

A stencil of the finite-difference method for the 2D convection diffusion equation and its new iterative scheme

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The paper gives the numerical stencil for the two-dimensional convection diffusion equation and the technique of elimination, and builds up the new iterative scheme to solve the implicit difference equation. The scheme's convergence and its higher rate of convergence than the Jacobi iteration are proved. And the numerical example indicates that the new scheme has the same parallelism and a higher rate of convergence than the Jacobi iteration.

Keywords: convection diffusion equation; finite difference; iterative method; mathematics stencil; parallelism

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1. Introduction

In many fields of some nature sciences and engineering, the parabolic equation is always used to describe many phenomena; so the finite-difference method that solves the parabolic equation is always a focus of concern [3,5,8,9]. As the parallel computer comes into being and develops, people put more intention on the parallel computing of the finite-difference method [2,6,7,10,11]. The classic explicit scheme is obviously suitable for parallel computing, but it is conditionally stable; especially for the higher-dimensional problem, the time step is strictly limited. In many large scientific computations of the parabolic equation, the implicit scheme has high use in the solving process because it has good stability and can get a satisfactory approximation even if it is by the use of a big time step. In general, the implicit scheme can be solved by the iterative method, and the famous Jacobi method can be used for parallel computing, but its rate of convergence is very slow. So, constructing a fast iterative method with parallelism has become a popular topic.

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In the investigation of the numerical method of differential equations, many authors described the finite-difference approximation for a differential operator $U_{xx} + U_{yy}$ in a two-dimensional mesh region by a stencil [1,4,9]. Hui *et al.* introduced the concept of the stencil and the elimination technique to the finite-difference approximation for Poisson equation, and established a new iterative method that had a higher rate of convergence.

In this paper, the technique of mathematics stencil is used for solving the convection diffusion equation. The stencil for parabolic equation is constructed by elimination among different points' stencil, and the corresponding iteration scheme can be used for parallel computing. By analysing the iteration error, we prove that the scheme has a higher rate of convergence than the Jacobi method and is suitable for parallel computing. From numerical experiments, we know that the iteration scheme has better convergence than the Jacobi method under the same error limit. And the CPU time indicates that the calculation is reduced while the convergence is improved.

The paper is organised as follows. Section 2 puts forward the definition of stencil according to the characteristic of the finite-difference scheme for two-dimensional convection diffusion equation, and constructs a fast iterative scheme with parallelism by making use of the local elimination; Section 3 proves the convergence of the new iteration and its high rate of convergence. Section 4 examines the performance of the new iterative method by some experiments; finally, we do the same numerical experimentation for a convection diffusion equation that has different coefficient of convection in two directions.

We consider the finite-difference approximation about the following convection diffusion problem

$$\frac{\partial u}{\partial t} + a \left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \right) = \varepsilon \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right), \quad (x, y) \in D, \quad t \in (0, T], \quad (1)$$

$$u|_{t=0} = \Phi(x, y), \quad (x, y) \in D, \quad (2)$$

$$u(x, y, t) = \Psi(x, y, t), \quad (x, y) \in \partial D, \quad t \in (0, T], \quad (3)$$

where D is a bounded region and ∂D is piecewise smooth curve.

First, we construct the equidistance grid D_h in D with mesh spacing as h in both the x and y directions,

$$D_h = \{(x_i, y_j) \in D | x_i = ih, y_j = jh\},$$

and D_h is the interior points set.

The boundary point set Γ_h is

$$\Gamma_h = \{(x_k, y_l) \in \partial D | x_k = kh \quad \text{or} \quad y_l = lh\}.$$

So $\overline{D_h} = D_h \cup \Gamma_h$ is the points set instead of $\overline{D} = D \cup \partial D$.

Denote the adjacent point set of an interior point (x_i, y_j) as

$$A(i, j) = \{(x_k, y_l) | |k - i| + |l - j| = 1\}.$$

Let τ be a step in time, $\tau = T/n$, $t_k = k\tau$. The net point (x_i, y_j, t_k) is called the regular interior point if $A(i, j) \in D_h$, otherwise it is called an irregular interior point. And the

other points $(x_i, y, t_k)((x_i, y) \in \partial D; k \geq 0)$ and $(x, y_j, t_k)((x, y_j) \in \partial D; k \geq 0)$ are called the boundary points. The point (x_i, y_j, t_k) can be abridged as (i, j, k) .

Let $r = \tau/h^2$, $r_1 = \varepsilon\tau/h^2$, $u_{i,j}^k$ be the approximations of the solution $u(x, y, t)$ at the point (i, j, k) . So we consider the following difference approximation for Equation (1) [3].

Five-point implicit difference scheme

$$\begin{aligned} & \frac{u_{i,j}^{k+1} - u_{i,j}^k}{\tau} + a \left(\frac{u_{i+1,j}^{k+1} - u_{i-1,j}^{k+1}}{2h} + \frac{u_{i,j+1}^{k+1} - u_{i,j-1}^{k+1}}{2h} \right) \\ &= \varepsilon \left[\frac{u_{i+1,j}^{k+1} - 2u_{i,j}^{k+1} + u_{i-1,j}^{k+1}}{h^2} + \frac{u_{i,j+1}^{k+1} - 2u_{i,j}^{k+1} + u_{i,j-1}^{k+1}}{h^2} \right], \end{aligned} \quad (4)$$

or rewrite it as

$$\begin{aligned} & - \left(r_1 - \frac{1}{2}arh \right) u_{i+1,j}^{k+1} - \left(r_1 + \frac{1}{2}arh \right) u_{i-1,j}^{k+1} + (1 + 4r_1) u_{i,j}^{k+1} - \left(r_1 - \frac{1}{2}arh \right) u_{i,j+1}^{k+1} \\ & - \left(r_1 + \frac{1}{2}arh \right) u_{i,j-1}^{k+1} = u_{i,j}^k, \quad i, j = 1, 2, \dots, m; \quad k = 1, 2, \dots, n. \end{aligned} \quad (5)$$

Dividing Equation (5) by r_1 and obtaining by reorganisation:

$$\begin{aligned} & -(1 - bh) u_{i+1,j}^{k+1} - (1 + bh) u_{i-1,j}^{k+1} + c_1 u_{i,j}^{k+1} - (1 - bh) u_{i,j+1}^{k+1} - (1 + bh) u_{i,j-1}^{k+1} = \frac{1}{r_1} u_{i,j}^k, \\ & i, j = 1, 2, \dots, m; \quad k = 1, 2, \dots, n. \end{aligned} \quad (6)$$

where $b = a/2\varepsilon$, $c_1 = 4 + 1/r_1$, it is obvious that Equation (5) is equivalent to Equation (6).

2. Numerical stencil and stencil elimination

We introduce the concept of numerical stencil and stencil elimination policy in order to construct the new iterative method of Equation (6) [1,4].

Similarly, we consider the case that the right-hand side of Equation (6) is zero

$$\begin{aligned} & -c_{01} u_{i+1,j}^{k+1} - c_{02} u_{i-1,j}^{k+1} + c_1 u_{i,j}^{k+1} - c_{01} u_{i,j+1}^{k+1} - c_{02} u_{i,j-1}^{k+1} = 0, \\ & i, j = 1, 2, \dots, m; \quad k = 1, 2, \dots, n, \end{aligned} \quad (7)$$

where $c_{01} = 1 - bh$, $c_{02} = 1 + bh$.

There is a mathematics stencil at any inner point (i, j, k) associated with the difference equation (7). The definition of the numerical stencil is as follows.

As shown in Figure 1(a), there is a stencil contacting with Equation (7), which has geometry and concludes five mesh points as O, A, B, C, D, whose coordinates are $(i, j, k + 1)$, $(i - 1, j, k + 1)$, $(i + 1, j, k + 1)$, $(i, j - 1, k + 1)$, $(i, j + 1, k + 1)$ and four line segments as $\overline{OA}$, $\overline{OB}$, $\overline{OC}$, $\overline{OD}$; where the mesh point $(i, j, k + 1)$ is called the centre of the stencil, and another four points are called neighbor points of the centre.

We mark the five points of the stencil centred at $(i, j, k + 1)$ with the respective coefficients $c_1, -c_{02}, -c_{01}, -c_{02}, -c_{01}$ of the unknown quantity $u_{i,j}^{k+1}, u_{i-1,j}^{k+1}, u_{i+1,j}^{k+1}, u_{i,j-1}^{k+1}, u_{i,j+1}^{k+1}$, then every point of the geometry stencil corresponds to a coefficient; and we call the geometry stencil with a coefficient as a numerical stencil (Figure 1(b)), and it has one-to-one correspondence with the

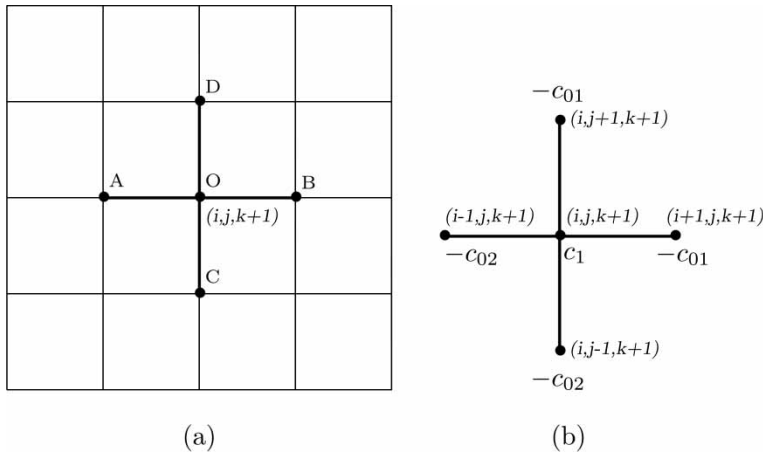


Figure 1. (a) The grid D_h and its geometry stencil, (b) numerical stencil I.

difference equation (7). We call the numerical stencil in Figure 1(b) numerical stencil I. We divide Equation (7) by c_1 and it is equivalent to Equation (7), thus change the number at the centre and the neighbor points as 1 and c_{12} , c_{11} , c_{12} , c_{11} (Figure 2(a)). The new numerical stencil is called stencil II, and it is equal to stencil I (where $c_{11} = -c_{01}/c_1$, $c_{12} = -c_{02}/c_1$).

The stencil elimination is as follows.

The first elimination. For the stencil in Figure 1(b), in order to eliminate the number $-c_{01}$ at the points $(i, j + 1, k + 1)$ and $(i + 1, j, k + 1)$, we multiply the two similar stencils II centred at the points in Figure 2(a) with c_{01} and add it to the stencil in Figure 1(b), by which the same point is at the same position and the numbers of the same point can be added. So the number becomes zero at the points $(i, j + 1, k + 1)$ and $(i + 1, j, k + 1)$. Similarly, we can make the numbers at the points $(i - 1, j, k + 1)$ and $(i, j - 1, k + 1)$ become zero by using the two stencils II centred at the two points. So we obtain the new stencil form as Figure 2(b).

The second elimination. Similar to the first elimination, we add the four stencils I that are centred at the points as $(i - 1, j - 1, k + 1)$, $(i - 1, j + 1, k + 1)$, $(i + 1, j - 1, k + 1)$, $(i + 1, j + 1, k + 1)$ and the numbers at the centres are labelled by $2(1 + bh)^2/c_1$, $2(1 - b^2h^2)/c_1$, $2(1 - b^2h^2)/c_1$, $2(1 - bh)^2/c_1$ onto the stencil in Figure 2(b) according to the same points being at the same position, and we can obtain the stencil as Figure 2(c).

The third elimination. Finally, we multiply the stencil as Figure 1(b) with $4(1 - b^2h^2)/c_1^2$ and change their sign, then add it to the stencil in Figure 2(c). At this time we obtain the stencil as Figure 2(d) and stop the elimination.

For the Equation (6), we assign the geometry stencil's centre with the coefficient $(c_1, -1/r_1)$ of the items $u_{i,j}^{(k+1)}$, $u_{i,j}^{(k)}$, and assign the neighbor points with the coefficients $(-c_{02}, 0)$, $(-c_{02}, 0)$, $(-c_{01}, 0)$, $(-c_{01}, 0)$ of the items $u_{i-1,j}^{(k+1)}$ and $u_{i-1,j}^{(k)}$, $u_{i,j-1}^{(k+1)}$ and $u_{i,j-1}^{(k)}$, $u_{i+1,j}^{(k+1)}$ and $u_{i+1,j}^{(k)}$, $u_{i,j+1}^{(k+1)}$ and $u_{i,j+1}^{(k)}$, respectively, so that the geometry stencil becomes a stencil with two-dimensional array (Figure 3(a)).

For similarity, the difference equation (6) has the one-to-one transformation with the vector stencil. We can obtain a new stencil form by the same process.

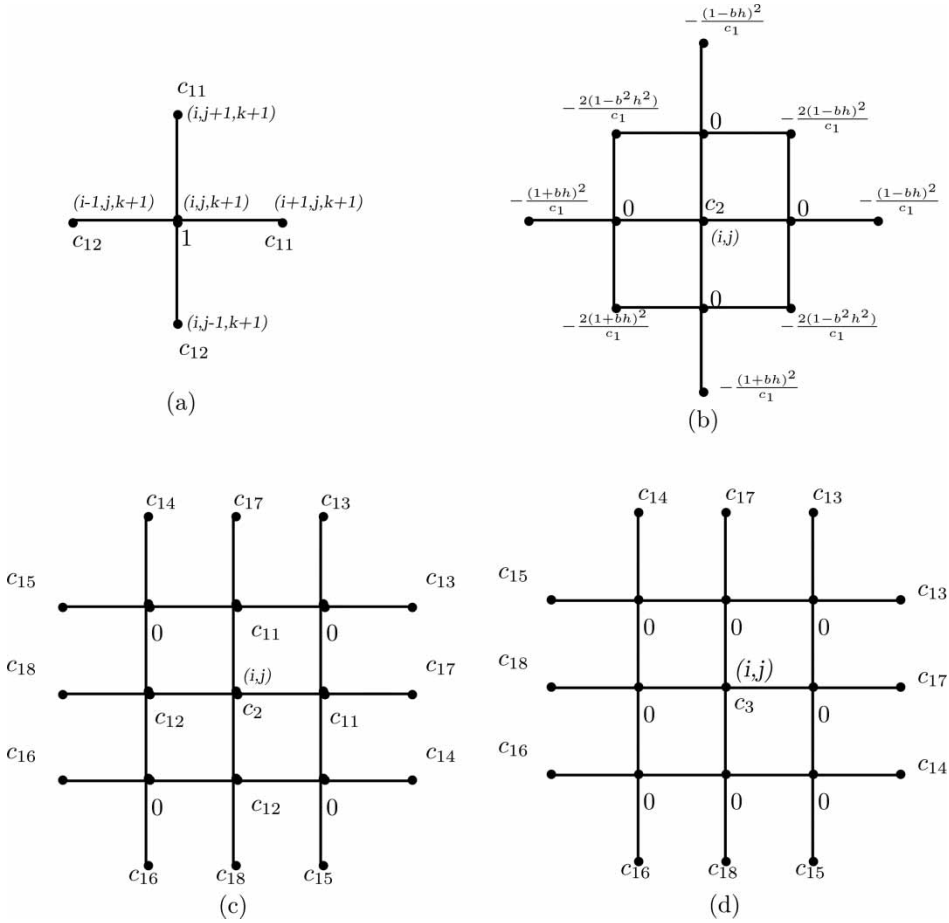


Figure 2. (a) Numerical stencil II; (b) the stencil after the first elimination; (c) the stencil after the second elimination; (d) the stencil after the third elimination, where $c_2 = c_1 - 4 - 4b^2h^2/c_1$, $c_3 = c_1 - 8(1 - b^2h^2)/c_1$, $c_{11} = -4(1 - bh)^2(1 + bh)/c_1^2$, $c_{12} = -4(1 + bh)^2(1 - bh)/c_1^2$, $c_{13} = -2(1 - bh)^3/c_1^2$, $c_{14} = -2(1 - bh)^2(1 + bh)/c_1^2$, $c_{15} = -2(1 + bh)^2(1 - bh)/c_1^2$, $c_{16} = -2(1 + bh)^3/c_1^2$, $c_{17} = -(1 - bh)^2/c_1$, $c_{18} = -2(1 + bh)^2/c_1$.

From the five-point implicit scheme (6), we do many stencil eliminations and evolve it to be the form as in Figure 3(b). The following equation is equal to the stencil in Figure 3(b)

$$\begin{aligned}
 & c_3 u_{i,j}^{(k+1)} + c_{17}(u_{i,j+2}^{(k+1)} + u_{i+2,j}^{(k+1)}) + c_{18}(u_{i-2,j}^{(k+1)} + u_{i,j-2}^{(k+1)}) + c_{13}(u_{i+2,j+1}^{(k+1)} \\
 & + u_{i+1,j+2}^{(k+1)}) + c_{14}(u_{i-1,j+2}^{(k+1)} + u_{i+2,j-1}^{(k+1)}) + c_{15}(u_{i+1,j-2}^{(k+1)} + u_{i-2,j+1}^{(k+1)}) + c_{16}(u_{i-1,j-2}^{(k+1)} \\
 & + u_{i-2,j-1}^{(k+1)}) = F_{i,j}, \quad i, j = 2, 3, \dots, m-1,
 \end{aligned} \quad (8)$$

where

$$\begin{aligned}
 F_{i,j} = & -c_4 u_{i,j}^{(k)} - c_{22}(u_{i-1,j}^{(k)} + u_{i,j-1}^{(k)}) - c_{21}(u_{i,j+1}^{(k)} + u_{i+1,j}^{(k)}) - c_{24}(u_{i-1,j+1}^{(k)} + u_{i+1,j-1}^{(k)}) \\
 & - c_{25} u_{i-1,j-1}^{(k)} - c_{23} u_{i+1,j+1}^{(k)}.
 \end{aligned}$$

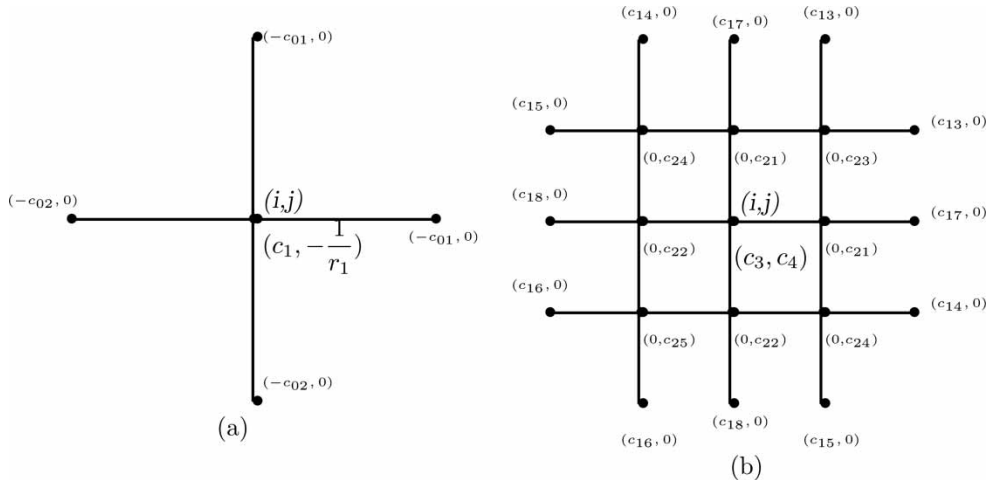


Figure 3. (a) The vector stencil and (b) the new stencil style for the layer of $k + 1$, where $c_3 = c_1 - 8(1 - b^2 h^2)/c_1$, $c_4 = 4(1 - b^2 h^2)/c_1^2 r_1 - 1/r_1$, $c_{21} = -1 - bh/c_1 r_1$, $c_{22} = -1 - bh/c_1 r_1$, $c_{23} = -2(1 - bh)^2/c_1^2 r_1$, $c_{24} = -2(1 - b^2 h^2)/c_1^2 r_1$, $c_{25} = -2(1 + bh)^2/c_1^2 r_1$.

3. The new iteration and its convergence

For simplicity in proving, suppose $D = (0, 1) \times (0, 1)$. We construct the grid D_h with mesh spacing as $h = 1/(m + 1)$ in both the x and y directions,

$$D_h = \{(x_i, y_j) | x_i = ih, y_j = jh, 0 \leq i, j \leq m + 1\}.$$

Let τ be a step in time, $\tau = T/n$, $t_k = k\tau$, then the net point (x_i, y_j, t_k) ($0 < i, j < m + 1, k > 0$) is called as the interior point, and the other points (x_i, y_j, t_k) ($i = 0, m + 1, j = 0, 1, \dots, m + 1; j = 0, m + 1, i = 0, 1, \dots, m + 1; k \geq 0$) are called the boundary points.

First, we introduce the definition of sub-class L_p for the mesh point in D_h and call L_p the layer of the grid D_h , and p is the serial number of the layer

$$L_p = \{(i, j, k + 1) | M_{ij} = ph\}, \quad p = 1, 2, \dots, \left\lceil \frac{m + 1}{2} \right\rceil, \quad (9)$$

where M_{ij} indicates the shortest distance from the mesh point $(i, j, k + 1)$ to the boundary ∂D . It is obvious that there must be a stencil as Figure 3 and a corresponding difference equation (8) for any point $(i, j, k + 1) \in L_p$ ($p = 2, \dots, [m + 1/2]$).

From Equations (6) and (8), we construct the new iterative method – stencil method as follows.

$$\begin{aligned} u_{i,j}^{(k+1,l+1)} &= \frac{1 - bh}{c_1} \left(u_{i+1,j}^{(k+1,l)} + u_{i,j+1}^{(k+1,l)} \right) + \frac{1 + bh}{c_1} \left(u_{i,j-1}^{(k+1,l)} + u_{i-1,j}^{(k+1,l)} \right) \\ &\quad + \frac{1}{c_1 r_1} u_{i,j}^{(k)}, \quad (i, j) \in L_1, \quad l = 0, 1, 2, \dots, \\ u_{i,j}^{(k+1,l+1)} &= -\frac{c_{17}}{c_3} \left(u_{i,j+2}^{(k+1,l)} + u_{i+2,j}^{(k+1,l)} \right) - \frac{c_{18}}{c_3} \left(u_{i-2,j}^{(k+1,l)} + u_{i,j-2}^{(k+1,l)} \right) \\ &\quad - \frac{c_{13}}{c_3} \left(u_{i+2,j+1}^{(k+1,l)} + u_{i+1,j+2}^{(k+1,l)} \right) - \frac{c_{14}}{c_3} \left(u_{i-1,j+2}^{(k+1,l)} + u_{i+2,j-1}^{(k+1,l)} \right) \end{aligned} \quad (10)$$

$$\begin{aligned}
& -\frac{c_{15}}{c_3} \left(u_{i+1,j-2}^{(k+1,l)} + u_{i-2,j+1}^{(k+1,l)} \right) - \frac{c_{16}}{c_3} \left(u_{i-1,j-2}^{(k+1,l)} + u_{i-2,j-1}^{(k+1,l)} \right) \\
& + \frac{F_{i,j}}{c_3}, \quad (i, j) \in L_p, \quad p = 2, 3, \dots, \left[\frac{m+1}{2} \right], \quad l = 0, 1, 2, \dots \quad (11)
\end{aligned}$$

In order to prove the convergence of the method easily, we adapt Equation (11) to the form as follows:

$$\begin{aligned}
u_{i,j}^{(k+1,l+1)} &= \frac{(1-bh)^2}{c_1^2 - 8(1-b^2h^2)} \left(u_{i,j+2}^{(k+1,l)} + u_{i+2,j}^{(k+1,l)} \right) \\
&+ \frac{(1+bh)^2}{c_1^2 - 8(1-b^2h^2)} \left(u_{i-2,j}^{(k+1,l)} + u_{i,j-2}^{(k+1,l)} \right) \\
&+ \frac{2(1+bh)^3}{c_1(c_1^2 - 8(1-b^2h^2))} \left(u_{i-1,j-2}^{(k+1,l)} + u_{i-2,j-1}^{(k+1,l)} \right) \\
&+ \frac{2(1-bh)^3}{c_1(c_1^2 - 8(1-b^2h^2))} \left(u_{i+2,j+1}^{(k+1,l)} + u_{i+1,j+2}^{(k+1,l)} \right) \\
&+ \frac{2(1-bh)^2(1+bh)}{c_1(c_1^2 - 8(1-b^2h^2))} \left(u_{i-1,j+2}^{(k+1,l)} + u_{i+2,j-1}^{(k+1,l)} \right) \\
&+ \frac{2(1+bh)^2(1-bh)}{c_1(c_1^2 - 8(1-b^2h^2))} \left(u_{i+1,j-2}^{(k+1,l)} + u_{i-2,j+1}^{(k+1,l)} \right) + \frac{F_{i,j}}{c_3}, \\
&(i, j) \in L_p, \quad p = 2, 3, \dots, \left[\frac{m+1}{2} \right], \quad l = 0, 1, 2, \dots \quad (12)
\end{aligned}$$

The method is convergence, namely when $l \rightarrow \infty$, $u_{i,j}^{(k+1,l)} \rightarrow u_{i,j}^{(k+1)}$, where $u_{i,j}^{(k+1)}$ is the solution of the difference equation (6), and $u_{i,j}^{(k+1,0)}$ is the iterative initial value. The proof of convergence is as follows.

Let

$$\begin{aligned}
\xi_{i,j}^{(k+1,l+1)} &= u_{i,j}^{(k+1,l+1)} - u_{i,j}^{(k+1)}, \\
E^{(l+1)} &= \max_{i,j} \left| \xi_{i,j}^{(k+1,l+1)} \right|. \quad (13)
\end{aligned}$$

Choosing the suitable h in order to guarantee $bh < \sqrt{2} - 1$, and from Equations (10) and (11) we can obtain:

$$\begin{aligned}
|\xi_{i,j}^{(k+1,l+1)}| &\leq \frac{1+bh}{c_1} \left(\left| \xi_{i-1,j}^{(k+1,l)} \right| + \left| \xi_{i,j-1}^{(k+1,l)} \right| \right) \\
&+ \frac{1-bh}{c_1} \left(\left| \xi_{i+1,j}^{(k+1,l)} \right| + \left| \xi_{i,j+1}^{(k+1,l)} \right| \right), \quad (i, j) \in L_1, \quad (14)
\end{aligned}$$

and

$$\begin{aligned}
 \left| \xi_{i,j}^{(k+1,l+1)} \right| &\leq \frac{(1-bh)^2}{c_1^2 - 8(1-b^2h^2)} \left(\left| \xi_{i,j+2}^{(k+1,l)} \right| + \left| \xi_{i+2,j}^{(k+1,l)} \right| \right) \\
 &+ \frac{(1+bh)^2}{c_1^2 - 8(1-b^2h^2)} \left(\left| \xi_{i-2,j}^{(k+1,l)} \right| + \left| \xi_{i,j-2}^{(k+1,l)} \right| \right) \\
 &+ \frac{2(1+bh)^3}{c_1(c_1^2 - 8(1-b^2h^2))} \left(\left| \xi_{i-1,j-2}^{(k+1,l)} \right| + \left| \xi_{i-2,j-1}^{(k+1,l)} \right| \right) \\
 &+ \frac{2(1-bh)^3}{c_1(c_1^2 - 8(1-b^2h^2))} \left(\left| \xi_{i+2,j+1}^{(k+1,l)} \right| + \left| \xi_{i+1,j+2}^{(k+1,l)} \right| \right) \\
 &+ \frac{2(1-bh)^2(1+bh)}{c_1(c_1^2 - 8(1-b^2h^2))} \left(\left| \xi_{i-1,j+2}^{(k+1,l)} \right| + \left| \xi_{i+2,j-1}^{(k+1,l)} \right| \right) \\
 &+ \frac{2(1+bh)^2(1-bh)}{c_1(c_1^2 - 8(1-b^2h^2))} \left(\left| \xi_{i+1,j-2}^{(k+1,l)} \right| + \left| \xi_{i-2,j+1}^{(k+1,l)} \right| \right), \\
 (i, j) &\in L_p, \quad p = 2, 3, \dots, \left\lfloor \frac{m+1}{2} \right\rfloor.
 \end{aligned} \tag{15}$$

Because of $c_1 = 4 + 1/r_1$, $1 - bh > 0$, then,

$$c_1^2 - 8(1 - b^2h^2) > 8 + 8b^2h^2 = 8(1 + b^2h^2).$$

Thus, we have

$$\begin{aligned}
 &\frac{(1-bh)^2}{c_1^2 - 8(1-b^2h^2)} + \frac{(1+bh)^2}{c_1^2 - 8(1-b^2h^2)} < \frac{2(1+b^2h^2)}{8(1+b^2h^2)} = \frac{1}{4}, \quad \frac{2(1+bh)^3}{c_1(c_1^2 - 8(1-b^2h^2))} \\
 &+ \frac{2(1-bh)^3}{c_1(c_1^2 - 8(1-b^2h^2))} + \frac{2(1-bh)^2(1+bh)}{c_1(c_1^2 - 8(1-b^2h^2))} + \frac{2(1+bh)^2(1-bh)}{c_1(c_1^2 - 8(1-b^2h^2))} \\
 &< \frac{2(1-bh)2(1+b^2h^2) + 2(1+bh)2(1+b^2h^2)}{32(1+b^2h^2)} = \frac{1}{4}.
 \end{aligned}$$

We can prove the formula as follows for every point

$$E^{l+1} \leq E^l.$$

In order to prove when $l \rightarrow \infty$, $E^l \rightarrow 0$, we give the estimating formula of $|\xi_{i,j}^{(k+1,l)}|$ in every layer.

When $(i, j) \in L_1$, by using the boundary condition, there is a zero value at least among the four values at the right-hand side of Equation (14), so we have

$$\begin{aligned}
 \left| \xi_{i,j}^{(k+1,l+1)} \right| &\leq \max \left(\frac{3-bh}{c_1}, \frac{3+bh}{c_1} \right) E^l \\
 &\leq \left(1 - \frac{1-bh}{4} \right) E^l \\
 &\leq \left(1 - \left(\frac{1}{2} - \frac{(1+bh)^2}{4} \right) \right) E^l.
 \end{aligned} \tag{16}$$

At this time we have $0 < 1/2 - (1+bh)^2/4 < 1/2$.

When $(i, j) \in L_2$, by the same means, there is a zero value at least among the front four values and two zero values among the other four at the right-hand side of Equation (15),

$$\begin{aligned} \left| \xi_{i,j}^{(k+1,l+1)} \right| &\leq \max \left(\frac{2(1+b^2h^2) + (1-bh)^2}{8(1+b^2h^2)}, \frac{2(1+b^2h^2) + (1+bh)^2}{8(1+b^2h^2)} \right) E^{(l)} \\ &\quad + \max \left(\frac{8(1+b^2h^2) + 4(1-bh)^2}{32(1+b^2h^2)}, \frac{8(1+b^2h^2) + 4(1+bh)^2}{32(1+b^2h^2)} \right) E^l \\ &\leq \left(\frac{1}{2} + \frac{(1+bh)^2}{4} \right) E^l = \left(1 - \left(\frac{1}{2} - \frac{(1+bh)^2}{4} \right) \right) E^l. \end{aligned} \quad (17)$$

Because of $1 \leq (1+bh)^2 \leq 2$, then $[(1+bh)^2 - 1][(1+bh)^2 - 2] \leq 0$, thus we have

$$1 - \frac{1}{4} \left(\frac{1}{2} - \frac{(1+bh)^2}{4} \right) \leq 1 - \left(\frac{1}{2} - \frac{(1+bh)^2}{4} \right)^2.$$

When $(i, j) \in L_3$, from Equations (15) and (17) we obtain:

$$\begin{aligned} \left| \xi_{i,j}^{(k+1,l+1)} \right| &\leq \frac{3}{4} E^{(l)} + \frac{1}{4} \left(1 - \left(\frac{1}{2} - \frac{(1+bh)^2}{4} \right) \right) E^{(l-1)} \\ &\leq \frac{3}{4} E^{(l-1)} + \frac{1}{4} \left(1 - \left(\frac{1}{2} - \frac{(1+bh)^2}{4} \right) \right) E^{(l-1)} \\ &\leq \left(1 - \frac{1}{4} \left(\frac{1}{2} - \frac{(1+bh)^2}{4} \right) \right) E^{(l-1)} \\ &\leq \left(1 - \left(\frac{1}{2} - \frac{(1+bh)^2}{4} \right)^2 \right) E^{(l-1)}. \end{aligned} \quad (18)$$

Similarly, when $(i, j) \in L_4$, we also have

$$\left| \xi_{i,j}^{(k+1,l+1)} \right| \leq \left(1 - \left(\frac{1}{2} - \frac{(1+bh)^2}{4} \right)^2 \right) E^{(l-1)}. \quad (19)$$

For $p > 4$, we can prove it by mathematical induction.

In general, when $(i, j) \in L_p$, we have

$$\left| \xi_{i,j}^{(k+1,l+1)} \right| \leq \left(1 - \left(\frac{1}{2} - \frac{(1+bh)^2}{4} \right)^{[(p+1)/2]} \right) E^{(l-[(p+1)/2]+1)}. \quad (20)$$

Suppose the total number of layers is r , then $r = [(m+1)/2]$, for all the inner points we have

$$\left| \xi_{i,j}^{(k+1,l+1)} \right| \leq \left(1 - \left(\frac{1}{2} - \frac{(1+bh)^2}{4} \right)^{[(r+1)/2]} \right) E^{(l-[(r+1)/2]+1)}, \quad (21)$$

then

$$E^{(l+1)} \leq \left(1 - \left(\frac{1}{2} - \frac{(1+bh)^2}{4} \right)^{[(r+1)/2]} \right) E^{(l-[(r+1)/2]+1)}, \quad (22)$$

so

$$E^{(N[(r+1)/2])} \leq \left(1 - \left(\frac{1}{2} - \frac{(1+bh)^2}{4} \right)^{[(r+1)/2]} \right)^N E^{(0)}. \quad (23)$$

From all the above, when $N \rightarrow \infty$, the right-hand side of Equation (23) $\rightarrow 0$, the new algorithm obtains convergence.

Similarly, we can obtain the convergence estimate of Jacobi method as

$$E^{(l+1)} \leq \left(1 - \left(\frac{1}{2} - \frac{(1+bh)^2}{4}\right)^r\right) E^{(l-r+1)}, \quad (24)$$

then

$$E^{(N_1 r)} \leq \left(1 - \left(\frac{1}{2} - \frac{(1+bh)^2}{4}\right)^r\right)^{N_1} E^{(0)}. \quad (25)$$

From Equations (23) and (25), we can see that when the iteration number is same, $N = 2N_1$ and $1 - (1/2 - (1+bh)^2/4)^{[(r+1)/2]} < 1 - (1/2 - (1+bh)^2/4)^r$, so we have

$$\left(1 - \left(\frac{1}{2} - \frac{(1+bh)^2}{4}\right)^{[(r+1)/2]}\right)^N \ll \left(1 - \left(\frac{1}{2} - \frac{(1+bh)^2}{4}\right)^r\right)^{N_1}.$$

It indicates the new iteration scheme is much better than the Jacobi iteration method.

From the formulas (10) and (11) and the idea of Seidel iteration, we can construct the iterative style as follows:

$$\begin{aligned} u_{i,j}^{(k+1,l+1)} &= \frac{c_{01}}{c_1} \left(u_{i+1,j}^{(k+1,l)} + u_{i,j+1}^{(k+1,l)}\right) + \frac{c_{02}}{c_1} \left(u_{i,j-1}^{(k+1,l+1)} + u_{i-1,j}^{(k+1,l+1)}\right) \\ &\quad + \frac{1}{c_1 r_1} u_{i,j}^{(k)}, \quad (i, j) \in L_1, \quad l = 0, 1, 2, \dots, \end{aligned} \quad (26)$$

$$\begin{aligned} u_{i,j}^{(k+1,l+1)} &= -\frac{c_{17}}{c_3} \left(u_{i-1,j+2}^{(k+1,l)} + u_{i+2,j}^{(k+1,l)}\right) - \frac{c_{18}}{c_3} \left(u_{i-2,j}^{(k+1,l+1)} + u_{i,j-2}^{(k+1,l+1)}\right) \\ &\quad - \frac{c_{16}}{c_3} \left(u_{i-1,j-2}^{(k+1,l+1)} + u_{i-2,j-1}^{(k+1,l+1)}\right) - \frac{c_{13}}{c_3} \left(u_{i+2,j+1}^{(k+1,l)} + u_{i+1,j+2}^{(k+1,l)}\right) \\ &\quad - \frac{c_{14}}{c_3} \left(u_{i-1,j+2}^{(k+1,l)} + u_{i+2,j-1}^{(k+1,l)}\right) - \frac{c_{15}}{c_3} \left(u_{i+1,j-2}^{(k+1,l+1)} + u_{i-2,j+1}^{(k+1,l+1)}\right) + \frac{F_{i,j}}{c_3}, \\ &\quad (i, j) \in L_p, \quad p = 2, 3, \dots, [m + 1/2], \quad l = 0, 1, 2, \dots \end{aligned} \quad (27)$$

For the irregular region, the difference scheme is as follows: the variable step five-point implicit difference for the irregular interior node, the five-point implicit difference for the first layer of regular interior nodes and stencil scheme or Equations (26) and (27) for all the other interior nodes. And the convergence can be proved by the similar way.

4. Numerical experiments

We consider the solution of the following convection-dominated diffusion equation:

$$\frac{\partial u}{\partial t} + 0.8 \left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \right) = 0.01 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right). \quad (28)$$

Its exact solution is

$$u(x, y, t) = \frac{1}{4t+1} \exp \left[-\frac{(x-0.8t-0.5)^2}{0.01(4t+1)} - \frac{(y-0.8t-0.5)^2}{0.01(4t+1)} \right], \quad (29)$$

where

$$(x, y) \in D = \{0 < x, y < 1\}, \quad t \in (0, T].$$

The initial condition and the boundary condition comes from Equation (29).

We do the numerical experiments under the condition: $h = 1/200$, $\tau = 0.0005$, $n = 200$ using the classic Jacobi method, the new Seidel iteration, Gauss–Seidel method and the formulas (26) and (27) based on the five-point difference style, then count the iterative times S , where $S = \sum_{k=1}^n I_k$,

$$I_k = \min \left\{ l; \max_{i,j} \left| u_{i,j}^{k,l+1} - u_{i,j}^{k,l} \right| < \varepsilon \right\}.$$

Do the experiments for $\varepsilon = 10^{-5}$, 10^{-6} , 10^{-7} , 10^{-8} and the result is as in Tables 1 and 2. The exact solution, the numerical solution and the contour lines of the two solution are shown in Figures 4–6 when the iterative error limit is 10^{-8} by using the stencil method based on the five-point difference scheme.

From Table 1 and Figures 4–6, we know that the new scheme has its advantage in convergence and parallelism. And Table 2 shows that the calculation has not increased while the iteration times have decreased. So the scheme is applicable to the convection diffusion equation.

From the scheme's construction, we know it cannot be used for solving variable coefficient parabolic equations, but finding an iteration scheme that has high-rate convergence and parallelism for solving variable coefficient parabolic equations is an interesting problem to resolve.

Table 1. The number of iteration for four method at the same error limit (for five-point difference scheme).

Method	10^{-5}	10^{-6}	10^{-7}	10^{-8}
Jacobi	1613	2200	2800	3362
Stencil	800	1000	1200	1400
Gauss–Seidel	1133	1400	1800	2061
Formulas (26) and (27)	600	800	848	1000

Table 2. The CPU time of the iteration for four methods at the same error limit(s) (for five-point difference scheme).

Method	10^{-5}	10^{-6}	10^{-7}	10^{-8}
Jacobi	10.796875	14.625	18.515625	22.078125
Stencil	8.5	10.96875	12.796875	15
Gauss–Seidel	8.171875	9.734375	12.640625	15.171875
Formulas (26) and (27)	5.65625	7.609375	7.8125	9.421875

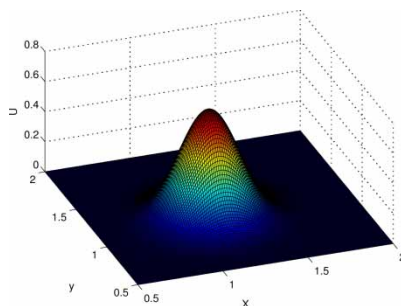


Figure 4. The exact solution of the example when the error limit is 10^{-8} .

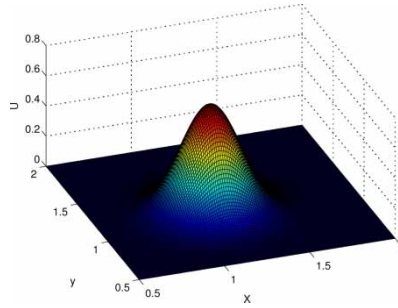


Figure 5. The numerical solution of the example when the error limit is 10^{-8} .

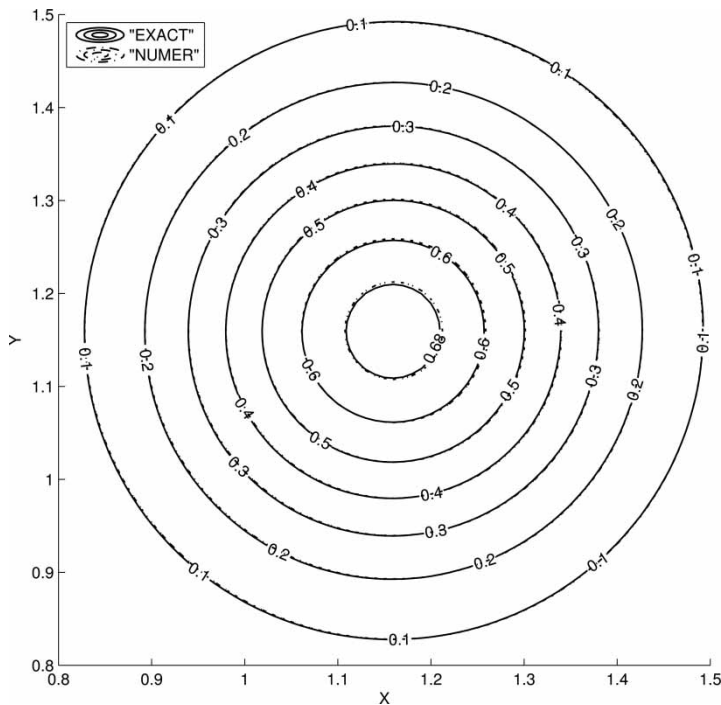


Figure 6. Contour lines of the exact solution and the numerical solution by stencil scheme in the sub-region $0.5 < x, y < 1.5$ at $t = 0.1$ s when the error limit is 10^{-8} .

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