Elementary partial differential equations: homework 9

Assigned 04/22/2014, due 04/29/2014.

Exercise 1

This exercise is partly reprinted from [Strauss], §5.6, Exercise 2.

The purpose of this exercise is to solve the heat equation over the interval $(0, \ell)$:

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0, \tag{1}$$

completed with *inhomogeneous* Dirichlet boundary conditions:

$$\forall t > 0, \;\; u(t, 0) = 0, \quad \text{and } u(t, \ell) = e^t, \tag{2}$$

and initial conditions:

$$\forall x \in [0, \ell], \;\; u(0, x) = 0. \tag{3}$$

To this end, we expand the (supposedly twice differentiable) function u and its partial derivatives $\frac{\partial u}{\partial t}$, $\frac{\partial^2 u}{\partial x^2}$ as sine Fourier series over $(0, \ell)$:

$$u(t, x) = \sum_{n=1}^{\infty} u_n(t) \sin\left(\frac{n\pi x}{\ell}\right),$$

$$\frac{\partial u}{\partial t}(t, x) = \sum_{n=1}^{\infty} v_n(t) \sin\left(\frac{n\pi x}{\ell}\right),$$

$$\frac{\partial^2 u}{\partial x^2}(t, x) = \sum_{n=1}^{\infty} w_n(t) \sin\left(\frac{n\pi x}{\ell}\right).$$

(1) According to you, why is it more relevant to consider the *sine* Fourier expansions of u, $\frac{\partial u}{\partial t}$ and $\frac{\partial^2 u}{\partial x^2}$ than, e.g. their cosine or full Fourier expansions?

(2) By using the formulae for Fourier coefficients, express the coefficients $u_n(t)$, $v_n(t)$ and $w_n(t)$ in terms of u and its derivatives.

(3) By relying on the same methodology as in the lectures (that is, by using either inversion of the $\frac{\partial}{\partial t}$ and $\int$ signs, or integration by parts), express $v_n(t)$ and $w_n(t)$ in terms of $u_n(t)$ and its derivative(s).

(4) From the fact that u solves (1), derive the following ODE for each of the coefficients $u_n(t)$, $n \in \mathbb{N}^*$:

$$\frac{du_n}{dt}(t) + \frac{n^2\pi^2}{\ell^2} u_n(t) = (-1)^{n+1} \frac{2n\pi}{\ell^2} e^t.$$

(5) For any fixed real numbers $a, b \in \mathbb{R}$, solve the following first-order ODE of an unknown function $y(t)$:

$$\frac{dy}{dt}(t) + ay(t) = be^t.$$

(6) Infer from the answers to Questions (5) and (6) that the coefficients $u_n(t)$ are of the form:

$$u_n(t) = \frac{(-1)^{n+1} 2n\pi}{\ell^2 + n^2\pi^2} e^t + c_n e^{-\frac{n^2\pi^2 t}{\ell^2}},$$

for some constants c_n yet to be found. Sketch the corresponding expansion for $u(t, x)$.

(7) Eventually, by using the initial condition (3), find the values of c_n and the expression of $u(t, x)$.

Exercise 2

This exercise is partly reprinted from [Haberman], §2.5, Exercise 12.

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain, n be its normal vector field, pointing outward Ω. Recall that, if $V = (V_x, V_y, V_z) : \Omega \to \mathbb{R}^3$ is a differentiable vector field, *Green's formula* reads:

$$\iiint_\Omega \operatorname{div}(V)\, dx = \iint_{\partial\Omega} V \cdot n\, ds,$$

where the *divergence* $\operatorname{div}(V)$ of V is defined as $\operatorname{div}(V) = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$.

(1) Recall that the *gradient* ∇u of a differentiable function $u : \Omega \to \mathbb{R}$ is the vector field over Ω: $\nabla u = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\right)$. Show by a direct calculation that, for any differentiable function $u : \Omega \to \mathbb{R}$ and any differentiable vector field $V : \Omega \to \mathbb{R}^3$,

$$\operatorname{div}(uV) = u \operatorname{div}(V) + \nabla u \cdot V.$$

(2) Deduce from Question (1) and Green's formula that, for any twice differentiable functions $u, v : \Omega \to \mathbb{R}$, the following holds:

$$\iiint_\Omega u\Delta v\, dx = \iint_{\partial\Omega} u\frac{\partial v}{\partial n}\, ds - \iiint_\Omega \nabla u \cdot \nabla v\, dx,$$

where the notation $\frac{\partial v}{\partial n}$ stands for $\nabla v \cdot n$.

(3) We now consider the Laplace equation:

$$\Delta u = f \text{ on } \Omega, \tag{4}$$

together with *inhomogeneous* Dirichlet boundary conditions

$$u = g_1 \text{ on } \partial\Omega. \tag{5}$$

Show unicity of the solution to (4,5) by using the energy method and the result of Question (2).

[Hint: consider two solutions u, v and take $w = u - v$. Express the PDE satisfied by w, multiply it by w and integrate over Ω.]

(4) We still consider Laplace equation (4), but now with *inhomogeneous* Neumann boundary conditions:

$$\frac{\partial u}{\partial n} = g_2 \text{ on } \partial\Omega. \tag{6}$$

Show unicity of the solution to (4,6) *up to a constant*, by the same method as for Question (3).

(5) We eventually consider Laplace equation (4) with *inhomogeneous* Robin boundary conditions:

$$\frac{\partial u}{\partial n} + \alpha u = g_3 \text{ on } \partial\Omega, \tag{7}$$

for a fixed parameter $\alpha > 0$. Show unicity of the solution to (4,7).

Exercise 3

This exercise is partly reprinted from [Strauss], §6.1, Exercise 9.

We consider the two-dimensional domain $\Omega := \left\{(x, y) \in \mathbb{R}^2,\ 1 < ||(x, y)|| < 2\right\}$, that is, the annulus of inner radius 1 and outer radius 2. Remember that the temperature $u(t, x, y)$ inside the annulus is driven by the heat equation $\frac{\partial u}{\partial t} - \kappa\Delta u = 0$. In this exercise, we are interested in the steady state of this equation - still denoted as $u(x, y)$ - which satisfies the Laplace equation:

$$\Delta u = 0 \text{ on } \Omega. \tag{8}$$

The corresponding boundary conditions are:

$$u = 100 \text{ on the inner radius,} \tag{9}$$

$$\frac{\partial u}{\partial n} = -\gamma \text{ on the outer radius,} \tag{10}$$

where $\gamma > 0$, and we search $u(x, y)$ as a radially symmetric function: $u(x, y) \equiv u(r)$.

(1) According to you, which properties of the problem may legitimate the search for a radially symmetric solution to (8,9,10)?
(2) Interpret the sign of γ in terms of energy flow: is energy flowing in or out of the outer radius of Ω?
(3) By using the expression of the Laplace operator in polar coordinates:

$$\Delta u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}.$$

Solve the system (8,9,10) for $u(r)$.
(4) Where are the coldest and hottest values of the temperature $u(r)$ located on $\partial\Omega$? Is this in agreement with the maximum and minimum principles?
(5) Is there a means to choose the value of γ so that the temperature at the outer radius of Ω is 20?

Exercise 4

This exercise is partly reprinted from [Strauss], §5.6, Exercise 10.

Consider a three-dimensional rod oriented along the x-axis, lying in the interval $0 < x < \ell$. The cross-sectional area $\mathcal{A}(x)$ of the rod varies with x and reads:

$$\mathcal{A}(x) = b\left(1 - \frac{x}{\ell}\right)^2,$$

for some fixed parameter $b > 0$. The rod is insulated at its lateral sides, and its temperature is kept at 0 at both ends. It is also assumed to be homogeneous, with coefficients c, ρ, κ equal to 1, and we assume that the temperature u in the rod only depends on time and the x-variable: $u \equiv u(t, x)$. The initial temperature distribution in the rod is $u(0, x) = \phi(x)$, $x \in [0, \ell]$, for some given function ϕ.

(1) Sketch a drawing of the situation.
(2) Show that the PDE governing the temperature $u(t, x)$ inside the rod is:

$$\mathcal{A}(x)\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(\mathcal{A}(x)\frac{\partial u}{\partial x}\right), \ t > 0, \ 0 < x < \ell, \tag{11}$$

with boundary conditions:

$$u(t, 0) = 0, \ u(t, \ell) = 0, \ t > 0, \tag{12}$$

$$u(0, x) = \phi(x), \ 0 < x < \ell. \tag{13}$$

[Hint: As we did in the lectures devoted to the derivation of PDE from physics, consider a slice of rod lying between x and $x + h$. Express the internal energy of the slice, as well as the energy flowing in and out of it, and express the energy balance for this slice.]

(3) We search to solve the above system by the method of separation of variables, and search first for separated solutions $u(t, x) = T(t)X(x)$. Show that there exists a real constant λ such that:

$$\begin{cases} T'(t) + \lambda T(t) = 0 & \text{for } t > 0, \\ \frac{d}{dx}\left(\mathcal{A}(x)X'(x)\right) + \lambda\mathcal{A}(x)X(x) = 0 & \text{for } 0 < x < \ell, \\ X(0) = X(\ell) = 0 \end{cases} . \tag{14}$$

(4) We now make the change of variable functions $v(x) = \left(1 - \frac{x}{\ell}\right)X(x)$. Show that $v(x)$ satisfies:

$$v''(x) + \lambda v(x) = 0, \ v(0) = v(\ell) = 0. \tag{15}$$

(5) Search for the negative eigenvalues $\lambda < 0$ of this last problem (15).
(6) Is $\lambda = 0$ an eigenvalue of this problem?
(7) Search for the positive eigenvalues λ_n of this problem, and write down the associated eigenfunctions $v_n(x)$, as well as the corresponding functions $X_n(x)$.
(8) Solve for the temporal part $T_n(t)$ of the system (14) for each of the eigenvalues λ_n, and show that $u(t, x)$ can be written as the expansion:

$$\sum_{n=1}^{\infty} b_n \frac{\sin\left(\frac{n\pi x}{\ell}\right)}{1 - \frac{x}{\ell}} e^{-\frac{n^2\pi^2 t}{\ell^2}},$$

for some constants b_n to be found.

(9) Eventually, by using the initial condition (13), identify b_n as:

$$b_n = \frac{2}{\ell}\int_0^\ell \phi(x)\left(1-\frac{x}{\ell}\right)\sin\left(\frac{n\pi x}{\ell}\right)\,dx.$$

Exercise 5

This exercise is reprinted from [Strauss], §5.6, Exercise 5.

The purpose of this exercise is to solve the inhomogeneous wave equation

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = e^t\sin(5x),\ \ t>0,\ \ 0<x<\pi, \tag{16}$$

with boundary conditions:

$$u(t,0)=0,\ \ u(t,\pi)=0,\ \ t>0, \tag{17}$$

and initial conditions

$$u(0,x)=0,\ \ \frac{\partial u}{\partial t}(0,x)=\sin(3x),\ \ 0<x<\pi. \tag{18}$$

(1) To go back into the framework of homogeneous PDE, we seek to make a change of unknown function u so that the new unknown function satisfies the homogeneous wave equation. In this particular case, we consider the new function $v(t,x)=u(t,x)+ae^t\sin(5x)$, for some constant a. Find the value of a such that v satisfies the homogeneous wave equation, and write down the whole system (i.e. with boundary and initial conditions) satisfied by v.

(2) By using the contents of the lectures over the method of separation of variables, show that v can be written as the following expansion:

$$v(t,x)=\sum_{n=1}^{\infty}\left(a_n\cos(nct)+b_n\sin(nct)\right)\sin(nx),$$

for some coefficients a_n, b_n, $n=1,\ldots$ to be found.

(3) By using the initial condition for $v(0,x)$, calculate the values of the coefficients a_n.

(4) Now, use the second initial condition over $\frac{\partial v}{\partial t}(0,x)$ to calculate the coefficients b_n.

[Hint: remember that you should never derivate under the $\sum$ sign! Express $\frac{\partial v}{\partial t}$ as a sine Fourier expansion - i.e. $\frac{\partial v}{\partial t}(t,x)=\sum_{n=1}^{\infty}v_n(t)\sin(nx)$, and calculate the coefficients $v_n(t)$ in terms of the b_n. Eventually evaluate $v_n(0)$.]

(5) Conclude as for the expression of $v(t,x)$, and that of $u(t,x)$.