Exercise 1 Strauss 5.2.11

Homework 7: partial correction

1) Find the full Fourier series of e^x on $(-\ell,\ell)$, first in their complex form, then in their real form.

We first search for the complex coefficients, i.e. those c_n appearing in $\phi(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{\ell}}$.

By definition:

$$c_n = \frac{1}{2\ell}\int_{-\ell}^{\ell} \phi(x) e^{-\frac{in\pi x}{\ell}}\,dx. \quad \text{Here: } c_n = \frac{1}{2\ell}\int_{-\ell}^{\ell} e^{x} e^{-\frac{in\pi x}{\ell}}$$

$$= \frac{1}{2\ell}\int_{-\ell}^{\ell} e^{x(1-\frac{in\pi}{\ell})}\,dx \quad \text{, and because } 1-\frac{in\pi}{\ell} \neq 0,$$

$$= \frac{1}{2\ell}\cdot\frac{1}{1-\frac{in\pi}{\ell}}\left[e^{x(1-\frac{in\pi}{\ell})}\right]_{-\ell}^{\ell}$$

$$= \frac{1}{2\ell-2in\pi}\left[e^{\ell}e^{-in\pi}-e^{-\ell}e^{in\pi}\right]$$

$$= \frac{(-1)^n}{2\ell-2in\pi}\left(e^{\ell}-e^{-\ell}\right)$$

$$= \frac{(-1)^n \sinh(\ell)}{\ell - in\pi}$$

$$\boxed{= \frac{(-1)^n(\ell+in\pi)}{\ell^2+n^2\pi^2}\sinh(\ell)}$$

Now we can find the real coefficients, by using their relations with the complex ones. If $\phi(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(\frac{n\pi x}{\ell}) + b_n \sin(\frac{n\pi x}{\ell})$, the coefficients write:

$$\forall n = 0, \dots, \quad a_n = \frac{1}{\ell}\int_{-\ell}^{\ell}\phi(x)\cos(\frac{n\pi x}{\ell})\,dx$$

$$= 2\,\mathrm{Re}\left(\frac{1}{2\ell}\int_{-\ell}^{\ell}\phi(x)e^{-i\frac{n\pi x}{\ell}}\,dx\right), \text{ that is}$$

$$= 2\,\mathrm{Re}(c_n)$$

$$\boxed{= \frac{(-1)^n 2\ell}{\ell^2+n^2\pi^2}\sinh(\ell)} \text{ by the previous formula}$$

Similarly, $\boxed{b_n = 2\,\mathrm{Im}(c_n) = \frac{(-1)^n 2n\pi}{\ell^2+n^2\pi^2}\sinh(\ell)}$

Exercise 2 Strauss 5.2.11

a) odd, periodic with period $\frac{2\pi}{a}$.
b) none of them
c) even if m is even, odd if m is odd.
d) even
e) even, periodic with period $\frac{\pi}{b}$.
f) odd.

Exercise 3 (Strauss ...)

a) It is the case if and only if $\phi(0)=0$.
b) It is always the case
c) The odd extension is naturally differentiable over $(0,\ell)$ and $(-\ell,0)$. A first condition to be diff. over $(-\ell,\ell)$ is of course to be continuous; i.e. by a), one must have $\phi(0)=0$.
The only possible problem arises at $x=0$.
At this point ϕ is left and right differentiable. The right derivative is only $\phi'(0)$.
The left derivative is $\lim_{h\to 0, h>0} \frac{\phi_{odd}(0-h)-\phi(0)}{-h} = \lim_{h\to 0} \frac{-\phi(h)}{-h}$ because of the odd property and because $\phi(0)=0$

$$= \lim_{h\to 0}\frac{\phi(h)-\phi(0)}{h} = \phi'(0).$$

Thus one must have also $\phi'(0) = \phi'(0)$, for ϕ_{odd} to be differentiable (i.e. the left and right derivatives must coincide) which is always the case.
The only condition is thus (as for continuity): $\phi(0)=0$

d) Same as above: the left and right derivatives of ϕ_{even} must coincide at $x=0$, and ϕ_{even} must be continuous at 0 (which we have seen to be always the case).

Thus, one must have: $\phi'(0) = \lim_{h\to 0, h>0}\frac{\phi_{even}(-h)-\phi(0)}{-h}$

$$= -\lim_{h\to 0, h>0}\frac{\phi(h)-\phi(0)}{h} = -\phi'(0).$$

Thus, one must have $2\phi'(0)=0 \;=\; \boxed{\phi'(0)=0}$

St. 5.2.15.

Exercise 4 $\phi(x) = |\sin x|$ is even. By the contents of the lectures, its Fourier series on $(-\ell,\ell)$ only features cosines, i.e. all the $\boxed{b_n=0}$.

Exercise 5 1) Compute the sine Fourier series of $\phi(x)=x$ on $(0,\ell)$.

One has: $b_n = \frac{2}{\ell}\int_0^{\ell}\phi(x)\sin(\frac{n\pi x}{\ell})\,dx$

$$= \frac{2}{\ell}\int_0^{\ell} x\sin(\frac{n\pi x}{\ell})\,dx$$

$$= \frac{2}{\ell}\cdot\frac{\ell}{n\pi}\left[-x\cos(\frac{n\pi x}{\ell})\right]_0^{\ell} + \frac{2}{\ell}\cdot\frac{\ell}{n\pi}\int_0^{\ell}\cos(\frac{n\pi x}{\ell})\,dx.$$

$$= \frac{-2\ell}{n\pi}\cos(n\pi) + \frac{2}{n\pi}\cdot\frac{\ell}{n\pi}\cancel{\left[\sin\frac{n\pi x}{\ell}\right]_0^{\ell}}$$

$$\boxed{= (-1)^{n+1}\frac{2\ell}{n\pi}}$$

2) We now assume that the Fourier sine series of a function over $(0,\ell)$ can be integrated term by term (i.e. that the symbols $\sum_{n=1}^{\infty}$ and $\int_0^{\ell}$ can be interchanged)
Compute the Fourier cosine series of $\frac{x^2}{2}$.

We have $x = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{2\ell}{n\pi}\sin(\frac{n\pi x}{\ell})$

By integration ($\int_0^x$), we have $\int_0^x t\,dt = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{2\ell}{n\pi}\int_0^x \sin(\frac{n\pi t}{\ell})\,dt$. ; here $\int_0^x \sin(\frac{n\pi t}{\ell})\,dt = \frac{\ell}{n\pi}\left[-\cos(\frac{n\pi t}{\ell})\right]_0^x$

$$= \frac{\ell}{n\pi}\left(1-\cos(\frac{n\pi x}{\ell})\right).$$

$$\frac{x^2}{2} = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{2\ell^2}{n^2\pi^2}\left(1-\cos(\frac{n\pi x}{\ell})\right). = \boxed{\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2\ell^2}{n^2\pi^2} + \sum_{n=1}^{\infty}(-1)^{n+2}\frac{2\ell^2}{n^2\pi^2}\cos(\frac{n\pi x}{\ell}).}$$

3) By identifying the constant a_0, find the sum of the series: $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$.

On the one hand: $a_0 = \frac{2}{\ell}\int_0^{\ell} \frac{x^2}{2}\,dx = \frac{\ell^2}{3}$

But, by directly reading the previous series: $\frac{a_0}{2} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2\ell^2}{n^2\pi^2}$. Thus $\boxed{\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}}$

Exercise 6 Find the complex eigenvalues of the operator $[f \mapsto f']$ over functions on $[0,1]$ such that $f(0) = f(1)$. Are the eigenfunctions orthogonal?

We find the complex values λ such that there exists a function $\phi \neq 0$ with $\begin{cases} \phi'(x) = \lambda\phi(x), \\ \phi(0) = \phi(1) \end{cases}$ — [illegible]

The necessary form for ϕ is: $\phi(x) = Ae^{\lambda x}$

we must then have $\phi(0) = \phi(1)$, i.e. $A = Ae^{\lambda}$. and for ϕ not to be trivial ($i.e.$ $\phi \neq 0$):

$e^{\lambda} = 1$

Now, we must be careful, because this equation has plenty of roots in the complex plane.

These roots are exactly the $\boxed{2in\pi = \lambda_n}$, $n = -\infty, \dots, +\infty$.

The associated eigenfunction is $\boxed{X_n(x) = e^{2in\pi x}}$

If $n \neq m$, $(X_n, X_m) = \int_0^1 e^{2in\pi x} \cdot e^{-2im\pi x}\,dx = \int_0^1 e^{2i\pi(n-m)x}\,dx$

$= \frac{1}{2i\pi(n-m)} [1 - 1] = 0.$

Exercise 7 A rod has length $\ell = 1$, and constant $k = 1$.

Its temperature obeys the heat equation $\frac{\partial u}{\partial t} - k\frac{\partial^2 u}{\partial x^2} = 0$, and the left end is at $T = 0$, the right hand at $T = 1$.

The initial temperature is: $\phi(x) = \begin{cases} \frac{5x}{2} & \text{for } 0 < x < \frac{2}{3} \\ 3 - 2x & \text{for } \frac{2}{3} < x < 1 \end{cases}$

1) Find the equilibrium solution U.

We have $\frac{\partial^2 U}{\partial x^2} = 0$, so that $U(x) = ax + b$.

By the boundary conditions $U(0) = b = 0$.

and $U(1) = a = 1$

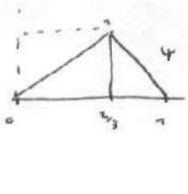
$\boxed{\text{So } U(x) = x}$

2) Now consider $v(t,x) = u(t,x) - U(x)$. What system does it satisfy?

Observe that $v(t,x)$ is again solution of (HE), with homogeneous Dirichlet BC,

and initial condition $v(0,x) = \phi(x) - U(x)$

$= \begin{cases} \frac{3x}{2} & \text{for } 0 < x < \frac{2}{3} \\ 3(1-x) & \text{for } \frac{2}{3} < x < 1. \end{cases} =: \psi(x)$



3) Decompose ψ as a Fourier sine series.

The coeffs reads: $b_n = \frac{2}{\ell}\int_0^{\ell} \psi(x)\sin(n\pi x)$, with $\ell = 1$.

$= \frac{2}{1}\int_0^{2/3} \frac{3x}{2}\sin(n\pi x)\,dx + 2\int_{2/3}^{1} 3(1-x)\sin(n\pi x)\,dx$

$= 3\left(\left[-\frac{x\cos(n\pi x)}{n\pi} \right]_0^{2/3} + \int_0^{2/3} \frac{\cos(n\pi x)}{n\pi}\,dx \right) + 6\left(\left[-\frac{(1-x)\cos(n\pi x)}{n\pi} \right]_{2/3}^{1} - \int_{2/3}^{1} \frac{\cos(n\pi x)}{n\pi}\,dx \right)$

$= \cancel{-\frac{2}{n\pi}\cos\left(\frac{2n\pi}{3}\right)} + \frac{3}{n\pi^2}\left[\sin(n\pi x)\right]_0^{2/3} + \cancel{\frac{2}{n\pi}\cos\left(\frac{2n\pi}{3}\right)} - \frac{6}{n\pi^2}\left[\sin(n\pi x)\right]_{2/3}^{1}$

$\boxed{= -\frac{3}{n\pi^2}\sin\left(\frac{2n\pi}{3}\right)}$

4) By using the method of separation of variables, using the contents of the lectures, decompose the solution $v(t,x)$ as a sine series.

We know indeed that the solution can be expressed as: $v(t,x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x)\, e^{-n^2\pi^2 t}$

5) Express the form of the coefficients, and conclude as for the form of $u(t,x)$.

By matching the coefficients of $v(0,x)$ and those of ψ:

$v(t,x) = -\frac{3}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{n}\sin\left(\frac{2n\pi}{3}\right)\sin(n\pi x)\, e^{-n^2\pi^2 t}$

$\boxed{\text{and } u(t,x) = x - \frac{3}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{n}\sin\left(\frac{2n\pi}{3}\right)\sin(n\pi x)\, e^{-n^2\pi^2 t}}$