Smaller and faster generators for quasi-separable matrices

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Introduction

Two main types of structured matrices

Rank displacement: Hankel, Toeplitz, Vandermonde, Cauchy, etc

- Arising from polynomial operations (evaluation, interpolation, etc)
- Rank displacement: $\text{rank}(A - PAQ)$ for $P, Q \in \{\text{Diag}, \text{Cyclic shift}\}$
- Structure: $\text{rank}(A - PAQ)$ small $\rightsquigarrow A - PAQ = LR$

Rank structure: Tridiagonal, Hessenberg, semiseparable, quasiseparable

- Structure: small rank in the upper/lower triangular part
- No closed form expression

Quasiseparable matrices

$$\begin{bmatrix} 4 & 2 & 3 & & & & & & & \\ 4 & 0 & 1 & 4 & & & & & & \\ 4 & 3 & 5 & 6 & 3 & & & & & \\ & 5 & 1 & 1 & 4 & 5 & & & & \\ & & 4 & 3 & 1 & 1 & 4 & & & \\ & & & 4 & 0 & 5 & 1 & 2 & & \\ & & & & 4 & 3 & 1 & 2 & 6 & \\ & & & & & 2 & 1 & 3 & 1 & 1 \\ & & & & & & 1 & 5 & 4 & 3 \\ & & & & & & & 3 & 0 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} 2 & 6 & 1 & 0 & 1 & 4 & 6 & 5 & 2 & 6 \\ 6 & 6 & 2 & 5 & 6 & 1 & 6 & 1 & 3 & 1 \\ 3 & 2 & 2 & 6 & 4 & 1 & 2 & 2 & 0 & 3 \\ 6 & 6 & 2 & 2 & 5 & 1 & 4 & 5 & 5 & 2 \\ 0 & 1 & 6 & 2 & 4 & 4 & 5 & 0 & 3 & 3 \\ 2 & 0 & 5 & 6 & 3 & 1 & 0 & 6 & 2 & 4 \\ 6 & 2 & 6 & 1 & 3 & 4 & 6 & 5 & 2 & 6 \\ 1 & 1 & 5 & 5 & 2 & 1 & 3 & 4 & 5 & 4 \\ 0 & 1 & 6 & 2 & 4 & 4 & 3 & 3 & 2 & 3 \\ 1 & 1 & 5 & 5 & 2 & 1 & 3 & 4 & 5 & 6 \end{bmatrix}$$

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rank $= 1$ rank $= 1$

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Definition

M $\in \mathrm{K}^{n\times n}$ is (r_L, r_U)-quasiseparable if $\begin{cases} \text{rank}(\mathrm{M}_{k+1..n,1..k}) & \leq & r_L \\ \text{rank}(\mathrm{M}_{1..k,k+1..n}) & \leq & r_U \end{cases}$

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Properties

- (s,t)-*quasiseparable* $\times$ (u,v)-*quasiseparable* $=$ $(s+u, t+v)$-*quasiseparable*
- $((s,t)$-*quasiseparable*$)^{-1}$ $=$ (s,t)-*quasiseparable*

Applications

Numerical linear algebra: naturally occur in solving

- generalized eigenvalue problems,
- Fast Multipole Method (N-body simulation)

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- normal forms of polynomial matrices over $\mathbb{Z}/p\mathbb{Z}$
- `LinBox`: black-box exact linear algebra library,

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Complexity

Displacement rank structure: space $O(sn)$

- [Pan90] $O\tilde{}(s^2 n)$ time [8]
- [Bostan & Al. 07,08] $O\tilde{}(s^{\omega-1} n)$ time [1]

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Quasiseparable structure

- [Eidelman Gohberg 99] $O(s^2n)$ space, $O(s^3n)$ time [6]
- [Chandrasekaran & Al. 07] $O(sn)$ space, $O(s^2n)$ time [2]
- Seeking to reduce the exponent in s

Structured representation of a quasiseparable matrix

Quasiseparable Generator (QG): generalized eigenvalue problems [6]

$$\mathsf{M}_{i,j} = \begin{cases} p(i)^T \mathsf{a}(i-1)\dots\mathsf{a}(j+1)q(j), & 1 \leq j < i \leq n \\ d(i), & 1 \leq i = j \leq n \\ g(i)^T \mathsf{b}(i+1)\dots\mathsf{b}(i-1)h(j), & 1 \leq i < j \leq n \end{cases}$$

where $p(i), q(i), g(i), h(i) \in \mathrm{K}^s$, $\mathsf{a}(i), \mathsf{b}(i) \in \mathrm{K}^{s\times s}$ for $1 \leq i \leq n-1$

Structured representation of a quasiseparable matrix

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Size	$O(s^2 n)$
Construction	$O(s^2 n^2)$

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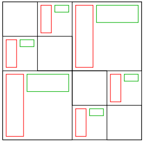
Structured representation of a quasiseparable matrix

Hierarchically SemiSeparable (HSS) representation: Fast Multipole Method



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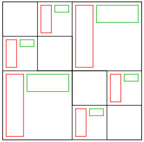

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QS $\times$ QS	$O(s^3n)$	$O(s^3n)$	$O(s^2n)$?
Inv/LinSys	$O(s^3n)$	$O(s^3n)$	$O(s^2n)$

Contribution

Connecting notions of **quasiseparability** and **rank profile matrix** [3]

Contribution

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1. Compute the quasiseparable orders in $O(s^{\omega-2}n^2)$

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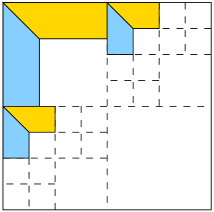
Connecting notions of **quasiseparability** and **rank profile matrix** [3]

1. Compute the quasiseparable orders in $O(s^{\omega-2}n^2)$
2. Two alternative structured representations
   1. Recursive Rank Revealing: RRR
   2. Compact Bruhat generator: CB

Structured representation of a quasiseparable matrix

Recursive Rank revealing (RRR) = simplified HSS



	QG	HSS [9]	HSS [2]	RRR
Size	$O(s^2n)$	$O(sn)$	$O(sn)$	
Construction	$O(s^2n^2)$	$O(sn^2)$	$O(sn^2)$	
QS $\times$ Vec	$O(s^2n)$	$O(s^2n)$	$O(sn)$	
QS $\times$ s Vec	$O(s^3n)$	$O(s^3n)$	$O(s^2n)$	
QS $\times$ QS	$O(s^3n)$	$O(s^3n)$	$O(s^2n)$?	
Inv/LinSys	$O(s^3n)$	$O(s^3n)$	$O(s^2n)$	

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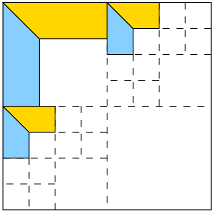

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QS × Vec	$O(s^2 n)$	$O(s^2 n)$	$O(sn)$	$O(sn \log \frac{n}{s})$
QS × s Vec	$O(s^3 n)$	$O(s^3 n)$	$O(s^2 n)$	
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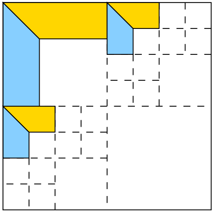

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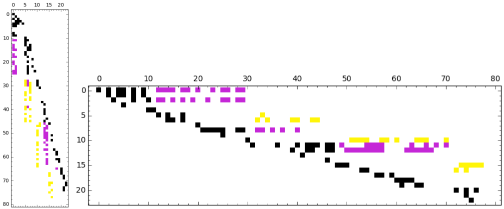
QS $\times$ s Vec	$O(s^3 n)$	$O(s^3 n)$	$O(s^2 n)$	$O(s^{\omega-1} n \log \frac{n}{s})$
QS $\times$ QS	$O(s^3 n)$	$O(s^3 n)$	$O(s^2 n)$?	$O(s^{\omega-1} n \log^2 \frac{n}{s})$
Inv/LinSys	$O(s^3 n)$	$O(s^3 n)$	$O(s^2 n)$	$O(s^{\omega-1} n \log^2 \frac{n}{s})$

Structured representation of a quasiseparable matrix

Compact Bruhat generator



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QS $\times$ s Vec	$O(s^3n)$	$O(s^3n)$	$O(s^2n)$	$O(s^{\omega-1}n\log\frac{n}{s})$	
QS $\times$ QS	$O(s^3n)$	$O(s^3n)$	$O(s^2n)$?	$O(s^{\omega-1}n\log^2\frac{n}{s})$	
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Structured representation of a quasiseparable matrix

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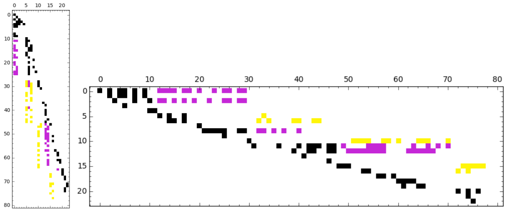

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Outline

1. The rank profile matrix
2. Computing the orders of quasiseparability
3. New generators
   - Recursive Rank Revealing
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Rank profiles

Definition (Row Rank Profile: RowRP)

Given $A \in \mathsf{K}^{m\times n}, r = \mathsf{rank}(A)$.

informally: *first* r linearly independent rows

formally: lexico-minimal list of r indices of linearly independant rows.

Example

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

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Rank = 3
RowRP = {1,2,4}

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ColRP = {1,2,3} $\rightarrow$ Generic ColRP.

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RowRP = {1,2,4}
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Generic Rank Profile: first r leading principal minors $\neq 0$

Generic rank profile $\not\Longleftarrow\!\Longrightarrow$ Generic Row RP and Genric ColRP

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

RowRP = ColRP = {1,2}

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Rank = 3
RowRP = {1,2,4}
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Generic Rank Profile: first r leading principal minors ≠ 0

Generic rank profile $\not\Leftarrow\Rightarrow$ Generic Row RP and Genric ColRP

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

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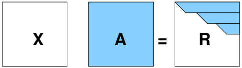
But $|A_{1,1}| = 0$

Relation to echelon forms

Transformation to echelon form

$\forall A\ \exists X$ non-singular s.t.



Relation to echelon forms:

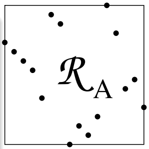
- ColRP unchanged by left multiplication with an invertible matrix

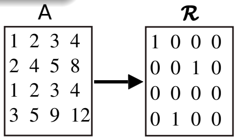
ColRP = pivot columns in the **row** echelon form

The rank profile matrix [Dumas, P. and Sultan'15]

Definition (Rank Profile matrix)

The unique $\mathcal{R}_A \in \{0,1\}^{m\times n}$ such that any pair of (i,j)-leading sub-matrix of $\mathcal{R}_A$ and of A have the same rank.





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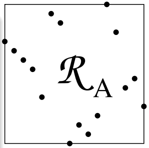

Theorem

- *RowRP and ColRP read directly on $\mathcal{R}_A$*
- *Same holds for any (i,j)-leading submatrix.*

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 5 & 8 \\ 1 & 2 & 3 & 4 \\ 3 & 5 & 9 & 12 \end{bmatrix} \longrightarrow \mathcal{R} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

RowRP $= \{1\}$
ColRP $= \{1\}$

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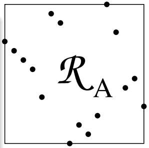

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A

1	2	3	4
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3	5	9	12

→

$\mathcal{R}$

1	0	0	0
0	0	1	0
0	0	0	0
0	1	0	0

RowRP = {1,2}
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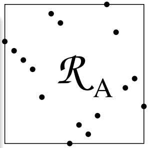

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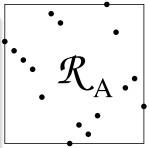

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RowRP = {1,4}
ColRP = {1,2}

$$A = PLUQ = P \begin{bmatrix} L & 0 \\ M & I_{m-r} \end{bmatrix} \begin{bmatrix} I_r & \\ & 0 \end{bmatrix} \begin{bmatrix} U & V \\ & I_{n-r} \end{bmatrix} Q$$

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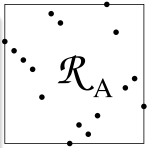

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RowRP = {1,4}
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$$A = PLUQ = P\begin{bmatrix} L & 0 \\ M & I_{m-r} \end{bmatrix} P^T P \begin{bmatrix} I_r & \\ & 0 \end{bmatrix} QQ^T \begin{bmatrix} U & V \\ & I_{n-r} \end{bmatrix} Q$$

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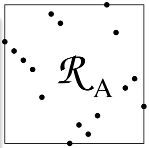

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→ $\mathcal{R}$

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$$A = PLUQ = \underbrace{P\begin{bmatrix} L & 0 \\ M & I_{m-r}\end{bmatrix}P^T}_{\overline{L}} \underbrace{P\begin{bmatrix} I_r & \\ & 0\end{bmatrix}Q}_{\mathcal{R}^A} \underbrace{Q^T\begin{bmatrix} U & V \\ & I_{n-r}\end{bmatrix}Q}_{\overline{U}}$$

Pivoting strategies revealing rank profiles

For any type of PLUQ algorithm: iterative / block iterative / recursive

Search	**Row perm.**	**Col. perm.**	RowRP	ColRP	$\mathcal{R}^A$	**Instance**
Row order Col. order						
Lexico.						
Rev. lex.						
Product						

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Search	**Row perm.**	**Col. perm.**	RowRP	ColRP	$\mathcal{R}^A$	**Instance**
Row order Col. order	Transposition	Transposition	✓			[IMH82] [JPS13]
Lexico.						
Rev. lex.						
Product						

- RowRP $= \begin{bmatrix} 1 & 2 & \dots & m \end{bmatrix} P \begin{bmatrix} I_r \\ 0 \end{bmatrix}$

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Row order	Transposition	Transposition	✓			[IMH82] [JPS13]
Col. order	Transposition	Transposition		✓		[KG85] [JPS13]
Lexico.						
Rev. lex.						
Product						

- RowRP $= [1\ 2\ \dots\ m]\, P \begin{bmatrix} I_r \\ 0 \end{bmatrix}$
- ColRP $= [I_r\ 0]\, Q\, [1\ 2\ \dots\ m]^T$

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Row order	Transposition	Transposition	✓			[IMH82] [JPS13]
Col. order	Transposition	Transposition		✓		[KG85] [JPS13]
Lexico.	Transposition	Transposition	✓			[Sto00]
Rev. lex.	Transposition	Transposition		✓		[Sto00]
Product						

- RowRP $= \begin{bmatrix} 1 & 2 & \dots & m \end{bmatrix} P \begin{bmatrix} I_r \\ 0 \end{bmatrix}$
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Row order	Transposition	Transposition	✓			[IMH82] [JPS13]
Col. order	Transposition	Transposition		✓		[KG85] [JPS13]
Lexico.	Transposition	Transposition	✓			[Sto00]
Rev. lex.	Transposition	Transposition		✓		[Sto00]
Product	Rotation	Rotation	✓	✓	✓	[DPS13]

- RowRP $= [\,1\ 2\ \dots\ m\,]\, P \begin{bmatrix} I_r \\ 0 \end{bmatrix}$
- ColRP $= [\,I_r\ 0\,]\, Q\, [\,1\ 2\ \dots\ m\,]^T$
- $\mathcal{R}^A = P \begin{bmatrix} I_r & \\ & 0 \end{bmatrix} Q$

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Search	Row perm.	Col. perm.	RowRP	ColRP	$\mathcal{R}^A$	Instance
Row order	Transposition	Transposition	✓			[IMH82] [JPS13]
Col. order	Transposition	Transposition		✓		[KG85] [JPS13]
Lexico.	Transposition	Transposition	✓			[Sto00]
Rev. lex.	Transposition	Transposition		✓		[Sto00]
Product	Rotation	Transposition	✓			[DPS15]
Product	Transposition	Rotation		✓		[DPS15]
Product	Rotation	Rotation	✓	✓	✓	[DPS13]

- RowRP $= \begin{bmatrix} 1 & 2 & \dots & m \end{bmatrix} P \begin{bmatrix} I_r \\ 0 \end{bmatrix}$
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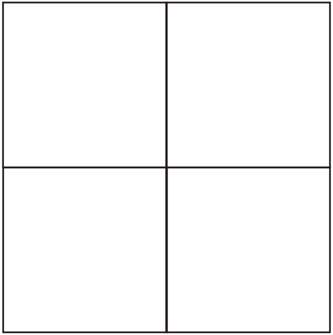
Lexico.	Transposition	Transposition	✓			[Sto00]
Lexico.	Transposition	Rotation	✓	✓	✓	[DPS15]
Lexico.	Rotation	Rotation	✓	✓	✓	[DPS15]
Rev. lex.	Transposition	Transposition		✓		[Sto00]
Rev. lex.	Rotation	Transposition	✓	✓	✓	[DPS15]
Rev. lex.	Rotation	Rotation	✓	✓	✓	[DPS15]
Product	Rotation	Transposition	✓			[DPS15]
Product	Transposition	Rotation		✓		[DPS15]
Product	Rotation	Rotation	✓	✓	✓	[DPS13]

- RowRP $= \begin{bmatrix} 1 & 2 & \dots & m \end{bmatrix} P \begin{bmatrix} I_r \\ 0 \end{bmatrix}$
- ColRP $= \begin{bmatrix} I_r & 0 \end{bmatrix} Q \begin{bmatrix} 1 & 2 & \dots & m \end{bmatrix}^T$
- $\mathcal{R}^A = P \begin{bmatrix} I_r & \\ & 0 \end{bmatrix} Q$

The tile recursive algorithm

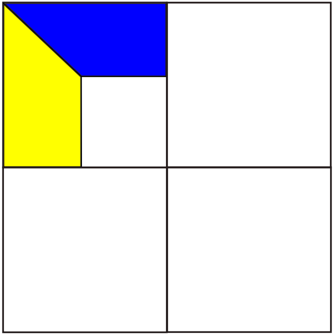
[Dumas P. Sultan 13][5]



2×2 block splitting

The tile recursive algorithm

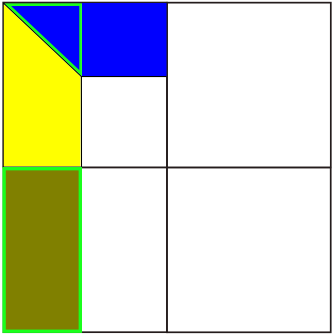
[Dumas P. Sultan 13][5]



Recursive call

The tile recursive algorithm

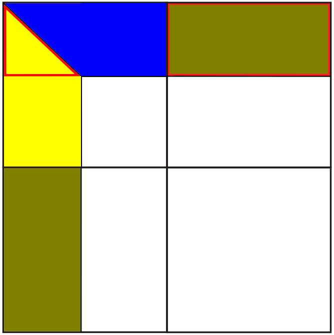
[Dumas P. Sultan 13][5]



TRSM: $B \leftarrow BU^{-1}$

The tile recursive algorithm

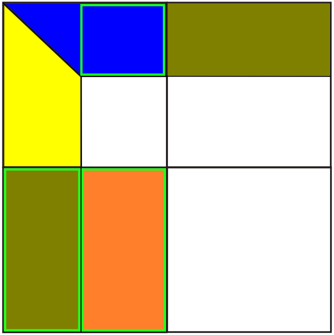
[Dumas P. Sultan 13][5]



TRSM: $B \leftarrow L^{-1}B$

The tile recursive algorithm

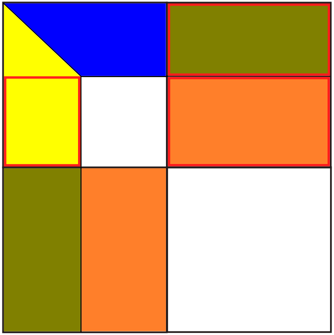
[Dumas P. Sultan 13][5]



`MatMul`: $C \leftarrow C - A \times B$

The tile recursive algorithm

[Dumas P. Sultan 13][5]



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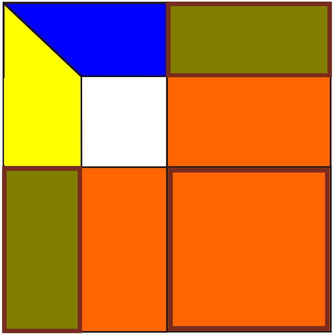
The tile recursive algorithm

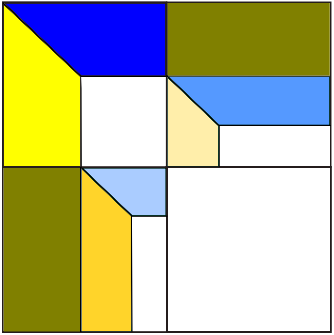
[Dumas P. Sultan 13][5]



MatMul: $C \leftarrow C - A \times B$

The tile recursive algorithm

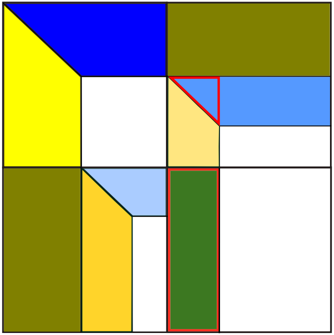
[Dumas P. Sultan 13][5]



2 independent recursive calls (compatible with the **product order**)

The tile recursive algorithm

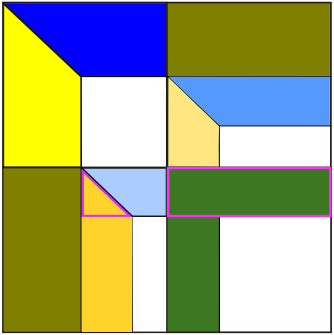
[Dumas P. Sultan 13][5]



TRSM: $B \leftarrow BU^{-1}$

The tile recursive algorithm

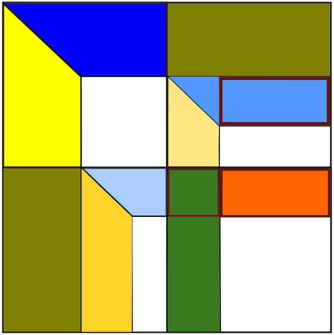
[Dumas P. Sultan 13][5]



TRSM: $B \leftarrow L^{-1}B$

The tile recursive algorithm

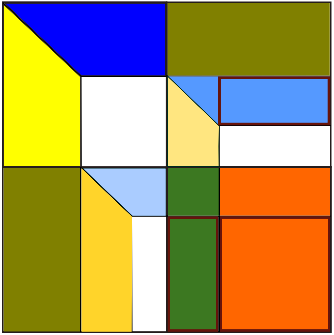
[Dumas P. Sultan 13][5]



MatMul: $C \leftarrow C - A \times B$

The tile recursive algorithm

[Dumas P. Sultan 13][5]



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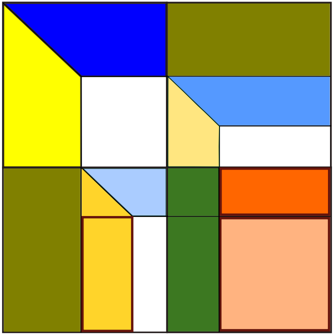
The tile recursive algorithm

[Dumas P. Sultan 13][5]



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The tile recursive algorithm

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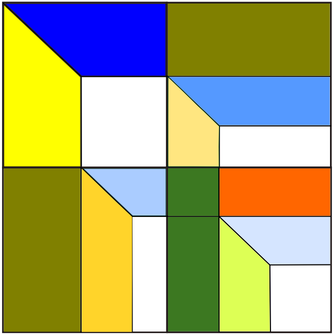
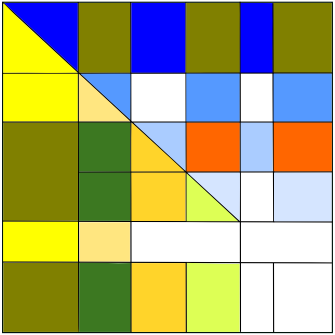


Recursive call

The tile recursive algorithm

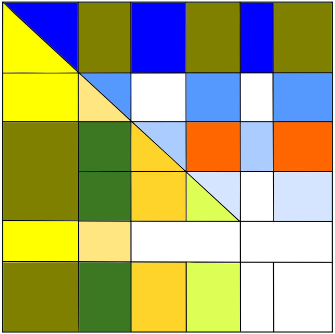
[Dumas P. Sultan 13][5]



Puzzle game (block **rotations**)

The tile recursive algorithm

[Dumas P. Sultan 13][5]



- $O(mnr^{\omega-2})$ ($2/3n^3$ for $\omega = 3$)
- fewer modular reductions than slab algorithms
- rank deficiency introduces parallelism

Outline

1. The rank profile matrix
2. Computing the orders of quasiseparability
3. New generators
   - Recursive Rank Revealing
   - The Bruhat generator
   - The Compact Bruhat generator

Finding the quasiseparability orders

Left triangular matrices

Of the form: $\begin{bmatrix} * & * & * & * & 0 \\ * & * & * & 0 & \\ * & * & 0 & & \\ * & 0 & & & \\ 0 & & & & \end{bmatrix}$

Examples: $J_n \times$ Lower, Upper $\times J_n$

$$\begin{bmatrix} 2 & 6 & 1 & 0 & 1 & 4 & 6 & 5 & 2 & 6 \\ 6 & 6 & 2 & 5 & 6 & 1 & 6 & 1 & 3 & 1 \\ 3 & 2 & 2 & 6 & 4 & 1 & 2 & 2 & 0 & 3 \\ 6 & 6 & 2 & 2 & 5 & 1 & 4 & 5 & 5 & 2 \\ 0 & 1 & 6 & 2 & 4 & 4 & 5 & 0 & 3 & 3 \\ 2 & 0 & 5 & 6 & 3 & 1 & 0 & 6 & 2 & 4 \\ 6 & 2 & 6 & 1 & 3 & 4 & 6 & 5 & 2 & 6 \\ 1 & 1 & 5 & 5 & 2 & 1 & 3 & 4 & 5 & 4 \\ 0 & 1 & 6 & 2 & 4 & 4 & 3 & 3 & 2 & 3 \\ 1 & 1 & 5 & 5 & 2 & 1 & 3 & 4 & 5 & 6 \end{bmatrix}$$

Finding the quasiseparability orders

Left triangular matrices

Of the form: $\begin{bmatrix} * & * & * & * & 0 \\ * & * & * & 0 \\ * & * & 0 \\ * & 0 \\ 0 \end{bmatrix}$

Examples: $J_n \times$ Lower, Upper $\times J_n$

$$\begin{bmatrix} 0 \\ 6 & 0 \\ 3 & 2 & 0 \\ 6 & 6 & 2 & 0 \\ 0 & 1 & 6 & 2 & 0 \\ 2 & 0 & 5 & 6 & 3 & 0 \\ 6 & 2 & 6 & 1 & 3 & 4 & 0 \\ 1 & 1 & 5 & 5 & 2 & 1 & 3 & 0 \\ 0 & 1 & 6 & 2 & 4 & 4 & 3 & 3 & 0 \\ 1 & 1 & 5 & 5 & 2 & 1 & 3 & 4 & 5 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 6 & 1 & 0 & 1 & 4 & 6 & 5 & 2 & 6 \\ & 0 & 2 & 5 & 6 & 1 & 6 & 1 & 3 & 1 \\ & & 0 & 6 & 4 & 1 & 2 & 2 & 0 & 3 \\ & & & 0 & 5 & 1 & 4 & 5 & 5 & 2 \\ & & & & 0 & 4 & 5 & 0 & 3 & 3 \\ & & & & & 0 & 0 & 6 & 2 & 4 \\ & & & & & & 0 & 5 & 2 & 6 \\ & & & & & & & 0 & 5 & 4 \\ & & & & & & & & 0 & 3 \\ & & & & & & & & & 0 \end{bmatrix}$$

Finding the quasiseparability orders

Left triangular matrices

Of the form: $\begin{bmatrix} * & * & * & * & 0 \\ * & * & * & 0 & \\ * & * & 0 & & \\ * & 0 & & & \\ 0 & & & & \end{bmatrix}$

Examples: $J_n \times$ Lower, Upper $\times J_n$

$$\begin{bmatrix} 1 & 1 & 5 & 5 & 2 & 1 & 3 & 4 & 5 & 0 \\ 0 & 1 & 6 & 2 & 4 & 4 & 3 & 3 & 0 & \\ 1 & 1 & 5 & 5 & 2 & 1 & 3 & 0 & & \\ 6 & 2 & 6 & 1 & 3 & 4 & 0 & & & \\ 2 & 0 & 5 & 6 & 3 & 0 & & & & \\ 0 & 1 & 6 & 2 & 0 & & & & & \\ 6 & 6 & 2 & 0 & & & & & & \\ 3 & 2 & 0 & & & & & & & \\ 6 & 0 & & & & & & & & \\ 0 & & & & & & & & & \end{bmatrix}, \begin{bmatrix} 6 & 2 & 5 & 6 & 4 & 1 & 0 & 1 & 6 & 0 \\ 1 & 3 & 1 & 6 & 1 & 6 & 5 & 2 & 0 & \\ 3 & 0 & 2 & 2 & 1 & 4 & 6 & 0 & & \\ 2 & 5 & 5 & 4 & 1 & 5 & 0 & & & \\ 3 & 3 & 0 & 5 & 4 & 0 & & & & \\ 4 & 2 & 6 & 0 & 0 & & & & & \\ 6 & 2 & 5 & 0 & & & & & & \\ 4 & 5 & 0 & & & & & & & \\ 3 & 0 & & & & & & & & \\ 0 & & & & & & & & & \end{bmatrix}$$

Finding the quasiseparability orders

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Of the form: $\begin{bmatrix} * & * & * & * & 0 \\ * & * & * & 0 & \\ * & * & 0 & & \\ * & 0 & & & \\ 0 & & & & \end{bmatrix}$

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Finding the quasiseparability orders

Left triangular matrices

Of the form: $\begin{bmatrix} * & * & * & * & 0 \\ * & * & * & 0 & \\ * & * & 0 & & \\ * & 0 & & & \\ 0 & & & & \end{bmatrix}$

Examples: $J_n\times$ Lower, Upper $\times J_n$

$$\begin{bmatrix} 1 & 1 & 5 & 5 & 2 & 1 & 3 & 4 & 5 & 0 \\ 0 & 1 & 6 & 2 & 4 & 4 & 3 & 3 & 0 & \\ 1 & 1 & 5 & 5 & 2 & 1 & 3 & 0 & & \\ 6 & 2 & 6 & 1 & 3 & 4 & 0 & & & \\ 2 & 0 & 5 & 6 & 3 & 0 & & & & \\ 0 & 1 & 6 & 2 & 0 & & & & & \\ 6 & 6 & 2 & 0 & & & & & & \\ 3 & 2 & 0 & & & & & & & \\ 6 & 0 & & & & & & & & \\ 0 & & & & & & & & & \end{bmatrix}, \begin{bmatrix} 6 & 2 & 5 & 6 & 4 & 1 & 0 & 1 & 6 & 0 \\ 1 & 3 & 1 & 6 & 1 & 6 & 5 & 2 & 0 & \\ 3 & 0 & 2 & 2 & 1 & 4 & 6 & 0 & & \\ 2 & 5 & 5 & 4 & 1 & 5 & 0 & & & \\ 3 & 3 & 0 & 5 & 4 & 0 & & & & \\ 4 & 2 & 6 & 0 & 0 & & & & & \\ 6 & 2 & 5 & 0 & & & & & & \\ 4 & 5 & 0 & & & & & & & \\ 3 & 0 & & & & & & & & \\ 0 & & & & & & & & & \end{bmatrix}$$

Finding the quasiseparability orders

Left triangular matrices

Of the form: $\begin{bmatrix} * & * & * & * & 0 \\ * & * & * & 0 & \\ * & * & 0 & & \\ * & 0 & & & \\ 0 & & & & \end{bmatrix}$

Examples: $J_n \times$ Lower, Upper $\times J_n$

$$\begin{bmatrix} 1 & 1 & 5 & 5 & 2 & 1 & 3 & 4 & 5 & 0 \\ 0 & 1 & 6 & 2 & 4 & 4 & 3 & 3 & 0 & \\ 1 & 1 & 5 & 5 & 2 & 1 & 3 & 0 & & \\ 6 & 2 & 6 & 1 & 3 & 4 & 0 & & & \\ 2 & 0 & 5 & 6 & 3 & 0 & & & & \\ 0 & 1 & 6 & 2 & 0 & & & & & \\ 6 & 6 & 2 & 0 & & & & & & \\ 3 & 2 & 0 & & & & & & & \\ 6 & 0 & & & & & & & & \\ 0 & & & & & & & & & \end{bmatrix}, \begin{bmatrix} 6 & 2 & 5 & 6 & 4 & 1 & 0 & 1 & 6 & 0 \\ 1 & 3 & 1 & 6 & 1 & 6 & 5 & 2 & 0 & \\ 3 & 0 & 2 & 2 & 1 & 4 & 6 & 0 & & \\ 2 & 5 & 5 & 4 & 1 & 5 & 0 & & & \\ 3 & 3 & 0 & 5 & 4 & 0 & & & & \\ 4 & 2 & 6 & 0 & 0 & & & & & \\ 6 & 2 & 5 & 0 & & & & & & \\ 4 & 5 & 0 & & & & & & & \\ 3 & 0 & & & & & & & & \\ 0 & & & & & & & & & \end{bmatrix}$$

Quasiseparability orders on the rank profile matrix

$$\begin{bmatrix} 2 & 2 & 1 & 0 & 2 & 1 & 1 & 1 & 0 & 0 \\ 2 & 2 & 1 & 0 & 2 & 0 & 1 & 2 & 0 & 0 \\ 2 & 2 & 1 & 0 & 2 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 1 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & & & \\ 0 & 0 & 0 & 0 & 1 & & & & & \\ 0 & 1 & 0 & 0 & & & & & & \\ 0 & 0 & 0 & & & & & & & \\ 0 & 0 & 1 & & & & & & & \\ 0 & & & & & & & & & \end{bmatrix}$$

Rank $= 1$ Quasiseparability order $= 1$

Quasiseparability orders on the rank profile matrix

$$\begin{bmatrix} 2 & 2 & 1 & 0 & 2 & 1 & 1 & 1 & 0 & 0 \\ 2 & 2 & 1 & 0 & 2 & 0 & 1 & 2 & 0 & 0 \\ 2 & 2 & 1 & 0 & 2 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 1 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & & & \\ 0 & 0 & 0 & 0 & 1 & & & & & \\ 0 & 1 & 0 & 0 & & & & & & \\ 0 & 0 & 0 & & & & & & & \\ 0 & 0 & 1 & & & & & & & \\ 0 & & & & & & & & & \end{bmatrix}$$

Rank $= 2$ Quasiseparability order $= 2$

Quasiseparability orders on the rank profile matrix

$$\begin{bmatrix} 2 & 2 & 1 & 0 & 2 & 1 & 1 & 1 & 0 & 0 \\ 2 & 2 & 1 & 0 & 2 & 0 & 1 & 2 & 0 & 0 \\ 2 & 2 & 1 & 0 & 2 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 1 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & & & \\ 0 & 0 & 0 & 0 & 1 & & & & & \\ 0 & 1 & 0 & 0 & & & & & & \\ 0 & 0 & 0 & & & & & & & \\ 0 & 0 & 1 & & & & & & & \\ 0 & & & & & & & & & \end{bmatrix}$$

Rank $= 2$ Quasiseparability order $= 2$

Quasiseparability orders on the rank profile matrix

$$\begin{bmatrix} 2 & 2 & 1 & 0 & 2 & 1 & 1 & 1 & 0 & 0 \\ 2 & 2 & 1 & 0 & 2 & 0 & 1 & 2 & 0 & 0 \\ 2 & 2 & 1 & 0 & 2 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 1 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & & & \\ 0 & 0 & 0 & 0 & 1 & & & & & \\ 0 & 1 & 0 & 0 & & & & & & \\ 0 & 0 & 0 & & & & & & & \\ 0 & 0 & 1 & & & & & & & \\ 0 & & & & & & & & & \end{bmatrix}$$

Rank $= 1$ Quasiseparability order $= 2$

Quasiseparability orders on the rank profile matrix

$$\begin{bmatrix} 2 & 2 & 1 & 0 & 2 & 1 & 1 & 1 & 0 & 0 \\ 2 & 2 & 1 & 0 & 2 & 0 & 1 & 2 & 0 & 0 \\ 2 & 2 & 1 & 0 & 2 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 1 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & & & \\ 0 & 0 & 0 & 0 & 1 & & & & & \\ 0 & 1 & 0 & 0 & & & & & & \\ 0 & 0 & 0 & & & & & & & \\ 0 & 0 & 1 & & & & & & & \\ 0 & & & & & & & & & \end{bmatrix}$$

Rank $= 1$ Quasiseparability order $= 2$

Major difficulty

Complexity in $O(n^2 r^{\omega-2})$ where $r = \text{rank}(A) \gg r_L, r_U$. But

Quasiseparability orders on the rank profile matrix

$$\begin{bmatrix} 2&2&1&0&2&1&1&1&0&0\\ 2&2&1&0&2&0&1&2&0&0\\ 2&2&1&0&2&2&1&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 2&2&1&0&2&0&0&0&0&0\\ 2&2&1&0&0&0&0&0&0&0\\ 2&1&0&0&0&0&0&0&0&0\\ 1&2&0&0&0&0&0&0&0&0\\ 2&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0 \end{bmatrix}, \begin{bmatrix} 1&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&\\ 0&0&0&0&0&0&0&0&&\\ 0&0&0&0&0&0&0&&&\\ 0&0&0&0&0&0&1&&&\\ 0&0&0&0&1&&&&&\\ 0&1&0&0&&&&&&\\ 0&0&0&&&&&&&\\ 0&0&1&&&&&&&\\ 0&&&&&&&&& \end{bmatrix}$$

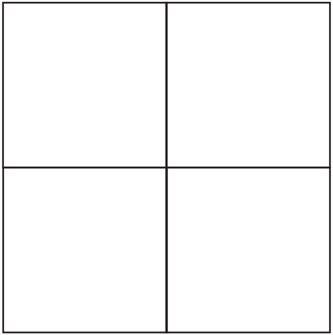
Rank $= 1$ Quasiseparability order $= 2$

Major difficulty

Complexity in $O(n^2 r^{\omega-2})$ where $r = \text{rank}(A) \gg r_L, r_U$. But

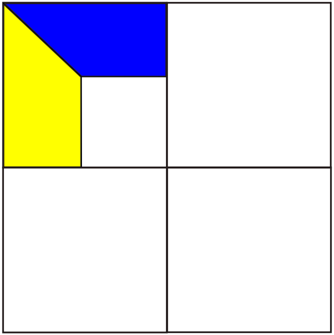
- only a few pivots $(O(r_L, r_U))$ near the top left corner
- numerous pivots must lie near the anti-diagonal $\rightsquigarrow$ cheaper elimination

Computing the left-triangular part of a rank profile matrix



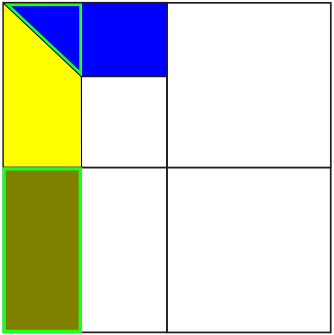
2×2 block splitting

Computing the left-triangular part of a rank profile matrix



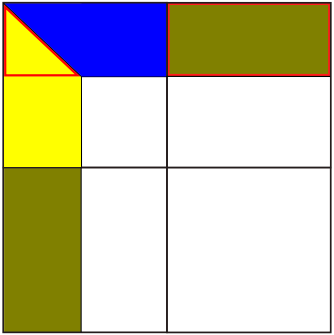
Recursive call

Computing the left-triangular part of a rank profile matrix



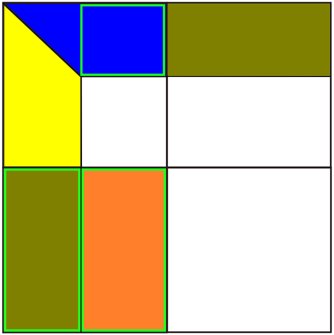
TRSM: $B \leftarrow BU^{-1}$

Computing the left-triangular part of a rank profile matrix



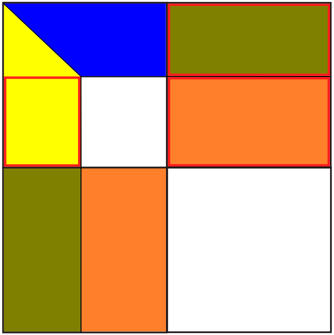
TRSM: $B \leftarrow L^{-1}B$

Computing the left-triangular part of a rank profile matrix



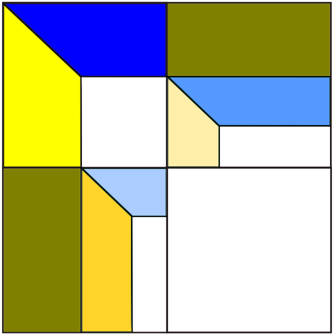
MatMul: $C \leftarrow C - A \times B$

Computing the left-triangular part of a rank profile matrix



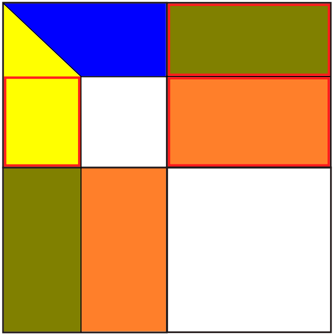
MatMul: $C \leftarrow C - A \times B$

Computing the left-triangular part of a rank profile matrix



2 recursive calls : not possible as no longer left-triangular

Computing the left-triangular part of a rank profile matrix



2 recursive calls : not possible as no longer left-triangular

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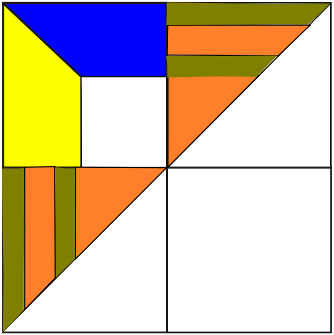
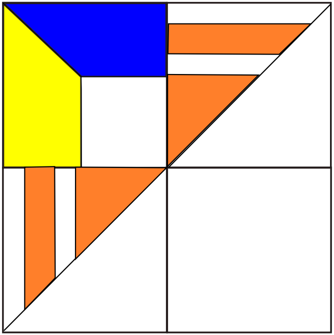


2 recursive calls : not possible as no longer left-triangular
$\rightsquigarrow$ permute back to original position

Computing the left-triangular part of a rank profile matrix

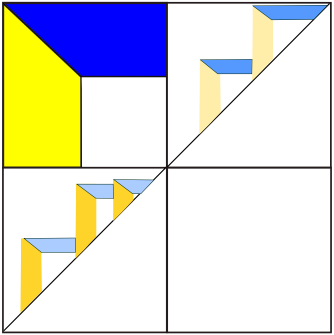


2 recursive calls : not possible as no longer left-triangular

$\leadsto$ permute back to original position

$\leadsto$ clear out cols/rows already processed

Computing the left-triangular part of a rank profile matrix



2 recursive calls : not possible as no longer left-triangular
$\rightsquigarrow$ permute back to original position
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recursive call

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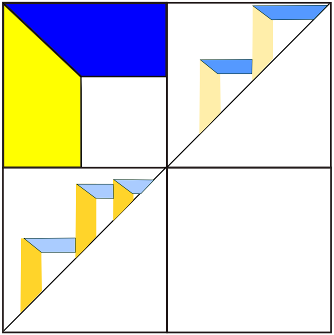


2 recursive calls : not possible as no longer left-triangular

⇝ permute back to original position

⇝ clear out cols/rows already processed

recursive call

- $O(n^2 s^{\omega-2})$
- in place

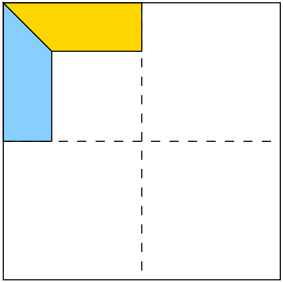
Outline

1. The rank profile matrix
2. Computing the orders of quasiseparability
3. New generators
   - Recursive Rank Revealing
   - The Bruhat generator
   - The Compact Bruhat generator

The Recursive Rank Revealing

Principle: *à la* Hierarchically Semiseparable representations

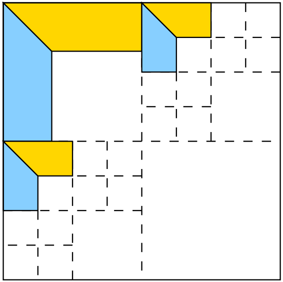
- Split in 4 quadrants
- Store the top left corner as a PLUQ decomposition: $O(nr)$



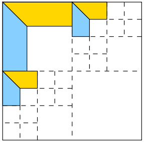
The Recursive Rank Revealing

Principle: *à la* Hierarchically Semiseparable representations

- Split in 4 quadrants
- Store the top left corner as a PLUQ decomposition: $O(nr)$
- Apply recursively on the anti-diagonal blocks



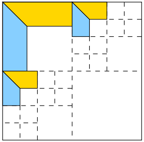
The Recursive Rank Revealing generator



For a quasiseparability order s:

- $s \geq$ all ranks of the PLUQ

The Recursive Rank Revealing generator



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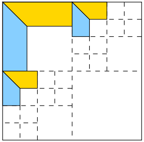

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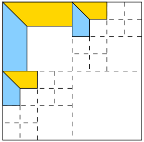

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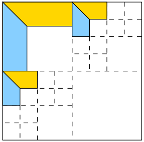

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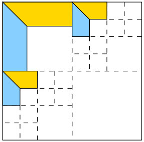

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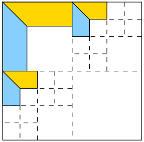
The Recursive Rank Revealing generator



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- ✓Inverse $O(ns^{\omega-1} \log^2 \frac{n}{s})$

RRR × RRR

if $n \leq s+t$ **then**

  return $\mathsf{C} \leftarrow \texttt{RRRExpand}(\mathsf{A}) \times \texttt{RRRExpand}(\mathsf{B})$

end if

Split the matrices as $\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \leftarrow \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$.

$C_{11} \leftarrow \texttt{RRRxRRR}\ (A_{11}, B_{11})$ ▷ $C_{11} \leftarrow A_{11}B_{11}$

$C_{22} \leftarrow \texttt{RRRxRRR}\ (A_{22}, B_{22})$ ▷ $C_{22} \leftarrow A_{22}B_{22}$

RRR × RRR

if $n \le s + t$ **then**
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$X \leftarrow \texttt{RRxRR}\ (A_{12}, B_{21})$ ▷ $X \leftarrow A_{12}B_{21}$
$Y \leftarrow \texttt{RRxRR}\ (A_{21}, B_{12})$ ▷ $Y \leftarrow A_{21}B_{12}$

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$Y \leftarrow \texttt{RRxRR}\ (A_{21}, B_{12})$ ▷ $Y \leftarrow A_{21}B_{12}$

$C_{11} \leftarrow \texttt{RRR+RR}\ (C_{11}, X)$ ▷ $C_{11} \leftarrow C_{11} + X$

$C_{22} \leftarrow \texttt{RRR+RR}\ (C_{22}, Y)$ ▷ $C_{22} \leftarrow C_{22} + Y$

RRR × RRR

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$C_{22} \leftarrow \texttt{RRR+RR}\ (C_{22}, Y)$ ▷ $C_{22} \leftarrow C_{22} + Y$
$L^X \leftarrow \texttt{RRRxTS}(A_{11}, L^B_{12})$; $R^X \leftarrow R^B_{12}$ ▷ $X \leftarrow A_{11}B_{12}$ in rank a revealing facto
$L^Y \leftarrow L^A_{12}$; $R^Y \leftarrow \texttt{TSxRRR}(R^A_{12}, B_{22})$ ▷ $Y \leftarrow A_{12}B_{22}$ in rank a revealing facto
$C_{12} \leftarrow \texttt{RR+RR}(X, Y)$ ▷ $C_{12} \leftarrow X + Y$

RRR × RRR

if $n \leq s+t$ **then**
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end if

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$C_{21} \leftarrow \texttt{RR+RR}(X, Y)$ ▷ $C_{21} \leftarrow X + Y$

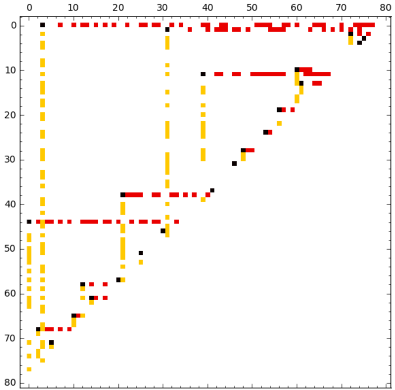
return $C \leftarrow \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}$

The Bruhat generator

Idea: *Numerous pivots (if any) must lie near the anti-diagonal.*

From a PLUQ decomposition revealing the rank profile matrix E, define:

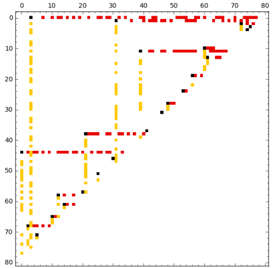
$$\mathcal{L} = \text{Left}(P \begin{bmatrix} L & 0 \end{bmatrix} Q),$$
$$\mathcal{E} = \text{Left}(E),$$
$$\mathcal{U} = \text{Left}(P \begin{bmatrix} U \\ 0 \end{bmatrix} Q).$$



The Bruhat Generator

For a quasiseparability order s:

- ✓Size: $O(ns)$



Lemma

There is less than ns non zero elements in $\mathcal{L}$.

Proof.

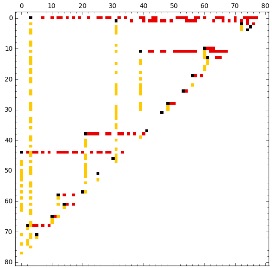
$\mathcal{L}$ is left triangular of QS order s.
The k-th row of $\mathcal{L}$ has $\leq s$ pivots above $\rightsquigarrow$ $\leq s$ non-zeros. □

The Bruhat Generator

For a quasiseparability order s:

- ✓Size: $O(ns)$
- ✓Generator: $A = \text{Left}(\mathcal{L}\mathcal{E}^T\mathcal{U})$

$$A = P \begin{bmatrix} L & 0 \end{bmatrix} Q Q^T \begin{bmatrix} U \\ 0 \end{bmatrix} Q$$



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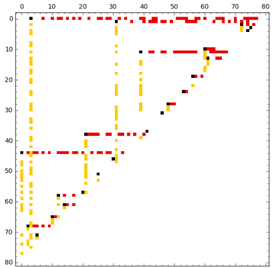

Lemma

If $A = BU$ with U upper triangular, then Left(A) = Left(Left$(B)U)$

The Bruhat Generator

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- ✓Size: $O(ns)$
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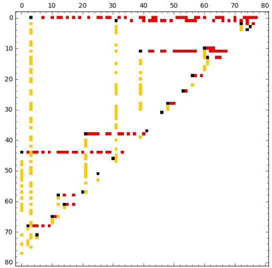
$$\mathcal{A} = \text{Left}\left(\mathcal{L}Q^T \begin{bmatrix} U \\ 0 \end{bmatrix} Q\right)$$



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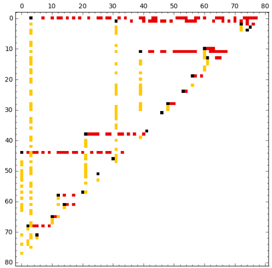
$$E = P \begin{bmatrix} I_r \\ 0 \end{bmatrix} Q$$



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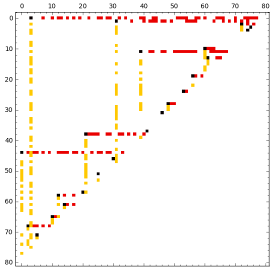

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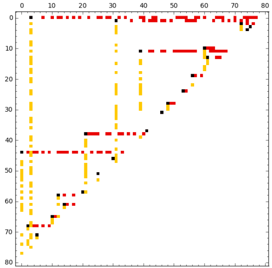

Lemma

If $A = LB$ with L lower triangular, then Left(A) = Left$(L\text{Left}(B))$

The Bruhat Generator

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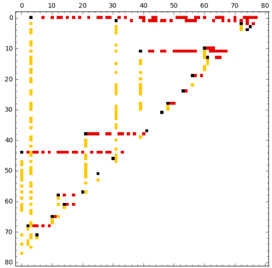

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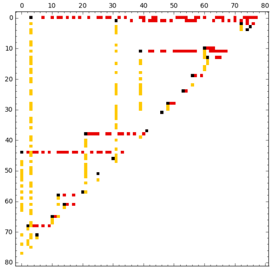

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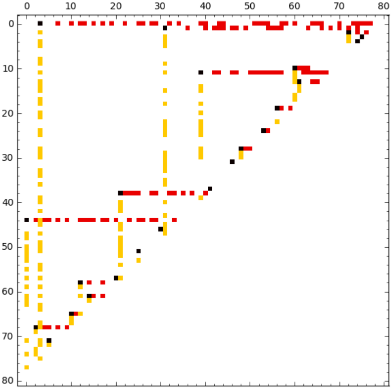
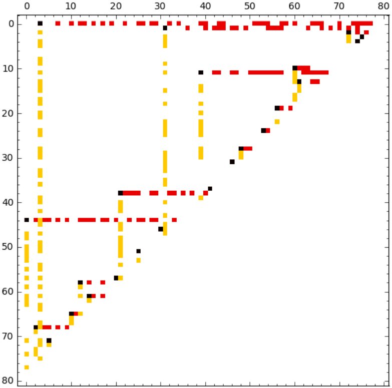


- ✓Computed in $O(n^2 s^{\omega-2})$
  $\rightsquigarrow$ Adapted from the tile rec. PLUQ algo (same as for the QS orders)



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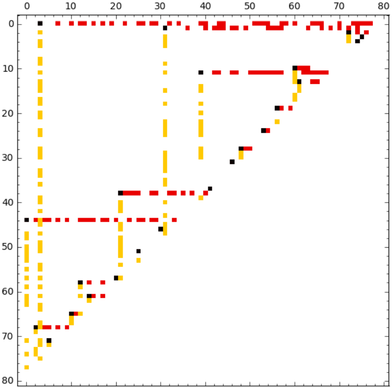
- ✓QuasiSep × Vector in $O(\text{Size}) = O(ns)$



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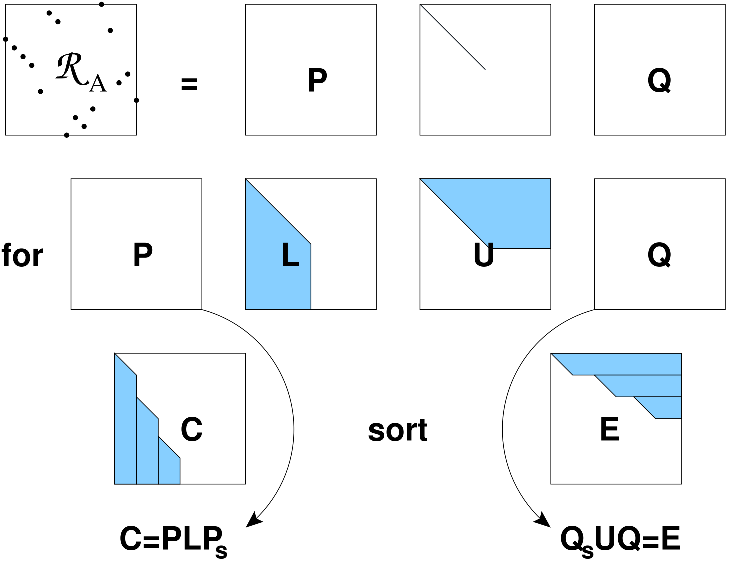
- ✓QuasiSep × Vector in $O(\text{Size}) = O(ns)$
- ✗Scattered⇝ no fast QuasiSep × QuasiSep

Row and column echelon forms from PLUQ



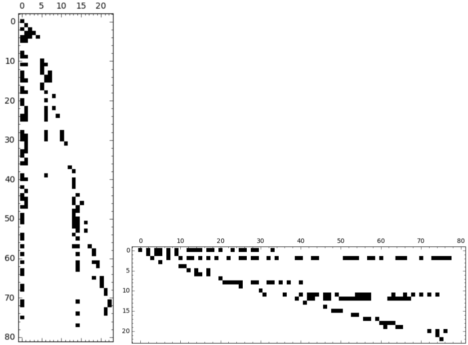
The compact Bruhat Generator

- C $= \mathcal{LQ}$: Col. permutation to column echelon form
- E $= \mathcal{PU}$: Row permutation to row echelon form

$$A = \text{Left}(\text{CRE})$$

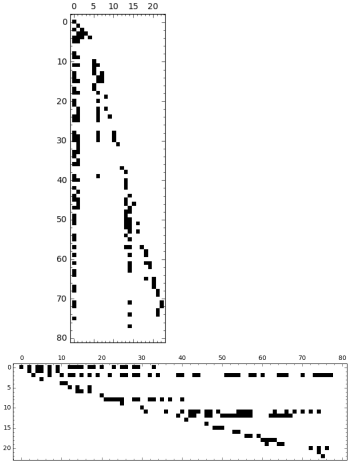
for an $r \times r$ permutation R.

(generalized Bruhat decomposition [Manthey Helmke 07][7]).



Major difficulty

Again $r = \text{rank}(A) \gg s$



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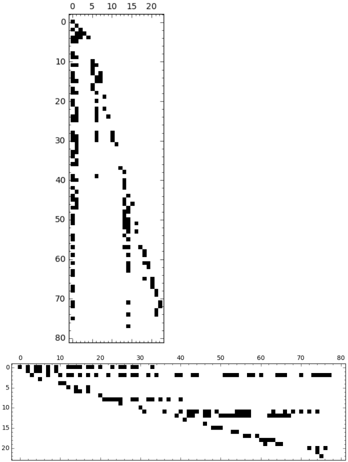
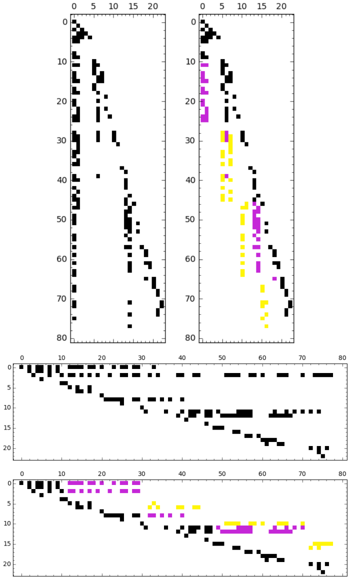
- But double structure: either echelon or Left-triangular



Major difficulty

Again $r = \text{rank}(A) \gg s$

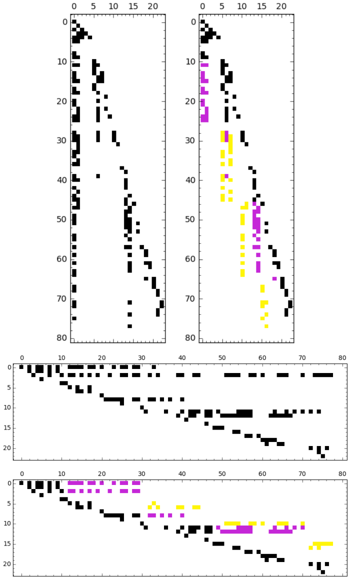
- But double structure: either echelon or Left-triangular
- Fold them in blocks of size s



Major difficulty

Again $r = \text{rank}(A) \gg s$

- But double structure: either echelon or Left-triangular
- Fold them in blocks of size s
- forming a **dense** $O(ns)$ block bi-diagonal structure



Multiplying two quasiseparable matrices

Quasiseparable $\times$ Quasiseparable

$$A \times B = (L_A + D_A + U_A)(L_B + D_B + U_B).$$

$\times$	Lower	Upper
Lower	$J_n \times$ Left $\times J_n \times$ Left	$J_n \times$ Left $\times$ Left $\times J_n$
Upper	Left $\times J_n \times J_n \times$ Left	Left $\times J_n \times$ Left $\times J_n$

$\rightsquigarrow$ reduce to the complexity of

- Left $\times$ Left
- Left $\times J_n \times$ Left

Left × Left

Ideally: Left × Left

$$A \times B = C_A R_A E_A \times C_B R_B E_B.$$

Left $\times$ Left

Ideally: Left $\times$ Left

$$A \times B = C_A(R_A(E_A \times C_B)R_B)E_B.$$

$$E_A \times C_B = (D_{\mathcal{U}_A} + T_{\mathcal{U}_A} S_{\mathcal{U}_A})(D_{\mathcal{L}_B} + S_{\mathcal{L}_B} T_{\mathcal{L}_B})$$

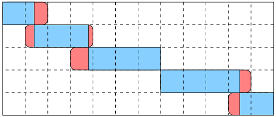
Left × Left

Ideally: Left × Left

$$\mathsf{A} \times \mathsf{B} = \mathsf{C_A}(\mathsf{R_A}(\mathsf{E_A} \times \mathsf{C_B})\mathsf{R_B})\mathsf{E_B}.$$

$\mathsf{E_A} \times \mathsf{C_B} = (\mathsf{D}_{\mathcal{U}_\mathsf{A}} + \mathsf{T}_{\mathcal{U}_\mathsf{A}}\mathsf{S}_{\mathcal{U}_\mathsf{A}})(\mathsf{D}_{\mathcal{L}_\mathsf{B}} + \mathsf{S}_{\mathcal{L}_\mathsf{B}}\mathsf{T}_{\mathcal{L}_\mathsf{B}})$

$\rightsquigarrow$ products of block (sub)-diagonal $r_A \times n$ and $r_B \times n$



$\rightsquigarrow O(s^{\omega-1}n)$

Left × Left

Ideally: Left × Left

$$\mathsf{A} \times \mathsf{B} = \mathsf{C_A}(\mathsf{R_A}(\mathsf{E_A} \times \mathsf{C_B})\mathsf{R_B})\mathsf{E_B}.$$

$\mathsf{E_A} \times \mathsf{C_B} = (\mathsf{D}_{\mathcal{U}_\mathsf{A}} + \mathsf{T}_{\mathcal{U}_\mathsf{A}}\mathsf{S}_{\mathcal{U}_\mathsf{A}})(\mathsf{D}_{\mathcal{L}_\mathsf{B}} + \mathsf{S}_{\mathcal{L}_\mathsf{B}}\mathsf{T}_{\mathcal{L}_\mathsf{B}})$
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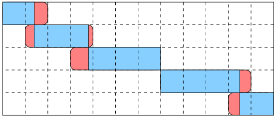

$\rightsquigarrow O(s^{\omega-1}n)$

But: Left × Left

$$\mathsf{A} \times \mathsf{B} = \mathrm{Left}(\mathsf{C_A}\mathsf{R_A}\mathsf{E_A}) \times \mathrm{Left}(\mathsf{C_B}\mathsf{R_B}\mathsf{E_B}).$$

Flavour of the workaround

Left $\times$ TallSkinny $\mathsf{C} = \mathrm{Left}(\mathsf{C}^{\mathsf{A}}\mathsf{R}^{\mathsf{A}}\mathsf{E}^{\mathsf{A}})\mathsf{B}$

$$\mathsf{C}_i = \mathsf{C}_i^{\mathsf{A}}\mathsf{R}^{\mathsf{A}} \sum_{j=1}^{N-i} \mathsf{E}_j^{\mathsf{A}}\mathsf{B}_j + \mathrm{Left}\left(\mathsf{C}_i^{\mathsf{A}}\mathsf{R}^{\mathsf{A}}\mathsf{E}_{N-i+1}^{\mathsf{A}}\right)\mathsf{B}_{N-i+1}.$$

Flavour of the workaround

Left $\times$ TallSkinny $C = \text{Left}(C^A R^A E^A) B$

$$C_i = C_i^A R^A \sum_{j=1}^{N-i} E_j^A B_j + \text{Left}\left(C_i^A R^A E_{N-i+1}^A\right) B_{N-i+1}.$$

1: compute each point-wise block-product, $X_j = E_j^A B_j$, in a compact storage $X_j = D_j^X + T^E S_j^X$;

2: merge the storage and the permutation R^A: $Y_j = R^A(D_j^X + T^E S_j^X)$;

3: compute all partial sums of these blocks: $Z_i = \sum_{j=1}^{N-i} Y_j$;

4: apply C_i^A to the left: $V_i = C_i^A Z_i$;

5: add the trailing term, $C_i = V_i + \text{Left}\left(C_i^A R^A E_{N-i+1}^A\right) B_{N-i+1}$.

6: **return** $C \leftarrow \begin{bmatrix} C_1 \\ \vdots \\ C_N \end{bmatrix}$

Flavour of the workaround

Left × TallSkinny $\mathsf{C} = \text{Left}(\mathsf{C}^\mathsf{A}\mathsf{R}^\mathsf{A}\mathsf{E}^\mathsf{A})\mathsf{B}$

$$\mathsf{C}_i = \mathsf{C}_i^\mathsf{A}\mathsf{R}^\mathsf{A}\sum_{j=1}^{N-i}\mathsf{E}_j^\mathsf{A}\mathsf{B}_j + \text{Left}\left(\mathsf{C}_i^\mathsf{A}\mathsf{R}^\mathsf{A}\mathsf{E}_{N-i+1}^\mathsf{A}\right)\mathsf{B}_{N-i+1}.$$

1: compute each point-wise block-product, $\mathsf{X}_j = \mathsf{E}_j^\mathsf{A}\mathsf{B}_j$, in a compact storage $\mathsf{X}_j = \mathsf{D}_j^\mathsf{X} + \mathsf{T}^\mathsf{E}\mathsf{S}_j^\mathsf{X}$;
2: merge the storage and the permutation R^A: $\mathsf{Y}_j = \mathsf{R}^\mathsf{A}(\mathsf{D}_j^\mathsf{X} + \mathsf{T}^\mathsf{E}\mathsf{S}_j^\mathsf{X})$;
3: compute all partial sums of these blocks: $\mathsf{Z}_i = \sum_{j=1}^{N-i}\mathsf{Y}_j$;
4: apply C_i^A to the left: $\mathsf{V}_i = \mathsf{C}_i^\mathsf{A}\mathsf{Z}_i$;
5: add the trailing term, $\mathsf{C}_i = \mathsf{V}_i + \text{Left}\left(\mathsf{C}_i^\mathsf{A}\mathsf{R}^\mathsf{A}\mathsf{E}_{N-i+1}^\mathsf{A}\right)\mathsf{B}_{N-i+1}$.
6: **return** $\mathsf{C} \leftarrow \begin{bmatrix}\mathsf{C}_1\\ \vdots \\ \mathsf{C}_N\end{bmatrix}$

$\rightsquigarrow O(s^{\omega-1}n)$

Perspectives

QS × QS complexity

- HSS: $O(s^2 n)$
- RRR: $O(s^{\omega-1} n \log^2 \frac{n}{s})$
- CB: $O(s^{\omega-1} n)$ for a factorization of C but not in CB form

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- [Storjohann Yang 15][10] computes RowRP in Las-Vegas $O\tilde{}(|A| + r^{\omega})$
- [Dumas P. Sultan 16][4] extends it to RPM

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Inverse/LinSys

HSS: $O(s^2 n)$

RRR: $O(s^{\omega-1} n \log^2 \frac{n}{s})$

CB: ? (target $O(s^{\omega-1} n)$)

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