\varepsilon})$ *MatMul*

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More relevant model

Rely on a given MatMul algorithm, and reach the same exponent

- No cheating w.r.t. the presence of $\log$
- Underlying hard question:

> *"How to compute an object related to a Krylov iteration without* $\log$ *?"*

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In practice:

- Only a few subcubic time MatMul are available (mainly Strassen's)
- [P. Storjohann'07] randomized algorithm inefficient with small fields

Charpoly via linear algebra over $\mathbb{K}$

Traces of Powers: [LeVerrier1840] [Faddeev'49, Souriau'48, ...], $O(m^4)$ or $O(m^{\omega+1})$
used by [Csanky'75] to prove $\mathcal{NC}^2$ membership.

Determinant expansion: [Samuelson'42, Berkowitz'84] $O(m^4)$
suited for division free algorithms with later developments in
[Abdlejaoued and Malaschonok'01, Kaltofen and Villard'05]

Krylov methods: [Danilevskij'37,Keller-Gehrig'85, P. and Storjohann'07]

- Deterministic $O(m^3)$ or $O(m^\omega \log m)$
- Generic $O(m^\omega)$
- Las-Vegas probabilistic for large fields ($|\mathbb{K}| \geqslant 2m^2$) $O(m^\omega)$

Charpoly via linear algebra over $\mathbb{K}[x]$

Determinant of a matrix $\mathbf{A} \in \mathbb{K}[x]^{m \times m}$ of degree d $\quad d = 1$

Evaluation-Interpolation: [folklore] $\quad O(m^{\omega+1})$

Diagonalization: Smith Form [Storjohann 2003] $\quad O(m^{\omega} \log(m)^2)$

Las Vegas randomized + additional logs for small fields

Triangularization:

- Generic: [Giorgi-Jeannerod-Villard 2003]: $\quad O(m^{\omega})$
  diagonal of Hermite form must be $1, \ldots, 1, \det(\mathbf{A})$
- [Neiger-Labahn-Zhou 2017] via triangularization: $\quad \tilde{O}(m^{\omega})$
  logarithmic factors in m and d

For polynomials of degree $\leqslant d$ in $\mathbb{K}[x]$:

PolMul in $\leqslant M(d)$ f.ops. $\qquad$ GCD in $\leqslant M'(d) \in O(M(d)\log(d))$ f.ops.

Where do the log come from?

Explicit Krylov iteration: to compute a Krylov basis

$$\begin{pmatrix} v & Av & \dots & A^{m-1}v \end{pmatrix}$$

$\rightarrow$ Fast matrix exponentiation: $\log m \times O(m^{\omega})$

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Polynomial matrix determinants: If Divide and conquer is applied

- on matrix dimension $\rightarrow$ no $\log(m)$
- on degree (Polynomial arithmetic) $\rightarrow$ no $\log(m)$
- on both dimension and degrees:
  $\rightarrow$ manageable if the total degree remains controlled.
  Hard case: long Krylov chains with small dimension drop
  $\rightarrow$ reminiscent of the failure to de-randomize [P. Storjohann'07]

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Partial block triangularization

[Giorgi-Jeannerod-Villard 2003, Zhou 2012, Neiger-Labahn-Zhou 2017]

Triangularization of $\mathrm{m} \times \mathrm{m}$ matrix $\mathbf{A}$ using $\mathrm{m}/2 \times \mathrm{m}/2$ blocks

not computed

$$\begin{bmatrix} * & * \\ \mathbf{K}_1 & \mathbf{K}_2 \end{bmatrix} \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 \\ \mathbf{A}_3 & \mathbf{A}_4 \end{bmatrix} = \begin{bmatrix} \mathbf{R} & * \\ \mathbf{0} & \mathbf{B} \end{bmatrix}$$

kernel basis of $\begin{bmatrix} \mathbf{A}_1 \\ \mathbf{A}_3 \end{bmatrix}$

$\mathbf{K}_1\mathbf{A}_2 + \mathbf{K}_2\mathbf{A}_4$

row basis of $\begin{bmatrix} \mathbf{A}_1 \\ \mathbf{A}_3 \end{bmatrix}$

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Property: $\det(\mathbf{A}) = \det(\mathbf{R}) \det(\mathbf{B})$

Generic case without log factor

[Giorgi-Jeannerod-Villard 2003, Zhou 2012, Neiger-Labahn-Zhou 2017]

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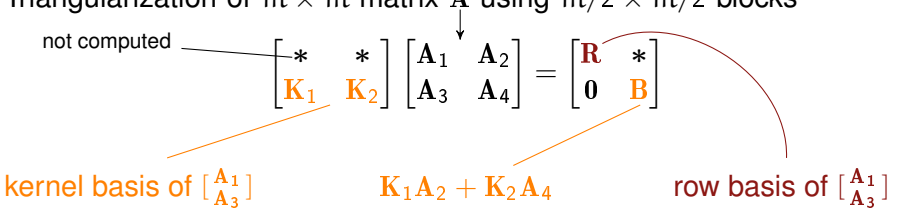


Property: $\det(\mathbf{A}) = \det(\mathbf{R})\det(\mathbf{B})$

Generic input $\Rightarrow$ $\det(\mathbf{A})$ without $\log(m)$ [Giorgi-Jeannerod-Villard 2003]

$\mathbf{A}_1$ and $\mathbf{A}_3$ are coprime $\Rightarrow \mathbf{R} = \mathbf{I}_{m/2} \Rightarrow \det(\mathbf{A}) = \det(\mathbf{B})$

- Compute kernel $[\mathbf{K}_1 \ \ \mathbf{K}_2]$; deduce $\mathbf{B}$ by MatMul $\quad O(m^\omega M'(d))$
- Recursively, compute $\det(\mathbf{B})$, return it

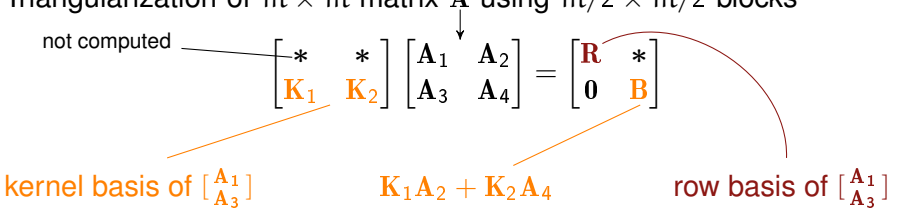
$\mathbf{A}$ and $[\mathbf{K}_1 \ \ \mathbf{K}_2]$ have degree $d \Rightarrow \mathbf{B}$ has degree $2d$: controlled total degree

total cost $O(m^\omega M'(d) + (m/2)^\omega M'(2d) + \cdots + M'(md)) \subset O(m^\omega M'(d))$

General case with log factor

[Giorgi-Jeannerod-Villard 2003, Zhou 2012, Neiger-Labahn-Zhou 2017]

Triangularization of $m \times m$ matrix $\mathbf{A}$ using $m/2 \times m/2$ blocks



Property: $\det(\mathbf{A}) = \det(\mathbf{R})\det(\mathbf{B})$

Matrix degree not controlled: degree of $\mathbf{B}$ up to $D = \sum \mathrm{rdeg}(\mathbf{A}) \leqslant md$
but controlled average row degree: at most $\frac{D}{m}$

General input $\Rightarrow \det(\mathbf{A})$ in $\tilde{O}(m^\omega \frac{D}{m})$ [Labahn-Neiger-Zhou 2017]

- Compute kernel $[\mathbf{K}_1 \ \ \mathbf{K}_2]$; deduce $\mathbf{B}$ by MatMul $O(m^\omega M'(\frac{D}{m}))$
- Compute row basis $\mathbf{R}$ $\tilde{O}(m^\omega \frac{D}{m})$ with $\log(m)$
- Recursively, compute $\det(\mathbf{R})$ and $\det(\mathbf{B})$, return $\det(\mathbf{R})\det(\mathbf{B})$

Be lazy: if hard to compute, don't compute

[Giorgi-Jeannerod-Villard 2003, Zhou 2012, Neiger-Labahn-Zhou 2017]

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Property: $\det(\mathbf{A}) = \det(\mathbf{R}) \det(\mathbf{B})$

Obstacle: removing log factors in row basis computation

$\Rightarrow$ solution: remove row basis computation

$$\begin{bmatrix} \mathbf{I}_{\mathrm{m}/2} & \mathbf{0} \\ \mathbf{K}_1 & \mathbf{K}_2 \end{bmatrix} \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 \\ \mathbf{A}_3 & \mathbf{A}_4 \end{bmatrix} = \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 \\ \mathbf{0} & \mathbf{B} \end{bmatrix}$$

Property: $\det(\mathbf{A}) = \det(\mathbf{A}_1) \det(\mathbf{B}) / \det(\mathbf{K}_2)$

Further obstacles (brought by laziness)

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- 👍 no $\log(m)$ in the computation of $\mathbf{A}_1$, $\mathbf{B}$, $\mathbf{K}_2$
- 👎 requires nonsingular $\mathbf{A}_1$, otherwise $\det(\mathbf{K}_2) = 0$
- 👎 3 recursive calls in matrix size $m/2$ is 👍, but requires $\sum \mathrm{rdeg}(\mathbf{A}_1) \leqslant D/2$ otherwise degree control is too weak. (this implies $\sum \mathrm{rdeg}(\mathbf{K}_2) \leqslant D/2$)

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Solution: require $\mathbf{A}$ in weak Popov form

(the characteristic matrix $\mathbf{A} = x\mathbf{I}_m - \mathbf{M}$ is in Popov form)

- 👍 implies $\mathbf{A}_1$ nonsingular and $\sum \mathrm{rdeg}(\mathbf{A}_1) \leqslant D/2$ up to easy transformations
- 👍 both $\mathbf{A}_1$ and $\mathbf{B}$ are also in weak Popov form ⇒ suitable for recursive calls
- 👎 $\mathbf{K}_2$ is in "shifted reduced" form… find weak Popov $\mathbf{P}$ with same determinant

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- 👎 $\mathbf{K}_2$ is in "shifted reduced" form… find weak Popov $\mathbf{P}$ with same determinant $\rightsquigarrow$ $\mathbf{P}$ = transpose of the $-\mathrm{rdeg}_{\mathrm{rdeg}(\mathbf{A}_4)}(\mathbf{K}_2)$-Popov form of $\mathbf{K}_2^{\mathsf{T}}$

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Complexity

$$\mathcal{C}(m, D) \leqslant 2\mathcal{C}\left(\frac{m}{2}, \left\lfloor \frac{D}{2} \right\rfloor\right) + \mathcal{C}\left(\frac{m}{2}, D\right) + O\left(m^{\omega} M'\left(\frac{D}{m}\right)\right)$$

where: $M'(d) = GCD(d) \in O(PolMul(d)\log(d))$

$\frac{D}{m} = \frac{\deg \det}{m} =$ avg row degree

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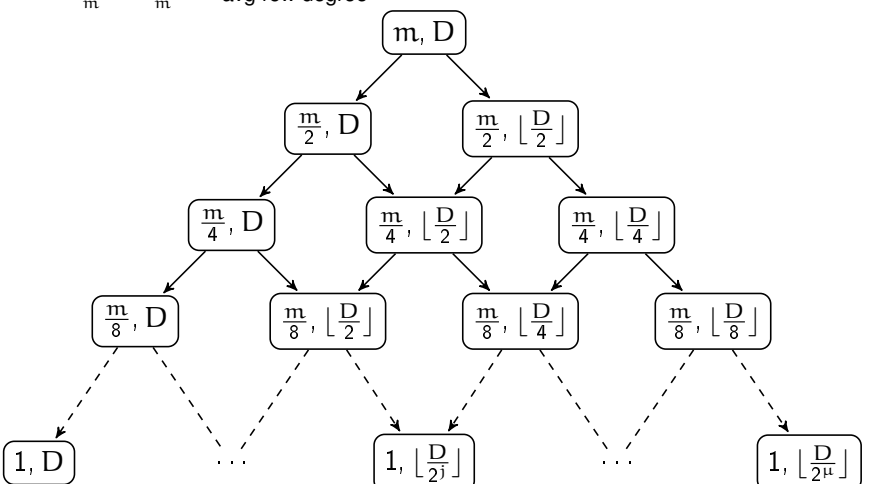

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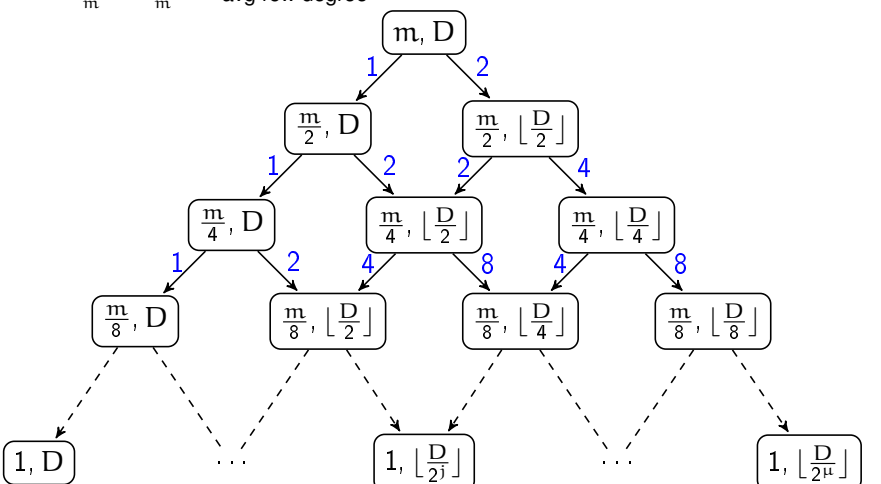

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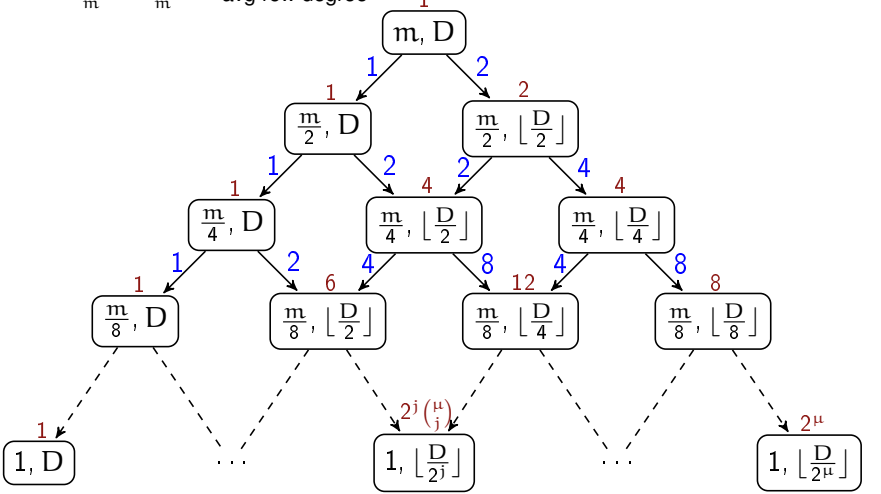

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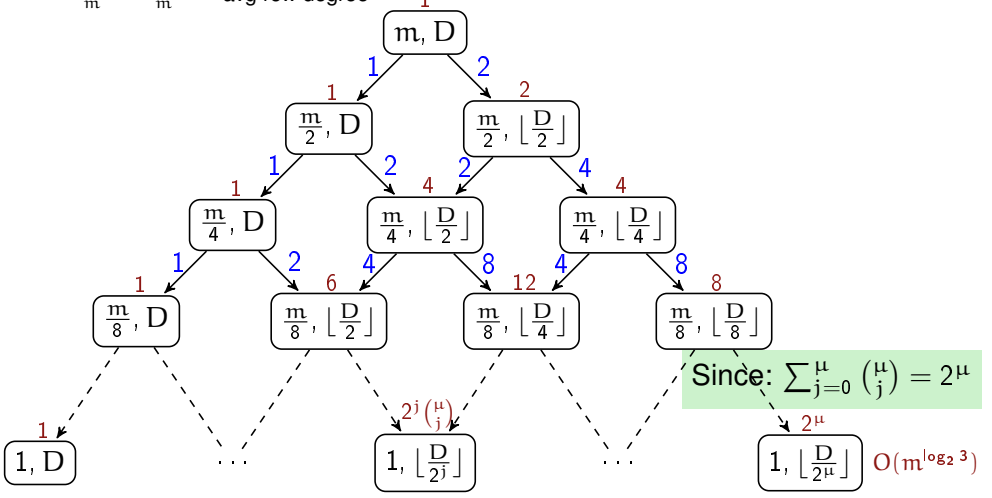

Since: $\sum_{j=0}^{\mu} \binom{\mu}{j} = 2^{\mu}$

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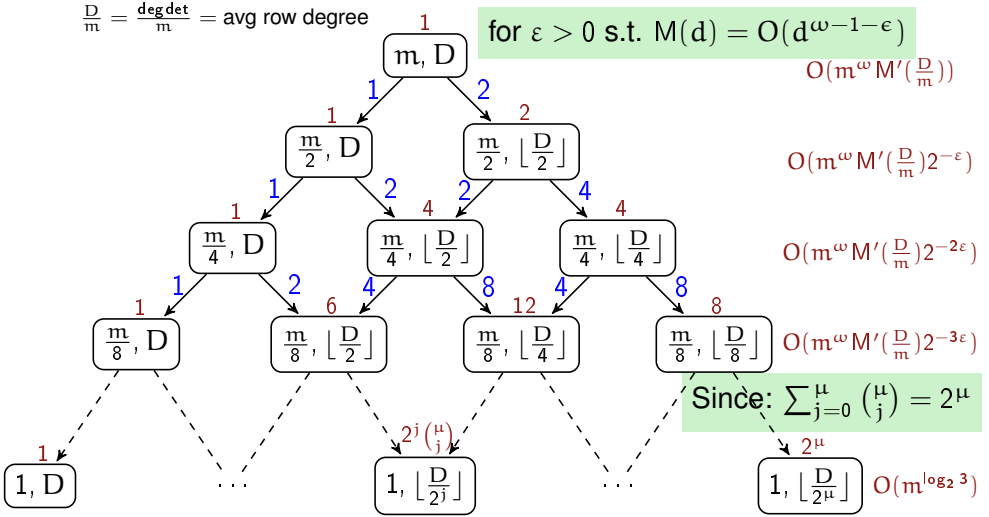

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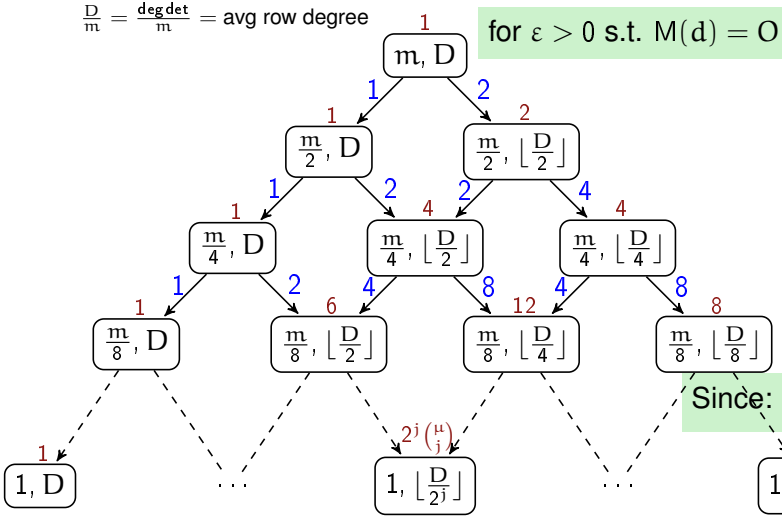
for $\varepsilon > 0$ s.t. $M(d) = O(d^{\omega-1-\epsilon})$



$O(m^{\omega} M'(\frac{D}{m}))$

$O(m^{\omega} M'(\frac{D}{m}) 2^{-\varepsilon})$

$O(m^{\omega} M'(\frac{D}{m}) 2^{-2\varepsilon})$

$O(m^{\omega} M'(\frac{D}{m}) 2^{-3\varepsilon})$

Since: $\sum_{j=0}^{\mu} \binom{\mu}{j} = 2^{\mu}$

$O(m^{\log_2 3})$

New polynomial matrix transformations

Weak Popov to Popov

Input:	$\mathbf{s} \in \mathbb{Z}^{\mathrm{m}}$, a shift,
	$\mathbf{A} \in \mathbb{K}[\mathrm{x}]^{\mathrm{m} \times \mathrm{m}}$, a matrix in $\mathbf{s}$-weak Popov form
Output:	the $\mathbf{s}$-Popov form of $\mathbf{A}$
Requirement:	$-\mathbf{s} \geqslant \mathsf{pivotDegree}(\mathbf{A})$
Complexity:	$\mathrm{O}(\mathrm{m}^{\omega}\mathrm{M}'(\frac{\mathrm{D}}{\mathrm{m}}))$, where $\mathrm{D} = \sum \mathbf{s}$

Improvement and generalization of [Sarkar-Storjohann 2011, Section 4]
$\rightsquigarrow$ support nonzero shifts and involve average degree $\frac{\mathrm{D}}{\mathrm{m}}$

- problem viewed as a change of shift with a priori known output degrees
- introduction of partial linearization techniques for kernel bases

New polynomial matrix transformations

Weak Popov to Popov

Input:	$\mathbf{s} \in \mathbb{Z}^m$, a shift,
	$\mathbf{A} \in \mathbb{K}[x]^{m \times m}$, a matrix in $\mathbf{s}$-weak Popov form
Output:	the $\mathbf{s}$-Popov form of $\mathbf{A}$
Requirement:	$-\mathbf{s} \geqslant \text{pivotDegree}(\mathbf{A})$
Complexity:	$O(m^\omega M'(\frac{D}{m}))$, where $D = \sum \mathbf{s}$

Improvement and generalization of [Sarkar-Storjohann 2011, Section 4]
$\rightsquigarrow$ support nonzero shifts and involve average degree $\frac{D}{m}$

- problem viewed as a change of shift with a priori known output degrees
- introduction of partial linearization techniques for kernel bases

Reduced to weak Popov

Input:	$\mathbf{s} \in \mathbb{Z}^n$, a shift
	$\mathbf{A} \in \mathbb{K}[x]^{m \times n}$, a matrix in $\mathbf{s}$-reduced form
Output:	an $\mathbf{s}$-weak Popov form of $\mathbf{A}$
Complexity:	$O(m^{\omega-1} n(\frac{D}{m} + 1))$, where $D = \sum \text{rdeg}_{\mathbf{s}}(\mathbf{A}) - m \min(\mathbf{s})$

Easy extension of [Sarkar-Storjohann 2011, Section 3] to shifted forms

Summary and perspectives

Summary

- $\mathsf{CharPoly} = \mathrm{O}(\mathsf{MatMul})$
- Determinant of reduced polynomial matrices in $\mathrm{O}(\mathrm{m}^{\omega}\mathrm{M}'(\frac{\mathrm{D}}{\mathrm{m}}))$
- Fast transformations between shifted forms of polynomial matrices

$\frac{\mathrm{D}}{\mathrm{m}} = \frac{\mathbf{degdet}}{\mathrm{m}} =$ average row degree

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Perspectives

- Implementation and practical efficiency (small fields, degenerate instances, ...)
- Approach without fast polynomial arithmetic
  $\rightarrow$ Exploit the quasiseparable struct. of linearized polynomial matrices
- Frobenius normal form & Smith normal form