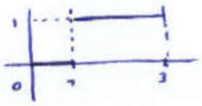
Exercise 1

Let $\phi(x) = \begin{cases} 0 & \text{for } 0 < x < 1 \\ 1 & \text{for } 1 < x < 3 \end{cases}$

Homework 8 : correction



1) Find the first four nonzero terms of its cosine Fourier series explicitly.

We start with the 0th order term: $a_0 = \frac{2}{3}\int_0^3 \phi(x)\,dx$

$$= \frac{2}{3}\int_1^3 1\,dx = \frac{4}{3}.$$

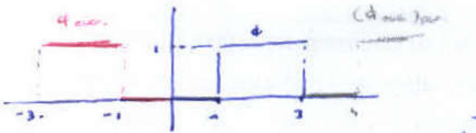
Then, for $n > 0$, $a_n = \frac{2}{3}\int_0^3 \phi(x)\cos\left(\frac{n\pi x}{3}\right)dx = \frac{2}{3}\int_1^3 \cos\left(\frac{n\pi x}{3}\right)dx$

$$= \frac{2}{n\pi}\left[\sin\left(\frac{n\pi x}{3}\right)\right]_1^3 = \boxed{-\frac{2}{n\pi}\sin\left(\frac{n\pi}{3}\right)}$$

In particular: $\boxed{a_1 = -\frac{2}{\pi}\sin\left(\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{\pi}.}$ $\boxed{a_2 = -\frac{1}{\pi}\sin\left(\frac{2\pi}{3}\right) = -\frac{\sqrt{3}}{2\pi}}$ $\boxed{a_3 = 0.}$ $\boxed{a_4 = -\frac{1}{2\pi}\sin\left(\frac{4\pi}{3}\right) = +\frac{\sqrt{3}}{4\pi}}$

b) For each $0 \leq x \leq 3$, what is the pointwise limit of the series?

We apply the pointwise convergence theory of Fourier series, which works for full Fourier series. The cosine series is a cosine series, so it is also the full Fourier series of the even extension ϕ_{even}, which is then to be extended by 6-periodicity.



By the pointwise convergence theorem (ϕ_{even} being piecewise continuous, as well as ϕ_{even}'), we have:

$$\boxed{\frac{a_0}{2} + \sum_{n=0}^{N} a_n\cos\left(\frac{n\pi x}{3}\right) \xrightarrow[N\to\infty]{} \begin{cases} 0 & \text{for } x = 0. \text{ (because } \phi_{even} \text{ is continuous at } 0\text{.)} \\ 0 & \text{for } x < 1 \quad \text{(idem)} \\ \frac{0+1}{2} = \frac{1}{2} & \text{for } x = 1 \text{ (there is a jump discontinuity)} \\ 1 & \text{for } 1 < x < 3 \\ \frac{1+0}{2} = \frac{1}{2} & \text{for } x = 3. \end{cases}}$$

3) Does the series converge in the L^2-sense?

Yes: we apply the second convergence theorem with $\int_0^3 \phi^2(x)\,dx = \int_1^3 1\,dx = 2 < +\infty$.

4) By using the pointwise convergence result with $x = 0$, compute the sum: $1 + \frac{1}{2} - \frac{1}{4} - \frac{1}{5} + \frac{1}{7} + \frac{1}{8} - \frac{1}{10} - \frac{1}{11} + \dots$

We have: $\frac{a_0}{2} + \sum_{n=0}^{N} a_n \xrightarrow[N\to+\infty]{} 0$, by the result of question 2).

Thus $\frac{2}{3} + \left(-\frac{\sqrt{3}}{\pi} - \frac{\sqrt{3}}{2\pi} + \frac{\sqrt{3}}{4\pi} + \frac{\sqrt{3}}{5\pi} - \frac{\sqrt{3}}{7\pi} - \dots\right) = 0.$

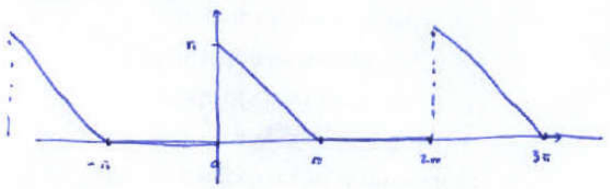
by iterating the computation of question 1).

Hence: $\boxed{1 + \frac{1}{2} - \frac{1}{4} - \frac{1}{5} + \frac{1}{7} + \frac{1}{8} - \frac{1}{10} - \frac{1}{10} + \dots = \frac{2\pi}{3\sqrt{3}}}$

Exercise 2

Let us consider the 2π-periodic function, defined over $[0, 2\pi[$ by: $f(x) = \begin{cases} \pi - x & \text{for } x \in [0, \pi[\\ 0 & \text{for } x \in [\pi, 2\pi[. \end{cases}$

1) Draw the graph of f over $\mathbb{R}$



2) Compute the coefficients of the full Fourier series of f.

We have $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi}(\pi - x)\,dx = \frac{1}{\pi}\left(\pi^2 - \frac{\pi^2}{2}\right) = \frac{\pi}{2}$

and for $n \geq 1$, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = \frac{1}{\pi}\int_0^{\pi}(\pi - x)\cos(nx)\,dx$

$$= \frac{1}{\pi}\left(\left[\cancel{(\pi - x)\frac{\sin(nx)}{n}}\right]_0^{\pi} + \frac{1}{n}\int_0^{\pi}\sin(nx)\,dx\right)$$

$$= \frac{1}{n^2\pi}\left[-\cos(nx)\right]_0^{\pi}$$

$$\boxed{= \frac{1}{n^2\pi}\left(1 - (-1)^n\right).}$$

$$= \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{2}{n^2\pi} & \text{if } n \text{ is odd.} \end{cases}$$

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx = \frac{1}{\pi}\int_0^{\pi}(\pi - x)\sin(nx)\,dx$

$$= \frac{1}{\pi}\left(\left[-(\pi - x)\frac{\cos nx}{n}\right]_0^{\pi} - \frac{1}{n}\int_0^{\pi}\cos(nx)\,dx\right)$$

$$= \frac{1}{n\pi}(\pi) - \cancel{\frac{1}{n^2}\left[\sin(nx)\right]_0^{\pi}}$$

$$= \frac{1}{n}.$$

The Fourier series of f is then: $\boxed{\frac{\pi}{4} + \sum_{p=0}^{+\infty}\frac{2}{(2p+1)^2\pi}\cos((2p+1)x) + \sum_{n=1}^{\infty}\frac{\sin(nx)}{n}.}$

3) What is the pointwise limit of the full Fourier series of f, for $x \in [-\pi,\pi]$?

We apply the theory for pointwise convergence of full Fourier series - which is possible since f is piecewise continuous and the derivative of f also.

The periodic extension of f to $\mathbb{R}$, say f_{per}, is continuous at any point $x \in [-\pi,\pi]$, except at $x=0$ where the one-sided limits are $f(0^-)=0$, $f(0^+)=\pi$.
Thus, the Fourier series converge to $\begin{cases} f(x) & \text{if } x \neq 0 \\ \frac{0+\pi}{2} = \frac{\pi}{2} & \text{at } x=0. \end{cases}$

4) By using the pointwise convergence series for $x=0$, evaluate the series $\sum_{p=0}^{\infty} \frac{1}{(2p+1)^2}$

We have: $\frac{\pi}{4} + \frac{2}{\pi}\sum_{p=0}^{\infty} \frac{1}{(2p+1)^2} = \frac{\pi}{2}$.

Thus: $\boxed{\sum_{p=0}^{\infty} \frac{1}{(2p+1)^2} = \frac{\pi^2}{8}}$

5) Calculate, depending on whether n is even or odd, the value of $\sin(n\frac{\pi}{2})$.
Use the result to compute the value of $\sum_{p=0}^{\infty} \frac{(-1)^p}{2p+1}$, by evaluating the previous series at a particular point.

- $\sin(n\frac{\pi}{2}) = 0$ if n is even.
- If n is odd, say $n = 2p+1$, then $\sin(n\frac{\pi}{2}) = \sin(p\pi + \frac{\pi}{2}) = (-1)^p$.

Now, let us evaluate the series at $x = \frac{\pi}{2}$. We know that $\cos\left(\frac{(2p+1)\pi}{2}\right) = 0$ for $p = 0, \dots, \infty$, and, by using the pointwise convergence result of question 2) at $x = \frac{\pi}{2}$:

$\frac{\pi}{2} = \frac{\pi}{4} + \sum_{n=1}^{\infty} \frac{\sin(n\frac{\pi}{2})}{n}$,

whence: $\sum_{p=0}^{\infty} \frac{\sin((2p+1)\frac{\pi}{2})}{2p+1} = \frac{\pi}{4}$, and eventually $\boxed{\sum_{p=0}^{\infty} \frac{(-1)^p}{2p+1} = \frac{\pi}{4}}$

6) Show that, for $x \in]0,\pi]$, the following equality holds: $\sum_{n=1}^{\infty} \frac{\sin(nx)}{n} = \frac{\pi}{4} + \frac{2}{\pi}\sum_{p=0}^{\infty} \frac{\cos((2p+1)x)}{(2p+1)^2}$ (

We know that then $-x \in [-\pi, 0[$, and that f_{per} is continuous at x, with value 0. Thus:

$0 = \frac{\pi}{4} + \frac{2}{\pi}\sum_{p=0}^{\infty} \frac{\cos((2p+1)(-x))}{(2p+1)^2} - \sum_{n=1}^{\infty} \frac{\sin(nx)}{n}$, which gives the desired equality.

Exercise 3

1) Compute the Fourier sine series of $\phi(x) = x$ over the interval $(0,\ell)$

$\forall n \geq 1$, $b_n = \frac{2}{\ell}\int_0^{\ell} x \sin(\frac{n\pi x}{\ell})\,dx$

$= \frac{2}{\ell}\left(\frac{\ell}{n\pi}\left[-x\cos(\frac{n\pi x}{\ell})\right]_0^{\ell} + \frac{\ell}{n\pi}\int_0^{\ell} \cos(\frac{n\pi x}{\ell})\,dx\right)$

$= \frac{2}{n\pi}(-\ell\cos(n\pi)) + \cancel{\frac{2}{\ell}\frac{\ell^2}{n^2\pi^2}\left[\sin(\frac{n\pi x}{\ell})\right]_0^{\ell}} = \boxed{\frac{2\ell}{n\pi}(-1)^{n+1}}$

2) Does the series converge in the L^2-sense?

We apply the L^2 theory for Fourier series: as $\int_0^{\ell} x^2\,dx = \frac{\ell^3}{3} < +\infty$, the sine Fourier series converges in the L^2-sense

3) Apply Parseval's equality to compute the sum $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

By Parseval's theorem, we have $\int_0^{\ell} \phi(x)^2\,dx = \sum_{n=1}^{\infty} b_n^2 \int_0^{\ell} \sin^2(\frac{n\pi x}{\ell})\,dx$; but we know that $\int_0^{\ell} \sin^2(\frac{n\pi x}{\ell})\,dx = \int_0^{\ell} \frac{1-\cos(\frac{2n\pi x}{\ell})}{2}\,dx$

$= \frac{\ell}{2} - \frac{1}{2}\cancel{\left[\frac{\ell}{2n\pi}\sin(\frac{2n\pi x}{\ell})\right]_0^{\ell}}$

$= \frac{\ell}{2}$.

so: $\frac{\ell^3}{3} = \sum_{n=1}^{\infty} \frac{4\ell^2}{n^2\pi^2}\cdot\frac{\ell}{2}$; i.e.: $\boxed{\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}}$

Exercise 4 Let $\phi(x) = |x|$ in $(-\pi,\pi)$. Show that, if we approximate ϕ by the function $f(x) = \frac{1}{2}a_0 + a_1\cos x + b_1\sin x + a_2\cos 2x + b_2\sin 2x$, in the L^2-sense, the best choice of constants is $a_0 = \pi$, $a_1 = -\frac{4}{\pi}$, $b_1 = a_2 = b_2 = 0$.

We go back to the lecture over the L^2-theory for Fourier series.
The expression of f makes us think of a truncated full Fourier series of ϕ.
The contents of the lectures state that: if $X_n : (-\pi,\pi)$ are any set of orthogonal functions, then the constants c_n that minimize the least-square error $\left\|\phi - \sum_{n=0}^{N} c_n X_n\right\|_{L^2}$ are given by $c_n = \frac{(\phi, X_n)}{\|X_n\|^2}$.

We apply this result with $X_0 = 1$, $X_1 = \cos x$, $X_2 = \sin x$, $X_3 = \cos 2x$, $X_4 = \sin 2x$, which are orthogonal, as we have seen during the lectures.
The best choice is then given by:

$c_0 = \frac{a_0}{2} = \frac{(\phi, 1)}{\|1\|^2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} |x| = \frac{1}{\pi}\int_0^{\pi} x\,dx = \frac{\pi}{2}$, whence $\boxed{a_0 = \pi}$

$a_1 = \frac{(\phi, \cos x)}{\|\cos x\|^2}$, but $\int_{-\pi}^{\pi} |x|\cos x\,dx = 2\int_0^{\pi} x\cos x\,dx$ by evenness.

$= 2\cancel{\left[x\sin x\right]_0^{\pi}} - 2\int_0^{\pi} \sin x\,dx = 2\left[\cos x\right]_0^{\pi} = -4$.

and $\int_{-\pi}^{\pi} \cos^2 x\,dx = \int_{-\pi}^{\pi} \frac{1}{2} + \frac{\cos(2x)}{2}\,dx = \pi$, so $\boxed{a_1 = -\frac{4}{\pi}}$

Now $b_1 = \frac{1}{\|\sin x\|^2} \int_{-\pi}^{\pi} |x| \sin x \, dx \boxed{= 0}$, because the integrand function is odd.

For the same reason, $\boxed{b_2 = 0}$.

· Eventually, we have:

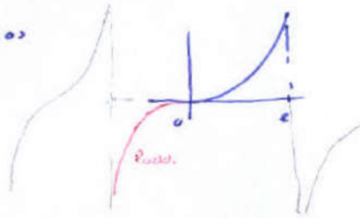
$$a_2 = \frac{1}{\|\cos 2x\|^2} \int_{-\pi}^{\pi} |x| \cos 2x \, dx, \quad \text{where } \|\cos(2x)\|^2 = \int_{-\pi}^{\pi} \cos^2(2x)\,dx = \frac{1}{2}\int_{-\pi}^{\pi} (1+\cos(4x))\,dx = \pi.$$

$$\text{and } \int_{-\pi}^{\pi} |x| \cos 2x \, dx = 2\int_0^{\pi} x\cos(2x)\,dx = 2\left[\frac{x \sin(2x)}{\cancel{2}}\right]_0^{\pi} - \int_0^{\pi} \sin(2x)\,dx$$

$$= \left[\frac{\cos(2x)}{2}\right]_0^{\pi} = 0, \quad \text{so that } \boxed{a_2 = 0}.$$

[Exercise 5] For each of the following functions, consider its sine Fourier series expansion on the specified interval (it is not asked to calculate the expansion!). Specify whether the series converge uniformly, pointwise (and in this case, calculate the pointwise limit at any point, including the endpoints), or in the L^2-sense.

1) $f(x) = x^3$ on $(0, \ell)$



. This function does not satisfy the BC associated to sine Fourier series on $(0,\ell)$ (because $f(\ell) \neq 0$).
Hence, the Fourier series does not converge uniformly on $(0,\ell)$.

. This function satisfies $\int_0^{\ell} f^2(x)\,dx = \int_0^{\ell} x^6\,dx = \frac{\ell^7}{7} < +\infty$.

Hence the series converges in the L^2-sense.

· For the pointwise convergence, remember that the theory applies to full Fourier series.
The sine Fourier series of f is the full Fourier series of its odd extension f_{odd}.

We then extend f_{odd} by 2ℓ-periodicity.

f_{odd} and f_{odd}' being piecewise continuous, the theory applies, and the sine Fourier series $\sum_{n=1}^{\infty} b_n \sin(\frac{n\pi x}{\ell})$ converges for any $x \in [0, \ell]$ (we could of course compute its limit at points $x \in [-\ell, \ell]$ or $\mathbb{R}$, but this is not the question).

to $\begin{cases} 0 \text{ if } x = 0, \text{ because } f_{odd} \text{ is continuous at } 0 \\ f(x) \text{ if } 0 < x < \ell \text{ because } f_{odd} \text{ is continuous at } x \\ \frac{f_{odd}(\ell^-) + f_{odd}(\ell^+)}{2} = 0 \text{ if } x = \ell. \end{cases}$

2) $f(x) = \ell x - x^2$

This function is such that f, f', and f'' are continuous on $(0,\ell)$.
What's more $f(0) = f(\ell) = 0$, so that f fulfills the BC of sine Fourier series.
Hence, $\sum_{n=0}^{\infty} b_n \sin(\frac{n\pi x}{\ell}) \to f(x)$ uniformly on $(0,\ell)$.

As a consequence, it also converges in the L^2-sense to f,
and pointwise to $f(x)$, i.e. $\forall x \in [0,\ell]$ (including the endpoints) $\sum_{n=0}^{\infty} b_n \sin(\frac{n\pi x}{\ell}) \to f(x)$.

3) $f(x) = \frac{1}{x^2}$

. This function does not satisfy the BC of sine Fourier series: the series do not converge uniformly to f.
· $\int_0^{\ell} \frac{dx}{x^2}$ is infinite! The sine Fourier series do not converge to f in the L^2-sense.

. The theorem about pointwise convergence cannot be applied either, because the odd extension of f is not piecewise continuous (the jump of f_{odd} at 0 is infinite).

[Exercise 6] 1) Compute the cosine Fourier series of $f(x) = x^2$ on $(0,\ell)$

We calculate: $a_0 = \frac{2}{\ell}\int_0^{\ell} x^2\,dx = \frac{2\ell^2}{3}$

Then: $\forall n \geq 1$,

$$a_n = \frac{2}{\ell}\int_0^{\ell} x^2 \cos(\tfrac{n\pi x}{\ell})\,dx = \frac{2}{\ell}\left(\cancel{\left[\frac{\ell x^2 \sin(\frac{n\pi x}{\ell})}{n\pi}\right]_0^{\ell}} - \frac{2\ell}{n\pi}\int_0^{\ell} x \sin(\tfrac{n\pi x}{\ell})\,dx\right)$$

$$= \frac{-4}{n\pi}\int_0^{\ell} x \sin(\tfrac{n\pi x}{\ell})\,dx$$

$$= \frac{-4}{n\pi}\left(\left[\frac{-\ell}{n\pi} x \cos(\tfrac{n\pi x}{\ell})\right]_0^{\ell} + \frac{\ell}{n\pi}\int_0^{\ell} \cos(\tfrac{n\pi x}{\ell})\,dx\right)$$

$$= \frac{-4}{n\pi}\left(-\frac{\ell^2}{n\pi}\cos(n\pi) + \cancel{\frac{\ell^2}{n\pi^2}\left[\sin(\tfrac{n\pi x}{\ell})\right]_0^{\ell}}\right)$$

$$\boxed{= \frac{4\ell^2}{n^2\pi^2}(-1)^n}$$

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2) Use the L^2-theory for Fourier series, and in particular Parseval's equality to calculate the sum $\sum_{n=1}^{\infty} \frac{1}{n^4}$.

The L^2-theory applies since: $\int_0^{\ell} f^2(x)\,dx = \int_0^{\ell} x^4\,dx = \frac{\ell^5}{5} < +\infty$.

Parseval equality then implies that

$$\int_0^{\ell} f^2(x)\,dx = \sum_{n=1}^{\infty} a_n^2 \int_0^{\ell} \cos^2(\tfrac{n\pi x}{\ell})\,dx + \left(\frac{a_0}{2}\right)^2 \int_0^{\ell} 1^2\,dx$$

$$= \sum_{n=1}^{\infty} \frac{16\ell^4}{n^4\pi^4}\cdot\frac{\ell}{2} + \frac{\ell^5}{9},$$

with $\int_0^{\ell} \cos^2(\frac{n\pi x}{\ell})\,dx = \int_0^{\ell} \frac{1}{2} + \cancel{\frac{\cos(2\frac{n\pi x}{\ell})}{2}}\,dx = \frac{\ell}{2}$.

i.e.: $\frac{\ell^5}{5} - \frac{\ell^5}{9} = \sum_{n=1}^{\infty} \frac{8\ell^5}{n^4\pi^4}$, and: $\boxed{\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}}$