Exercise 1

Consider $\phi : [0,1] \to \mathbb{R}$, defined by $\phi(x) = x^2$

1) Calculate the Fourier sine series of ϕ over $[0,1]$

The coefficients read: $b_n = 2\int_0^1 x^2 \sin(n\pi x)\,dx$, for $n = 1, \dots, \infty$.

We proceed by integration by parts:
$$b_n = 2\left[-x^2\frac{\cos(n\pi x)}{n\pi}\right]_0^1 + 2\int_0^1 \frac{2x\cos(n\pi x)}{n\pi}\,dx$$
$$= -\frac{2}{n\pi}\cos(n\pi) + \frac{4}{n\pi}\int_0^1 x\cos(n\pi x)\,dx$$
$$= \frac{2(-1)^{n+1}}{n\pi} + \frac{4}{n\pi}\left(\left[\frac{x\sin(n\pi x)}{n\pi}\right]_0^1 - \int_0^1 \frac{\sin(n\pi x)}{n\pi}\,dx\right)$$
$$= \frac{2(-1)^{n+1}}{n\pi} + \frac{4}{n\pi}\left(0 - \frac{1}{n\pi}\left[-\frac{\cos(n\pi x)}{n\pi}\right]_0^1\right)$$
$$= \frac{2(-1)^{n+1}}{n\pi} + \frac{4}{(n\pi)^3}\left(\cos(n\pi) - 1\right)$$
$$\boxed{= \frac{2(-1)^{n+1}}{n\pi} + \frac{4((-1)^n - 1)}{n^3\pi^3}}$$

2) Calculate the Fourier cosine coefficients of ϕ.

We have, for all $n = 1, \dots, \infty$: $a_n = 2\int_0^1 x^2\cos(n\pi x)\,dx$ and $a_0 = 2\int_0^1 x^2\,dx = 2\left[\frac{x^3}{3}\right]_0^1 = \frac{2}{3}$.
$$= 2\left[\cancel{x^2\frac{\sin(n\pi x)}{n\pi}}\right]_0^1 - \frac{4}{n\pi}\int_0^1 x\sin(n\pi x)\,dx$$
$$= -\frac{4}{n\pi}\left(\left[-x\frac{\cos(n\pi x)}{n\pi}\right]_0^1 + \frac{1}{n\pi}\int_0^1 \cos(n\pi x)\,dx\right)$$
$$= \frac{4}{n^2\pi^2}(-1)^n - \frac{4}{n^2\pi^2}\int_0^1 \cos(n\pi x)\,dx$$
$$= \frac{4(-1)^n}{n^2\pi^2} - \frac{4}{n^3\pi^3}\cancel{\left[\sin(n\pi x)\right]_0^1}$$
$$\boxed{= \frac{4(-1)^n}{n^2\pi^2}.}$$

3) We now assume that the Fourier cosine series of ϕ converge towards ϕ at any point of the interval $[0,1]$. Let us compute the sum $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$.

We write: $\phi(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(n\pi x)$, which is then assumed to hold in the sense that the series converge at any pt $x \in [0,1]$.

With what we computed: $\phi(x) = x^2 = \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos(n\pi x)$.

Evaluating at $x = 0$ (which is valid by assumption of the exercise), we get: $0 = \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$, whence the desired result: $\boxed{\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}}$

Exercise 2

1) Compute the full Fourier series of $\phi : [-\pi,\pi] \to \mathbb{R}$, $x \mapsto |\sin x|$.

We observe that ϕ is even. Then, all the coefficients $b_n = 0$.

Plus one has: $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}|\sin x|\,dx = \frac{2}{\pi}\int_0^{\pi}|\sin x|\,dx = \frac{2}{\pi}\int_0^{\pi}\sin x\,dx$
$$= \frac{2}{\pi}\left[-\cos x\right]_0^{\pi} = \frac{2}{\pi}(1-(-1)) = \boxed{\frac{4}{\pi}.}$$

and, for $n \neq 0$, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}|\sin x|\cos(nx)\,dx = \frac{2}{\pi}\int_0^{\pi}\sin x\cos(nx)\,dx$, and we use:

$\sin(a+b) = \sin a\cos b + \sin b\cos a$

$\sin(a-b) = \sin a\cos b - \sin b\cos a$

$\sin(a+b) + \sin(a-b) = 2\sin(a)\cos b.$

with $a = x$, $b = nx$.
$$= \frac{1}{\pi}\int_0^{\pi}\sin((n+1)x)\,dx + \frac{1}{\pi}\int_0^{\pi}\sin((1-n)x)\,dx$$
$$= \frac{1}{\pi}\left(\int_0^{\pi}\sin(n+1)x\,dx - \int_0^{\pi}\sin((n-1)x)\,dx\right)$$
$$= \frac{1}{\pi}\left(\left[-\frac{1}{n+1}\cos(n+1)x\right]_0^{\pi} - \left[-\frac{1}{n-1}\cos((n-1)x)\right]_0^{\pi}\right)$$
$$= \frac{1}{\pi}\left(\frac{1-(-1)^{n+1}}{n+1} - \frac{1-(-1)^{n-1}}{n-1}\right)$$
$$\boxed{= \begin{cases} 0 \text{ if } n \text{ is odd.} \\ \frac{1}{\pi}\left(\frac{2}{n+1} - \frac{2}{n-1}\right) = \frac{-4}{\pi(n^2-1)} \text{ if } n \text{ is even.}\end{cases}}$$

2) We now assume that the full Fourier series expansion of ϕ converges at any pt.

Then for any $x \in [-\pi,\pi]$, one has:
$$|\sin x| = \frac{a_0}{2} + \sum_{\substack{n=1 \\ n \text{ even}}}^{\infty} a_n\cos(nx)$$
$$= \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2nx)}{4n^2-1}.$$

Evaluating at $x = 0$ yields: $0 = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}$, i.e.: $\boxed{\sum_{n=1}^{\infty}\frac{1}{4n^2-1} = \frac{1}{2}}$

Next, we observe that $\cos(n\pi) = (-1)^n$.

Hence, evaluating the previous equality at $x = \frac{\pi}{2}$, we obtain:

$|\sin(\frac{\pi}{2})| = 1 = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{4n^2-1}$, and: $\boxed{\sum_{n=1}^{\infty}\frac{(-1)^n}{4n^2-1} = \frac{1}{2} - \frac{\pi}{4}}$

Exercise 3

We consider the heat equation $\frac{\partial u}{\partial t} - \kappa \frac{\partial^2 u}{\partial x^2} = 0$. with BC: $\forall t>0$, $\begin{cases} \frac{\partial u}{\partial x}(t,0) + u(t,0) = 0 \\ u(t,1) = 0. \end{cases}$

1) We use the method of separation of variables, and search for separated solutions $u(t,x) = T(t)X(x)$.

i.e.: $\frac{\partial u}{\partial t}(t,x) = T'(t)X(x)$
and: $\frac{\partial^2 u}{\partial x^2}(t,x) = T(t)X''(x)$.
$\Big\}$ whence: $T'(t)X(x) - \kappa T(t)X''(x) = 0$

which implies: $\forall t>0, \forall x \in [0,1]$: $-\frac{T'(t)}{\kappa T(t)} = -\frac{X''(x)}{X(x)}$

As this is an equality between a function of t only and a function of x only, which must hold for any t, x, we have the existence of $\lambda \in \mathbb{R}$ such that

$\forall t>0, \forall x \in (0,1)$: $\begin{cases} T'(t) + \lambda\kappa T(t) = 0 \\ X''(x) + \lambda X(x) = 0 \end{cases}$ with B.C: $\begin{cases} X'(0) + X(0) = 0 \\ X(1) = 0 \end{cases}$

2. We want to know whether $\lambda = 0$ is an eigenvalue of the problem.
We search for a non identically null function X s.t: $X''(x) = 0$ with $X'(0) + X(0) = 0$, $X(1) = 0$

Thus, $X(x)$ is of the form: $X(x) = ax + b$, for $a, b \in \mathbb{R}$ to be found.

$X'(0) + X(0) = 0 \Rightarrow a + b = 0$
$X(1) = 0 \Rightarrow a + b = 0.$
$\Big\}$ i.e. $b = -a$

Conclusion: an eigenfunction associated to $\lambda = 0$ (which is then an eigenvalue) is: $\boxed{X_0(x) = x - 1.}$

3) We now search for positive eigenvalues: $\lambda = \beta^2$, $\beta > 0$
We then search for $X \neq 0$ such that
$\begin{cases} X''(x) + \beta^2 X(x) = 0 \\ X'(0) + X(0) = 0 \\ X(1) = 0 \end{cases}$

and X is of the form: $X(x) = a\cos(\beta x) + b\sin(\beta x)$.
But: $X'(0) + X(0) = 0 \Rightarrow b\beta + a = 0$. i.e. $a = -b\beta$.
$X(1) = 0 \Rightarrow a\cos\beta + b\sin\beta = 0$.

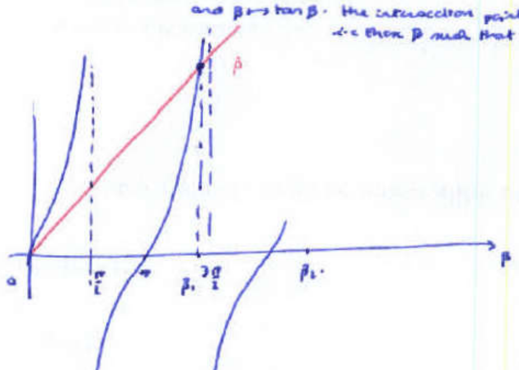
Thus $\boxed{-b\beta\cos\beta + b\sin\beta = 0}$

For β^2 to be an eigenvalue, one must have $b \neq 0$ (else, $X(x) = 0$ $\forall x$, and X is not an eigenfunction).

Thus: $-\beta\cos\beta + \sin\beta = 0$; i.e.: $\boxed{\tan\beta = \beta}$ (Note that we divided by $\cos\beta$, which should then be $\neq 0$).
Actually, if $\cos\beta = 0$, one has $\beta = \frac{(2n+1)\pi}{2}$,

for some $n \in \mathbb{N}$, and $\sin\beta = 0$, because of the above relation.
This is impossible for $\beta = \frac{(2n+1)\pi}{2}$, and this means that $\cos\beta \neq 0$ (if the above equality is satisfied).

4) Let us then show graphically that there are an infinite number of eigenvalues $\lambda = \beta^2$.
We can see it graphically by drawing the graphs of functions $\beta \mapsto \beta$
and $\beta \mapsto \tan\beta$. The intersection points give the solution to the equation $\beta = \tan\beta$, i.e. the β such that $\lambda = \beta^2$ is an eigenvalue of the problem.



The graphical resolution shows that there is a sequence β_n of solutions to this equation, with $n = 1, \dots, \infty$.
For each n: $\beta_n \in \left[\frac{\pi}{2} + (n-1)\pi, \frac{\pi}{2} + n\pi\right]$.

The positive eigenvalues of the problem are of the form $\lambda_n = \beta_n^2$, $n = 1 \dots \infty$,
with $\beta_n \in \left[\frac{\pi}{2} + (n-1)\pi, \frac{\pi}{2} + n\pi\right]$, with the associated eigenfunction:

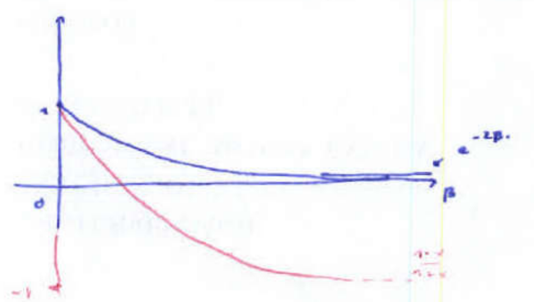
$\boxed{X_n(x) = -\beta_n\cos(\beta_n x) + \sin(\beta_n x)}$

5) Let us search for negative eigenvalues, $\lambda = -\beta^2$, $\beta > 0$.
Then X solves: $X''(x) - \beta^2 X(x) = 0$, and is of the form $X(x) = ae^{\beta x} + be^{-\beta x}$, for some $a, b \in \mathbb{R}$.
But $X'(0) + X(0) = 0 \Rightarrow \beta a - \beta b + (a+b) = 0$. i.e: $a(1+\beta) + b(1-\beta) = 0$.
and $X(1) = 0 \Rightarrow ae^{\beta} + be^{-\beta} = 0. \Rightarrow b = -ae^{2\beta}$.

Then, combining these two equalities, we have: $a(1+\beta) - ae^{2\beta}(1-\beta) = 0$.
For X to be $\neq 0$, one must have $a \neq 0$. and as $\beta > 0$, this implies:

$$e^{-2\beta} = \frac{1-\beta}{1+\beta}$$

We solve this equation graphically:



So that there is no negative eigenvalue.

6) The temporal equation reads: $T'(t) + \lambda_n k\, T(t) = 0$

where $T(t) = Ce^{-\lambda_n k t}$, for some constant C to be specified.

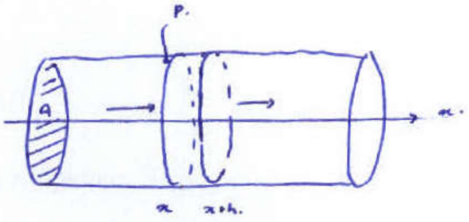
All in all, the series expansion of a solution to the considered system is: $\boxed{u(t,x) = \underbrace{a_0(x-1)}_{= X_0(x)} + \sum_{n=1}^{\infty} X_n(x)\, e^{-\lambda_n k t}.}$

$\boxed{\text{Exercise 4}}$

1) Let us consider a slice of the rod, lying between x and $x+h$.

The total energy in this slice reads $H(t) = A\int_x^{x+h} c\rho\, u(t,x)\,dx$.

This energy varies at time t as:
$$\begin{cases} \text{. a flux of energy } -Ak\frac{\partial u}{\partial x}(t,x) \text{ crosses the border at } x \text{ from the left to the right (and is thus added to this slice).} \\ \text{. a flux of energy } -Ak\frac{\partial u}{\partial x}(t,x+h) \text{ crosses the border at } x+h \text{ from the left to the right, and is thus leaving the slice.} \\ \text{. the system loses } P\alpha\int_x^{x+h} u(t,s)\,ds \end{cases}$$



Thus $\frac{d}{dt}H(t) = -Ak\frac{\partial u}{\partial x}(t,x) + Ak\frac{\partial u}{\partial x}(t,x+h) - Ph\alpha u(t,x)$

i.e. $A\int_x^{x+h} c\rho\frac{\partial u}{\partial t}(t,x)\,dx = -Ak\frac{\partial u}{\partial x}(t,x) + Ak\frac{\partial u}{\partial x}(t,x+h) - P\alpha\int_x^{x+h} u(t,s)\,ds$

and derivating with respect to h, at $h=0$ yields: $c\rho\frac{\partial u}{\partial t}(t,x) = k\frac{\partial^2 u}{\partial x^2}(t,x) - \frac{\alpha P}{A}u(t,x)$

$\boxed{\text{i.e.: } c\rho\frac{\partial u}{\partial t} - k\frac{\partial^2 u}{\partial x^2} + \frac{\alpha P}{A}u = 0}$

2) Suppose that an equilibrium temperature does exist.

Then: $-k\frac{\partial^2 u_{eq}}{\partial x^2} + \alpha u_{eq} = 0$.

i.e.: $\frac{\partial^2 u_{eq}}{\partial x^2} - \frac{\alpha}{k}u_{eq} = 0$. Thus u_{eq} is of the form: $u_{eq} = ce^{+\sqrt{\frac{\alpha}{k}}x} + de^{-\sqrt{\frac{\alpha}{k}}x}$, for some constants c, d to be found.

as $u_{eq}(0) = u_{eq}(\ell) = 0$, we have: $\begin{cases} c + d = 0 \\ c\sqrt{\frac{\alpha}{k}} - d\sqrt{\frac{\alpha}{k}} = 0 \end{cases}$, whence $c = d = 0$.

$\boxed{\text{The equilibrium temperature is } u_{eq} = 0}$

3) We search for separated solutions $u(t,x) = T(t)X(x)$.

Thus: $\left.\begin{array}{l} \frac{\partial u}{\partial t}(t,x) = T'(t)X(x) \\ \frac{\partial^2 u}{\partial x^2}(t,x) = T(t)X''(x) \end{array}\right\}$ this leads to: $T'(t)X(x) - kT(t)X''(x) + \alpha T(t)X(x) = 0$.

whence: $\frac{-T'(t)}{kT(t)} = -\frac{X''(x) - \frac{\alpha}{k}X(x)}{X(x)}$ $\forall t>0, \forall x \in (0,\ell)$.

As before, this implies the existence of a constant $\lambda \in \mathbb{R}$ such that: $\boxed{\begin{cases} T'(t) + k\lambda T(t) = 0. \\ -X''(x) + \frac{\alpha}{k}X(x) = \lambda X(x), \text{ and } X(0) = 0,\ X(\ell) = 0. \end{cases}}$

4. Is $\lambda = \frac{\alpha}{k}$ an eigenvalue of the problem?

For $\lambda = \frac{\alpha}{k}$ to be an eigenvalue of the problem, there should exist $X \neq 0$ such that $\begin{cases} -X''(x) + \cancel{\frac{\alpha}{k}X(x)} = \cancel{\frac{\alpha}{k}X(x)} \\ X(0) = 0 = X(\ell). \end{cases}$

But X is then necessarily of the form: $X(x) = ax + b$.

And because $X(0) = X(\ell) = 0$, one has $a = b = 0$, whence $X = 0$. Thus, $\lambda = \frac{\alpha}{k}$ is not an eigenvalue of the problem.

5) For $\lambda = \frac{\alpha}{k} + \beta^2$ to be an eigenvalue, one should have: $X \neq 0$ s.t.
$$\begin{cases} -X''(x) + \frac{\alpha}{k}X(x) = \frac{\alpha}{k}X(x) + \beta^2 X(x). \\ X(0) = X(\ell) = 0. \end{cases}, \text{ i.e. } \begin{cases} X''(x) + \beta^2 X(x) = 0. \\ X(0) = X(\ell) = 0. \end{cases}$$

As in the lectures, this leads to the fact that β should be of the form $\beta_n = \frac{n\pi}{\ell}$, $n = 1, \dots, \infty$; hence, the eigenvalues of the form $\lambda = \frac{\alpha}{k} + \beta^2$, are of the form:

$\boxed{\lambda_n = \frac{\alpha}{k} + \left(\frac{n\pi}{\ell}\right)^2, \text{ and the associated eigenfunctions are } X_n(x) = \sin\left(\frac{n\pi x}{\ell}\right).}$

6) For $\lambda = \frac{\alpha}{k} - \beta^2$ to be an eigenvalue, there should exist $X \neq 0$ s.t.:
$$\begin{cases} X''(x) - \beta^2 X(x) = 0. \\ X(0) = X(\ell) = 0. \end{cases}$$

As in the lectures, the unique solution to this system is $X = 0$.

Thus there is no eigenvalue of the form $\lambda = \frac{\alpha}{k} - \beta^2$.

7) For $n = 1, \dots, \infty$, the temporal part $T_n(t)$ associated to λ_n solves: $T_n'(t) + k\lambda_n T(t) = 0$,

i.e.: $T(t) = Ce^{-\lambda_n k t}$, for some constant $C \in \mathbb{R}$.

$\boxed{\text{i.e.: } T(t) = Ce^{-\alpha t}e^{-\frac{k\pi^2 n^2}{\ell^2}t}}$

The expansion for a general solution to the system reads: $\boxed{u(t,x) = e^{-\alpha t}\sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{\ell}\right)e^{-\frac{kn^2\pi^2}{\ell^2}t}}$

8) We observe that $\alpha > 0$ fastens the decrease of u to the equilibrium state $u_{eq} = 0$.

i.e. it implies that $\lim_{t \to \infty} u(t,x) = 0$ (actually the decrease is exponential).