ERRATUM TO THE ARTICLE: HOMOGENIZATION OF THE EIGENVALUES OF THE NEUMANN-POINCARÉ OPERATOR

An annoying mistake was found in the proof of Theorem 4.4 by L. Chesnel (INRIA DEFI, Centre de Mathématiques Appliquées École Polytechnique), whose careful reading is gratefully acknowledged. This note presents a revised, correct version of the proof.

Theorem 0.1. *Under the assumptions (4.1), there exists* ε_0 *such that, for* $0 < \varepsilon < \varepsilon_0$,

$$(\lambda \in \sigma(T_\varepsilon),\ \lambda \notin \{0,1\}) \quad \Rightarrow \quad m \leq \lambda \leq M,$$

where $0 < m < M < 1$ *are two constants, independent of* ε, *which only depend on the geometry of the rescaled inclusion* $\omega \Subset Y$.

Proof. Let us denote by λ_ε^- (resp. λ_ε^+) the lowest (resp. largest) eigenvalue of T_ε which is different from 0 (resp. different from 1).

Exploiting the min-max principle of Proposition 3.4 in combination with the characterization of $\text{Ker}(T_\varepsilon)$ given in Proposition 3.2, it comes:

$$\lambda_\varepsilon^- = \min_{\substack{u \in \mathfrak{h}_\varepsilon \\ u \neq 0}} \frac{\displaystyle\int_{\omega_\varepsilon} |\nabla u|^2 \, dx}{\displaystyle\int_\Omega |\nabla u|^2 \, dx}, \text{ and } \lambda_\varepsilon^+ = \max_{\substack{u \in \mathfrak{h}_\varepsilon \\ u \neq 0}} \frac{\displaystyle\int_{\omega_\varepsilon} |\nabla u|^2 \, dx}{\displaystyle\int_\Omega |\nabla u|^2 \, dx}, \tag{0.1}$$

where the space $\mathfrak{h}_\varepsilon$ is defined by (see (3.7)):

$$\mathfrak{h}_\varepsilon = \left\{ u \in H^1_0(\Omega),\ \Delta u = 0 \text{ on } \omega_\varepsilon \cup (\Omega \setminus \overline{\omega_\varepsilon}), \text{ and } \int_{\partial \omega_\varepsilon^\xi} \frac{\partial u^+}{\partial n} \, ds = 0,\ \xi \in \Xi_\varepsilon \right\}.$$

Our purpose is to prove that

$$m \leq \lambda_\varepsilon^-, \text{ and } \lambda_\varepsilon^+ \leq M, \tag{0.2}$$

for some constants $0 < m \leq M < 1$ depending only on the geometry of the inclusion $\omega \Subset Y$.

Proof of the right-hand inequality in (0.2): Let $u \in \mathfrak{h}_\varepsilon$, $u \neq 0$ be arbitrary. For any $\xi \in \Xi_\varepsilon$, define the rescaled function $u_\varepsilon^\xi(y) := u(\varepsilon\xi + \varepsilon y)$ in $H^1(Y)$. A simple change of variables yields:

$$\int_{\omega_\varepsilon} |\nabla u|^2 \, dx = \varepsilon^{d-2} \sum_{\xi \in \Xi_\varepsilon} \int_\omega |\nabla_y u_\varepsilon^\xi|^2 \, dy, \tag{0.3}$$

and similarly:

$$\int_\Omega |\nabla u|^2 \, dx = \int_{\mathcal{B}_\varepsilon} |\nabla u|^2 \, dx + \varepsilon^{d-2} \sum_{\xi \in \Xi_\varepsilon} \int_Y |\nabla_y u_\varepsilon^\xi|^2 \, dy. \tag{0.4}$$

We then obtain:

$$\frac{\displaystyle\int_{\omega_\varepsilon} |\nabla u|^2 \, dx}{\displaystyle\int_\Omega |\nabla u|^2 \, dx} = \frac{\displaystyle\varepsilon^{d-2} \sum_{\xi \in \Xi_\varepsilon} \int_\omega |\nabla_y u_\varepsilon^\xi|^2 \, dy}{\displaystyle\int_{\mathcal{B}_\varepsilon} |\nabla u|^2 \, dx + \varepsilon^{d-2} \sum_{\xi \in \Xi_\varepsilon} \int_Y |\nabla_y u_\varepsilon^\xi|^2 \, dy} \leq \max_{\xi \in \Xi_\varepsilon} \frac{\displaystyle\int_\omega |\nabla_y u_\varepsilon^\xi|^2 \, dy}{\displaystyle\int_Y |\nabla_y u_\varepsilon^\xi|^2 \, dy}, \tag{0.5}$$

where we have used the easy algebraic identity:

$$\min\left(\frac{p_1}{q_1}, \frac{p_2}{q_2}\right) \leq \frac{p_1 + p_2}{q_1 + q_2} \leq \max\left(\frac{p_1}{q_1}, \frac{p_2}{q_2}\right), \ \ p_1, p_2, q_1, q_2 \geq 0,\ \ q_1 q_2 \neq 0. \tag{0.6}$$

Now, since $u \in \mathfrak{h}_\varepsilon$, it follows that for every $\xi \in \Xi_\varepsilon$, $u_\varepsilon^\xi \in H^1(Y)$ satisfies:

$$-\Delta_y u_\varepsilon^\xi = 0 \text{ on } \omega, \text{ and so } \int_{\partial\omega} \frac{\partial u_\varepsilon^{\xi -}}{\partial n}\, ds = 0. \tag{0.7}$$

Hence, an integration by parts yields:

$$\begin{aligned} \int_\omega |\nabla_y u_\varepsilon^\xi|^2\, dy &= -\int_{\partial\omega} u_\varepsilon^\xi \frac{\partial u_\varepsilon^{\xi -}}{\partial n}\, ds \\ &= -\int_{\partial\omega} \left(u_\varepsilon^\xi - \frac{1}{|\partial\omega|} \int_{\partial\omega} u_\varepsilon^\xi\, ds \right) \frac{\partial u_\varepsilon^{\xi -}}{\partial n}\, ds, \end{aligned}$$

where the last line follows from (0.7). Using now the definition of the norm in $H^{-1/2}(\partial\omega)$ (together with (0.7) again) and combining the trace theorem with the Poincaré-Wirtinger inequality in $Y \setminus \overline{\omega}$, we obtain:

$$\begin{aligned} \int_\omega |\nabla_y u_\varepsilon^\xi|^2\, dy &\leq \left|\left| \frac{\partial u_\varepsilon^{\xi -}}{\partial n} \right|\right|_{H^{-1/2}(\partial\omega)} \left|\left| u_\varepsilon^\xi - \frac{1}{|\partial\omega|} \int_{\partial\omega} u_\varepsilon^\xi\, ds \right|\right|_{H^{1/2}(\partial\omega)} \\ &= C \left(\int_\omega |\nabla_y u_\varepsilon^\xi|^2\, dy \right)^{\frac{1}{2}} \left(\int_{Y\setminus\overline{\omega}} |\nabla_y u_\varepsilon^\xi|^2\, dy \right)^{\frac{1}{2}}, \end{aligned}$$

where the constant C depends only on the geometry of $\omega \Subset Y$. It follows that, for the same constant C,

$$||\nabla u_\varepsilon^\xi||_{L^2(\omega)^d} \leq C ||\nabla u_\varepsilon^\xi||_{L^2(Y\setminus\overline{\omega})^d}. \tag{0.8}$$

Finally, combining (0.5) with (0.8) yields the desired inequality.

Proof of the left-hand inequality in (0.2): It is enough to prove that there exists a constant $C > 0$, which depends only on the geometry of $\omega \Subset Y$ and is independent of ε such that:

$$\forall u \in \mathfrak{h}_\varepsilon,\ \int_{\Omega\setminus\overline{\omega_\varepsilon}} |\nabla u|^2\, dx \leq C \int_{\omega_\varepsilon} |\nabla u|^2\, dx.$$

To achieve this, let $u \in \mathfrak{h}_\varepsilon$ be arbitrary; an integration by parts yields:

$$\begin{aligned} \int_{\Omega\setminus\overline{\omega_\varepsilon}} |\nabla u|^2\, dx &= \int_{\partial\omega_\varepsilon} u \frac{\partial u^+}{\partial n}\, ds, \\ &= \sum_{\xi\in\Xi_\varepsilon} \int_{\partial\omega_\varepsilon^\xi} u \frac{\partial u^+}{\partial n}\, ds, \end{aligned}$$

where we recall that n stands for the unit normal vector to $\partial\omega_\varepsilon$, pointing outward ω_ε. Now, for a given $\xi \in \Xi_\varepsilon$, define the function $v(y) := u(\varepsilon\xi + \varepsilon y) \in H^1(Y)$. Using a change of variables, and taking advantage of the fact that $\int_{\partial\omega} \frac{\partial v^+}{\partial n}\, ds = 0$, one has:

$$\int_{\partial\omega_\varepsilon^\xi} u \frac{\partial u^+}{\partial n}\, ds = \varepsilon^{d-2} \int_{\partial\omega} v \frac{\partial v^+}{\partial n}\, ds = \varepsilon^{d-2} \int_{\partial\omega} \left(v - \frac{1}{|\partial\omega|} \int_{\partial\omega} v\, ds \right) \frac{\partial v^+}{\partial n}\, ds.$$

Now using the trace theorem and the Poincaré-Wirtinger inequality inside ω,

$$\left| \int_{\partial\omega_\varepsilon^\xi} u \frac{\partial u^+}{\partial n}\, ds \right| \leq C\varepsilon^{d-2} ||\nabla v||_{L^2(\omega)^d} \left|\left| \frac{\partial v^+}{\partial n} \right|\right|_{H^{-1/2}(\partial\omega)}.$$

Since $\omega \Subset Y$, and using the fact that $\Delta v = 0$ on $Y \setminus \overline{\omega}$ together with usual estimates for the Laplace equation, it holds:

$$\left|\left| \frac{\partial v^+}{\partial n} \right|\right|_{H^{-1/2}(\partial\omega)} \leq C ||\nabla v||_{L^2(Y\setminus\overline{\omega})^d}.$$

As a consequence, we obtain:

$$\left| \int_{\partial\omega_\varepsilon^\xi} u \frac{\partial u^+}{\partial n}\, ds \right| \leq C\varepsilon^{d-2} ||\nabla v||_{L^2(\omega)^d}\ ||\nabla v||_{L^2(Y\setminus\overline{\omega})^d};$$

then, rescaling (i.e. expressing the right-hand side of the above inequality in terms of u) yields:

$$\left| \int_{\partial\omega_\varepsilon^\xi} u \frac{\partial u^+}{\partial n}\, ds \right| \leq C ||\nabla u||_{L^2(\omega_\varepsilon^\xi)^d}\ ||\nabla u||_{L^2(Y_\varepsilon^\xi\setminus\overline{\omega_\varepsilon^\xi})^d}.$$

Eventually, summing over $\xi \in \Xi_\varepsilon$ and using the Cauchy-Schwarz inequality yields:

$$\begin{aligned}\int_{\Omega\setminus\overline{\omega_\varepsilon}} |\nabla u|^2\,dx = \left|\sum_{\xi\in\Xi_\varepsilon}\int_{\partial\omega_\varepsilon^\xi} u\frac{\partial u^+}{\partial n}\,ds\right| &\leq C\sum_{\xi\in\Xi_\varepsilon} ||\nabla u||_{L^2(\omega_\varepsilon^\xi)^d} ||\nabla u||_{L^2(Y_\varepsilon^\xi\setminus\overline{\omega_\varepsilon^\xi})^d} \\ &\leq C||\nabla u||_{L^2(\omega_\varepsilon)^d}\,||\nabla u||_{L^2(\Omega\setminus\overline{\omega_\varepsilon})^d},\end{aligned}$$

whence the expected result. □