Matrix Multiplication Based Computations of the Characteristic Polynomial

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Introduction

Dense Linear Algebra over a Field:

- one of the usual models for complexity in linear algebra
- applied to
  - $\mathbb{R}$: floating point linear algebra
  - $GF(q), Z_p$ and $\mathbb{Z}$ (using CRT)

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Applications in exact computation:

Cryptography :	number field sieves
Representation theory :	null space basis
Topology :	Smith normal forms
Graph theory :	characteristic polynomial

...

Approach

Problem

Compute the characteristic polynomial of a dense matrix over a field

- Deterministic or Las Vegas randomized algorithmns

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  $\Rightarrow$Balance between asymptotic complexity and practical efficiency considerations
- space complexity

Outline

1. Matrix multiplication based linear algebra
   - Matrix multiplication: a building block
   - Reductions to matrix multiplication
2. Computing the characteristic polynomial
   - State of the art
   - A new algorithm
   - Algorithm into practice

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Asymptotic complexity

Matrix multiplication:

Folklore:	$2n^3 - n^2$
Strassen 1969:	$7n^{2.807} + o(n^{2.807})$
Winograd 1971:	$6n^{2.807} + o(n^{2.807})$
...	
Coppersmith Winograd 1990:	$\mathcal{O}\left(n^{2.376}\right)$

$\Rightarrow \mathcal{O}\left(n^{\omega}\right)$, where ω denotes an admissible exponent

Efficiency in practice

The most efficient routine in linear algebra.
Several reasons:

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  - beware of unstability with floating point numbers
  - but improves efficiency over finite fields

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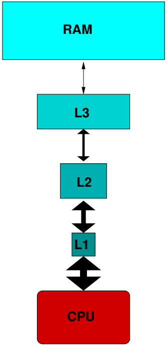
Memory management considerations

CPU-Memory communication: bandwidth gap
$\Rightarrow$Hierarchy of several cache memory levels



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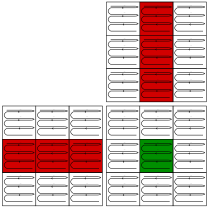
Imposes a structure for algorithms: operations must be blocked to increase data locality and fit in the cache



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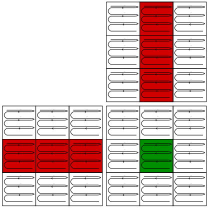

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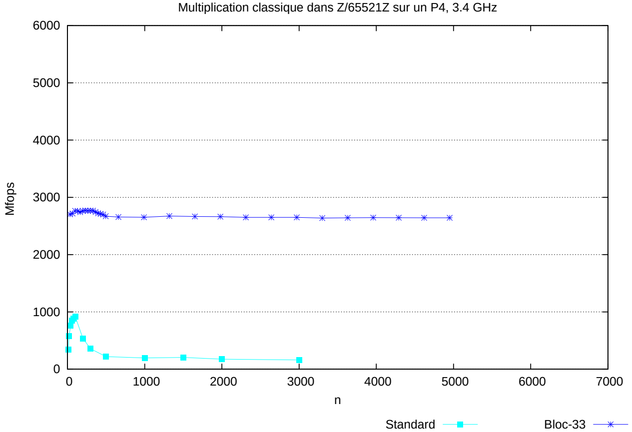
Reuse of the data

- `Work` $\gg$ `Data` to amortize memory transfer
  ⇒reach the peak performance of the CPU
- Matrix multiplication: $n^3 \gg n^2$
  ⇒well suited for block design

Practical implementation over finite fields

Matrix multiplication



Practical implementation over finite fields

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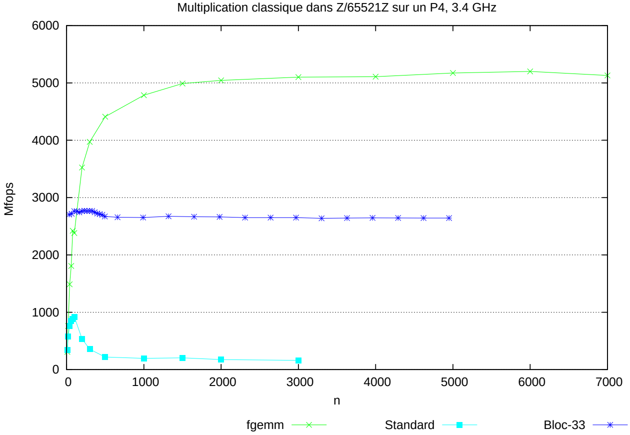

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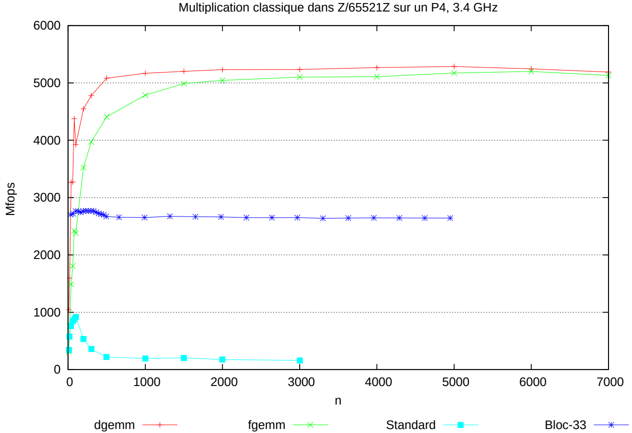

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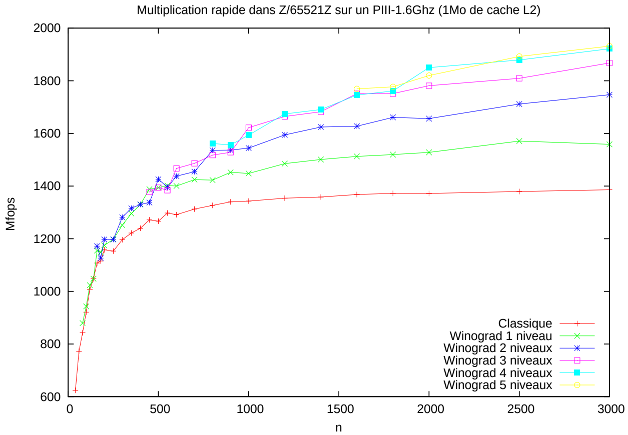

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Other linear algebra problems

Asymptotic complexity:

- Used to be in $\mathcal{O}\left(n^3\right)$
- Room for improvement: $\mathcal{O}\left(n^\omega\right)$ for everyone ?

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Practical efficiency: reuse the efficient matrix multiplication kernel

Reductions to matrix multiplication

Matrix Inversion [Strassen 69]

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} & \\ & I \end{bmatrix} \begin{bmatrix} I & -B \\ & I \end{bmatrix} \begin{bmatrix} I & \\ & (D - CA^{-1}B)^{-1} \end{bmatrix} \begin{bmatrix} I & \\ CA^{-1} & I \end{bmatrix}$$

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1: Compute $E = A^{-1}$ (Recursive call)
2: Compute $F = D - CEB$ (MM)
3: Compute $G = F^{-1}$ (Recursive call)
4: Compute $H = -EB$ (MM)
5: Compute $J = HG$ (MM)
6: Compute $K = CE$ (MM)
7: Compute $L = E + JK$ (MM)
8: Compute $M = GK$ (MM)
9: Return $\begin{bmatrix} E & J \\ M & G \end{bmatrix}$

Reductions to matrix multiplication

TRSM: Multiple triangular system solving

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2: Compute $G = D - BF$ (MM)
3: Compute $H = A^{-1}G$ (Recursive call)
4: Return $\begin{bmatrix} H \\ F \end{bmatrix}$

Reductions to matrix multiplication

LU decomposition

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} L_A & \\ CU_A^{-1} & L_E \end{bmatrix} \begin{bmatrix} U_A & L_A^{-1}B \\ & U_E \end{bmatrix}$$

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6: Return $\left(\begin{bmatrix} L_A & \\ F & L_E \end{bmatrix}, \begin{bmatrix} U_A & G \\ & U_E \end{bmatrix} \right)$

Reductions to matrix multiplication

Divide and conquer approach:

⇒ involves computations with dimensions $\frac{n}{2^i}$ for $i = 1 \dots \log_2 n$

⇒ overall time complexity by geometric progression

$$\sum_{i=1}^{\log_2 n} 2^i \left(\frac{n}{2^i}\right)^\omega = \mathcal{O}\left(n^\omega\right)$$

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- det, rank, rank profile,
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What about the characteristic polynomial ?

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Pre-Strassen age

Leverrier 1840: trace of powers of A, and Newton's formula

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But no trivial translation into a block algorithm with $\mathcal{O}\left(n^\omega\right)$ complexity.

Post-Strassen age

Preparata & Sarwate 1978: Update Csanky with fast matrix multiplication

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Keller-Gehrig 1985, alg.1: computes $(A^{2^i})_{i=1\dots\log_2 n}$ to form a Krylov basis.

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Keller-Gehrig 1985, alg.2: inspired by Danilevskii, block operations

- $\mathcal{O}\left(n^{\omega}\right)$
- Only valid with generic matrices

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Statement

Theorem

If A is a $n \times n$ matrix over a field having more than $2n^2$ elements, the characteristic polynomial of A can be computed in $\mathcal{O}(n^\omega)$ field operations by a Las Vegas randomized algorithm.

Definition (degree d Krylov matrix of one vector v)

$$K = \begin{bmatrix} v & Av & \dots & A^{d-1}v \end{bmatrix}$$

Property

$$A \times K = K \times \begin{bmatrix} 0 & & & * \\ 1 & & & * \\ & \ddots & & * \\ & & 1 & * \end{bmatrix}$$

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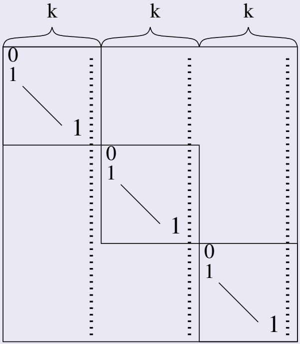
$\Rightarrow$ if $d = n$, $K^{-1}AK = C_{P^A_{car}}$
$\Rightarrow$ [Keller-Gehrig, alg. 2] computes K in $\mathcal{O}(n^\omega)$

Definition (degree k Krylov matrix of several vectors v_i)

$$K = \left[\begin{array}{ccc|ccc|c|ccc} v_1 & \dots & A^{k-1}v_1 & v_2 & \dots & A^{k-1}v_2 & \dots & v_l & \dots & A^{k-1}v_l \end{array} \right]$$

Property

$A \times K = K \times$



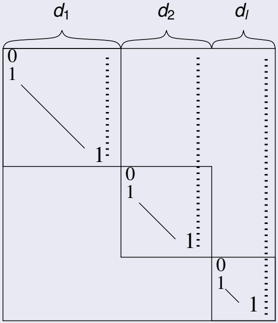
Fact

If $(d_1, \dots d_l)$ is lexicographically maximal such that

$$K = \left[\begin{array}{ccc|c|ccc} v_1 & \dots & A^{d_1-1}v_1 & \dots & v_l & \dots & A^{d_l-1}v_l \end{array} \right]$$

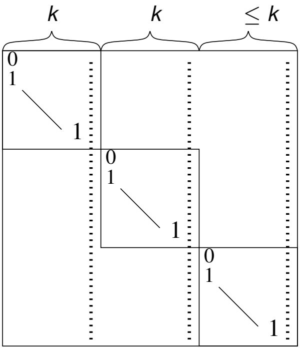
is non-singular, then

$K^{-1}AK =$



Principle

k-shifted form:



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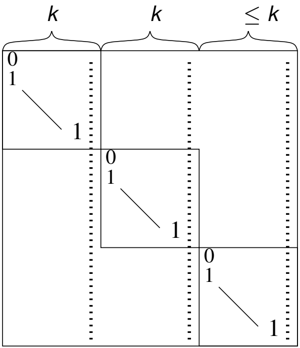
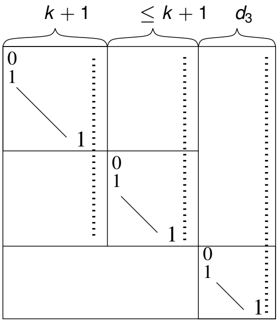


- try to inflate each slice by one ...

Principle

$k+1$-shifted form:



- try to inflate each slice by one ...
- ... to obtain the $k+1$-shifted form

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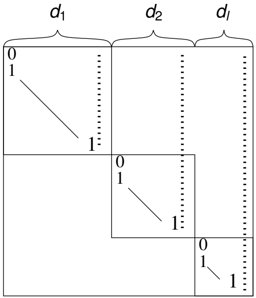
- until the shifted Hessenberg form is obtained:



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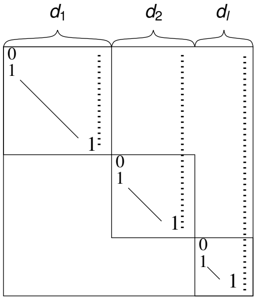
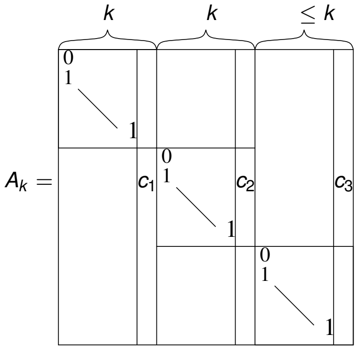


How to transform from k to $k+1$-shifted form ?

Krylov normal extension

for any k-shifted form



Krylov normal extension

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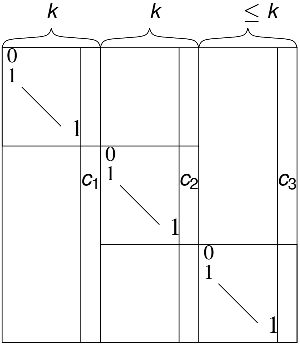
$A_k =$



compute the $n \times (n+k)$ matrix

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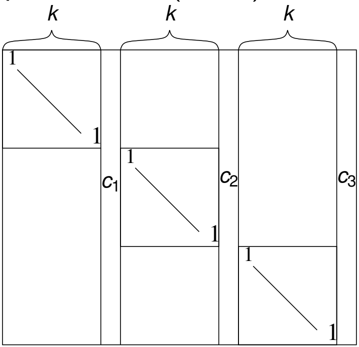

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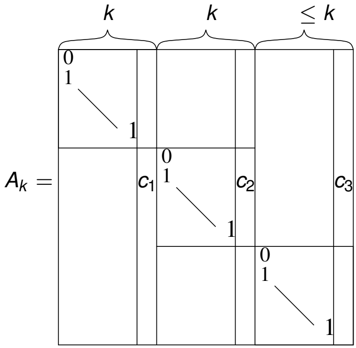

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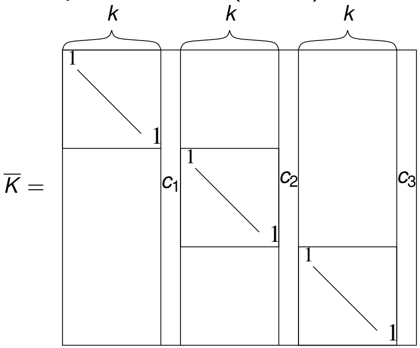


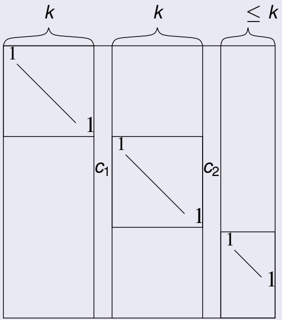
and form K by picking its first linearly independent columns.

Krylov normal extension

Lemma

If $\#F > 2n^2$, with high probability, the matrix K will have the form

$K =$



and $A_{k+1} = K^{-1} A_k K$ will be in $k+1$ shifted form

The algorithm

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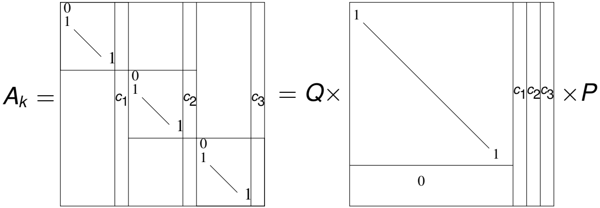
How to use matrix multiplication knowing the structure ?

Permutations: compressing the dense columns



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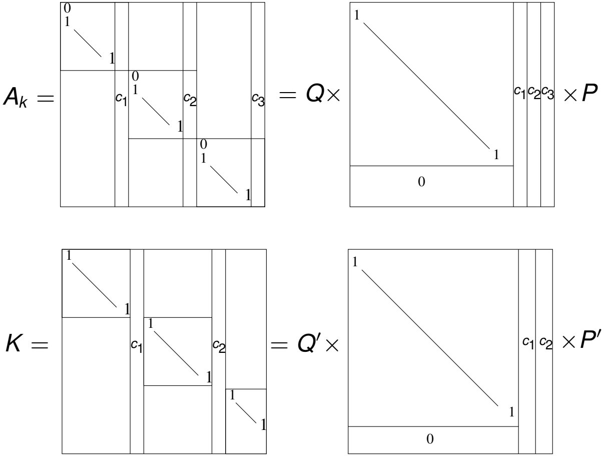

Reduction to Matrix multiplication

Rank profile: derived from LQUP

$\Rightarrow \mathcal{O}\left(k\left(\frac{n}{k}\right)^{\omega}\right)$

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Similarity transformation: parenthesing

$$K^{-1}AK = Q'^{T}\begin{bmatrix} I & * \\ 0 & * \end{bmatrix} P'^{T} Q \begin{bmatrix} I & * \\ 0 & * \end{bmatrix} PQ' \begin{bmatrix} I & * \\ 0 & * \end{bmatrix} P'$$

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Overall complexity: summing for each iteration:

$$\sum_{k=1}^{n} k\left(\frac{n}{k}\right)^{\omega} = n^{\omega}\sum_{k=1}^{n}\left(\frac{1}{k}\right)^{\omega-1} = \mathcal{O}\left(n^{\omega}\right)$$

Outline

Heuristic improvement

The randomization:

the iterate vectors at the first iteration must be random vectors.
or equivalently
the matrix has to be preconditioned: $M^{-1}AM$ for a random matrix M.

$\Rightarrow$as expensive as the rest of the algorithm

A block Krylov preconditoner

1: Pick n/c random vectors $U = \begin{bmatrix} u_1 & \dots & u_{n/c} \end{bmatrix}$.
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4: $\quad M^{-1}AM = H_c$ is in c-shifted form.

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if M is non singular **then**
    $M^{-1}AM = H_c$ is in c-shifted form.
else
    complete M into a non singular matrix $\overline{M}$ by adding some columns at the end
    then $\overline{M}^{-1}A\overline{M} = \begin{bmatrix} H_c & * \\ & R \end{bmatrix}$
end if

Efficiency balancing parameter

- c small: full square matrix multiplications, but more ops
- c large: tends to matrix-vector products, but less ops

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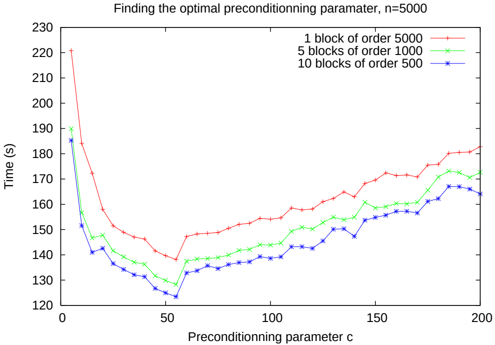

Experiments

n	`LU-Krylov`	`New algorithm`
200	0.024	0.032
300	0.06s	0.088s
500	0.248s	0.316s
750	1.084s	1.288s
1000	2.42s	2.296s
5000	267.6s	153.9s
10 000	1827s	991s
20 000	14 652s	7097s
30 000	48 887s	24 928s

Computation time for 1 Frobenius block matrices, Itanium2-64 1.3Ghz, 192Gb

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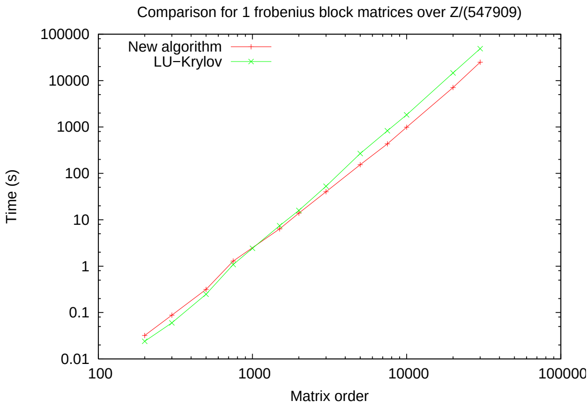

Timing comparison between the new algorithm and LU-Krylov, logarithmic scales, Itanium2-64 1.3Ghz, 192Gb

Comparison to Magma

n	magma-2.11	LU-Krylov	New algorithm
100	0.010s	0.005s	0.006s
300	0.830s	0.294s	0.105s
500	3.810s	1.316s	0.387s
800	15.64s	4.663s	1.387s
1000	29.96s	10.21s	2.755s
1500	102.1s	33.36s	7.696s
2000	238.0s	79.13s	17.91s
3000	802.0s	258.4s	61.09s
5000	3793s	1177s	273.4s
7500	MT	4209s	991.4s
10 000	MT	8847s	2080s

Computation time for 1 Frobenius block matrices, Athlon 2200, 1.8Ghz, 2Gb

MT: Memory thrashing

Conclusion and perspectives

Results:

- Las Vegas reduction to matrix multiplication,
- The Frobenius normal form is easily derivable in $\mathcal{O}(n^\omega)$...
- ...but no transformation matrix
- Adaptive combination with block Krylov in practice.

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Results:

- Las Vegas reduction to matrix multiplication,
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Still to be done:

- Condition on the size of the field is a limitation. Eberly's algorithm ?
- Ideally: derandomization ? (deterministic)