Elementary partial differential equations: homework 5

Assigned 03/11/2014, due 03/25/2014.

Exercise 1

(1) Find the solution to the PDE
$$\frac{\partial u}{\partial t} + x\frac{\partial u}{\partial x} + u = 3x$$
of a function $u \equiv u(t,x)$ of two variables, such that $u(0,x) = 2x$, $x \in \mathbb{R}$.

(2) *(Optional)* Find the solution to the PDE
$$x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} + y^2 u = y^2$$
of a function $u \equiv u(x,y)$ of two variables, such that $u(x,1) = 1$, $x \in \mathbb{R}$.

Exercise 2

This exercise is reprinted from [Strauss], §2.3*, Exercises* $6-7$.

(1) Consider the one-dimensional heat equation:

$$\frac{\partial u}{\partial t} - \kappa\frac{\partial^2 u}{\partial x^2} = 0, \tag{1}$$

over the space interval $x \in (0,\ell)$, and for $t > 0$. Prove the *comparison principle*: if u and v are two solutions to (1) such that:
$$u(t,x) \leq v(t,x) \text{ for } \begin{cases} (t,x) = (0,x) & x \in [0,\ell] \\ (t,x) = (t,0) & t > 0 \\ (t,x) = (t,\ell) & t > 0 \end{cases},$$
then $u(t,x) \leq v(t,x)$ for all $x \in [0,\ell]$, $t > 0$.

(2) Consider now two solutions u, v to the following heat equations:
$$\frac{\partial u}{\partial t} - \kappa\frac{\partial^2 u}{\partial x^2} = f, \quad \frac{\partial v}{\partial t} - \kappa\frac{\partial^2 v}{\partial x^2} = g,$$
posed for values $x \in (0,\ell)$, and for $t > 0$, associated to different sources f, g. We assume that $f \leq g$, and that u and v satsify:
$$u(t,x) \leq v(t,x) \text{ for } \begin{cases} (t,x) = (0,x) & x \in [0,\ell] \\ (t,x) = (t,0) & t > 0 \\ (t,x) = (t,\ell) & t > 0 \end{cases}.$$
Prove that $u(t,x) \leq v(t,x)$ for all $x \in [0,\ell]$, $t > 0$.

[Hint: during the lectures, we have proved the maximum principle for the heat equation without sources. Observe that it stays true if there is a negative source. The proof of this fact is exactly the same as the one seen during the lectures.]

(3) *(Optional)* Let $v(t,x)$ be a function that fulfills the inequality:
$$\frac{\partial v}{\partial t} - \frac{\partial v}{\partial x^2} \geq \sin(x), \text{ for } 0 \leq x \leq \pi,\ t > 0.$$
We also assume that:
$$\forall t > 0,\ v(t,0) \geq 0,\ v(t,\pi) \geq 0,$$
and:
$$\forall x \in (0,\ell),\ v(0,x) \geq \sin(x).$$

Use question (2) to show that:

$$\forall t > 0,\ \forall x \in (0, \ell),\ \ v(t,x) \geq (1 - e^{-t}) \sin(x).$$

Exercise 3

This exercise is reprinted from [Strauss], §2.4, Exercise 15.

Consider the one-dimensional heat equation with source f:

$$\frac{\partial u}{\partial t} - \kappa \frac{\partial^2 u}{\partial x^2} = f(t,x), \tag{2}$$

over the space interval $x \in (0, \ell)$, and for $t > 0$, with Neumann non homogeneous boundary conditions:

$$\forall t > 0,\ \ \frac{\partial u}{\partial x}(t,0) = g(t),\ \ \frac{\partial u}{\partial x}(t,\ell) = h(t), \tag{3}$$

and initial condition:

$$\forall x \in (0, \ell),\ \ u(0,x) = \phi(x). \tag{4}$$

Show that the system (2-3-4) has at most one solution (i.e. if the solution exists, it is unique), by using the energy method as during the lectures.

Exercise 4

This exercise is reprinted from [Strauss], §2.4, Exercise 16.

Consider the one-dimensional heat equation with *constant dissipation*:

$$\frac{\partial u}{\partial t} - \kappa \frac{\partial^2 u}{\partial x^2} + bu = 0,$$

over the whole real line $x \in \mathbb{R}$, for $t > 0$, where $b > 0$ is a constant, and with initial condition:

$$\forall x \in (0, \ell),\ \ u(0,x) = \phi(x).$$

(1) Define the function $v(t,x)$ by the relation:

$$\forall x \in \mathbb{R},\ \forall t > 0,\ u(t,x) = e^{-bt} v(t,x).$$

Compute the partial derivatives of v in terms of those of u, and show that v satisfies the 'classical' heat equation over the real line.

(2) By using the content of the lectures over the diffusion equation over the real line, find the expressions of $v(t,x)$ and $u(t,x)$.

(3) Use the same process as in Questions $(1-2)$ to find the expression of the solution to the heat equation with *variable dissipation*:

$$\frac{\partial u}{\partial t} - \kappa \frac{\partial^2 u}{\partial x^2} + bt^2 u = 0,$$

where $b > 0$ is a constant, and with initial conditions

$$\forall x \in (0, \ell),\ \ u(0,x) = \phi(x),$$

by performing the change of variable functions

$$\forall x \in \mathbb{R},\ \forall t > 0,\ u(t,x) = e^{-\frac{bt^3}{3}} v(t,x).$$

Can you figure how those change of variable functions have been devised ?

Exercise 5

This exercise is partly reprinted from [Strauss], §4.1, Exercise 4.

This exercise is devoted to the study of the wave equation in a resistant medium:

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} + r\frac{\partial u}{\partial t} = 0, \tag{5}$$

where $t > 0$, $x \in (0, \ell)$, and $0 < r < \frac{2\pi c}{\ell}$ is the *resistance coefficient*. This system is endowed with Dirichlet homogeneous boundary conditions:

$$\forall t > 0,\ u(t, 0) = 0,\ u(t, \ell) = 0, \tag{6}$$

and the initial conditions read:

$$\forall x \in (0, \ell),\ u(0, x) = \phi(x),\ \frac{\partial u}{\partial t}(0, x) = \psi(x), \tag{7}$$

for given functions ϕ, ψ.

(1) By using the energy method, show that the system (5 - 6 - 7) admits at most one solution.
[Hint: assume there exists two solutions u_1, u_2 to the system, and consider their difference $w = u_2 - u_1$, which satsifies (5) with null initial conditions. Multiply both sides of (5) by $\frac{\partial w}{\partial t}$ and perform integration by parts so as to show that the energy $E(t) := \frac{1}{2}\int_0^\ell \left(\left(\frac{\partial w}{\partial t}\right)^2 + c^2\left(\frac{\partial w}{\partial x}\right)\right) dx$ decreases.]

(2) We now study this system by the method of separation of variables. Let $u(t, x) = T(t)X(x)$ be a solution to (5-6) under separated form, if any. Show that:

$$\forall t > 0, \forall x \in (0, \ell),\ -\frac{T''(t) + rT'(t)}{c^2 T(t)} = -\frac{X''(x)}{X(x)} = \lambda, \tag{8}$$

for some constant λ, and:

$$X(0) = X(\ell) = 0.$$

(3) Show that a separated solution to (5 - 6) which differs from 0 exists only in the case:

$$\lambda = \lambda_n = \left(\frac{n\pi}{\ell}\right)^2,\ n = 1, 2, ...,$$

and calculate the associated eigenfunctions $X_n(x)$.

(4) Calculate the associated time function $T_n(t)$ for each value of λ_n by solving the related ODE, stemming from relation (8).

(5) By using the superposition principle, write down the general series expansion for the solutions to (5 - 6) provided by the above study. What are the initial conditions ϕ, ψ associated to these solutions, i.e. those initial conditions you have actually solved the system (5 - 6-7) for ?

Exercise 6

This exercise is reprinted from [Strauss], §4.2, Exercise 4.

In this exercise, we study the one-dimensional heat equation in a rod of length 2ℓ:

$$\frac{\partial u}{\partial t} - \kappa\frac{\partial^2 u}{\partial x^2} = 0, \tag{9}$$

where $t > 0$, $-\ell < x < \ell$. This system comes with *periodic* boundary conditions:

$$\forall t > 0,\ u(t, -\ell) = u(t, \ell),\ \frac{\partial u}{\partial x}(t, -\ell) = \frac{\partial u}{\partial x}(t, \ell). \tag{10}$$

(1) Show that the eigenvalues of the problem are:

$$\lambda_n = \left(\frac{n\pi}{\ell}\right)^2,\ n = \underline{0},1, 2, ...,$$

and compute the associated eigenfunctions $X_n(x)$.

(2) Show that the general form of solutions to (9 - 10) is provided by the following series:

$$u(t, x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi x}{\ell}\right) + b_n \sin\left(\frac{n\pi x}{\ell}\right)\right) e^{-\frac{n^2\pi^2\kappa t}{\ell^2}},$$

for some coefficients a_n, $n = 0, ...,$ b_n, $n = 1, ...$