Elementary partial differential equations: partial correction of Homework 4

Exercise 2

We consider the solution $u(t,x)$ to the wave equation:

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0.$$

(1) Let us define the function $v(t,x)$ as: $v(t,x) = u(t, x-y)$, and let us compute its partial derivatives, owing to the chain rule:

$$\frac{\partial v}{\partial t}(t,x) = \frac{\partial u}{\partial t}(t,x-y), \;\; \frac{\partial^2 v}{\partial t^2}(t,x) = \frac{\partial^2 u}{\partial t^2}(t,x-y),$$
$$\frac{\partial v}{\partial x}(t,x) = \frac{\partial u}{\partial x}(t,x-y), \;\; \frac{\partial^2 v}{\partial x^2}(t,x) = \frac{\partial^2 u}{\partial x^2}(t,x-y).$$

Thus:

$$\left(\frac{\partial^2 v}{\partial t^2} - \frac{\partial^2 v}{\partial x^2}\right)(t,x) = \left(\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2}\right)(t,x-y) = 0,$$

and v is also a solution to the wave equation (with different initial data, of course !).

(2) Let us define the function $v(t,x)$ as: $v(t,x) = \frac{\partial u}{\partial x}(t,x)$, we have:

$$\frac{\partial^2 v}{\partial t^2}(t,x) = \frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial t^2}(t,x)\right), \frac{\partial^2 v}{\partial x^2}(t,x) = \frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial x^2}(t,x)\right),$$

Thus:

$$\left(\frac{\partial^2 v}{\partial t^2} - \frac{\partial^2 v}{\partial x^2}\right)(t,x) = \frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2}\right)(t,x) = 0,$$

and v is also a solution to the wave equation.

(3) Let us define the function $v(t,x)$ as: $v(t,x) = u(at, ax)$, and let us compute its partial derivatives, owing to the chain rule:

$$\frac{\partial v}{\partial t}(t,x) = a\frac{\partial u}{\partial t}(at,ax), \;\; \frac{\partial^2 v}{\partial t^2}(t,x) = a^2\frac{\partial^2 u}{\partial t^2}(at,ax),$$
$$\frac{\partial v}{\partial x}(t,x) = a\frac{\partial u}{\partial x}(at,ax), \;\; \frac{\partial^2 v}{\partial x^2}(t,x) = a^2\frac{\partial^2 u}{\partial x^2}(at,ax).$$

Thus:

$$\left(\frac{\partial^2 v}{\partial t^2} - \frac{\partial^2 v}{\partial x^2}\right)(t,x) = a^2\left(\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2}\right)(at,ax) = 0,$$

and v is also a solution to the wave equation.

Exercise 4

Let us consider the PDE:

$$3\frac{\partial^2 u}{\partial t^2} + 10\frac{\partial^2 u}{\partial x \partial t} + 3\frac{\partial^2 u}{\partial x^2} = \sin(x+t), \tag{1}$$

(1) This is a simple computation which resembles very much the factorization of second-order polynomial equations; for any function $u(t,x)$, one has:

$$\begin{aligned} 3\tfrac{\partial^2 u}{\partial t^2} + 10\tfrac{\partial^2 u}{\partial x\partial t} + 3\tfrac{\partial^2 u}{\partial x^2} &= 3\left(\tfrac{\partial^2 u}{\partial t^2} + \tfrac{10}{3}\tfrac{\partial^2 u}{\partial x\partial t} + \tfrac{\partial u}{\partial x^2}\right) \\ &= 3\left(\tfrac{\partial^2 u}{\partial t^2} + \tfrac{10}{3}\tfrac{\partial^2 u}{\partial x\partial t} + \tfrac{25}{9}\tfrac{\partial^2 u}{\partial x^2} - \tfrac{16}{9}\tfrac{\partial u}{\partial x^2}\right), \end{aligned}$$

where the second line consists in making appear the end of the perfect square term corresponding to the first two terms. We can now write:

$$3\frac{\partial^2 u}{\partial t^2}+10\frac{\partial^2 u}{\partial x\partial t}+3\frac{\partial^2 u}{\partial x^2}=3\left(\left(\frac{\partial}{\partial t}+\frac{5}{3}\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t}+\frac{5}{3}\frac{\partial}{\partial x}\right)u-\frac{16}{9}\frac{\partial^2 u}{\partial x^2}\right),$$

which leads immediately to the desired equation:

$$\left(\frac{\partial}{\partial t}+\frac{5}{3}\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t}+\frac{5}{3}\frac{\partial}{\partial x}\right)u-\frac{16}{9}\frac{\partial^2 u}{\partial x^2}=\frac{1}{3}\sin(x+t).$$

(2) The purpose of this question is to find a new set of variables (ξ,η) such that, for *any* twice differentiable function u, one has:

$$\frac{\partial u}{\partial \xi}=\frac{\partial u}{\partial t}+\frac{5}{3}\frac{\partial u}{\partial x},\ \text{ and }\ \frac{\partial u}{\partial \eta}=\frac{4}{3}\frac{\partial u}{\partial x}. \tag{2}$$

But, we know that, because of chain rule, for *any* such change of variables, one has:

$$\frac{\partial u}{\partial \xi}=\frac{\partial u}{\partial t}\frac{\partial t}{\partial \xi}+\frac{\partial u}{\partial x}\frac{\partial x}{\partial \xi},\ \text{ and }\ \frac{\partial u}{\partial \eta}=\frac{\partial u}{\partial t}\frac{\partial t}{\partial \eta}+\frac{\partial u}{\partial x}\frac{\partial x}{\partial \eta}. \tag{3}$$

Consequently, so that (2) holds, by identification with the general expression (3), it is enough that the new set of variables (ξ,η) satisfies:

$$\frac{\partial t}{\partial \xi}=1,\ \frac{\partial x}{\partial \xi}=\frac{5}{3},$$

and

$$\frac{\partial t}{\partial \eta}=0,\ \frac{\partial x}{\partial \eta}=\frac{4}{3}.$$

These formulae bring only constants into play, which suggests a linear change of variables. Let us search for (ξ,η) under the form:

$$t=a\xi+b\eta,\ x=c\xi+d\eta,$$

for some constants a,b,c,d to be found. In view of the above equations, one simply has:

$$a=\frac{\partial t}{\partial \xi}=1,\ b=\frac{\partial t}{\partial \eta}=0,$$

and

$$c=\frac{\partial x}{\partial \xi}=\frac{5}{3},\ d=\frac{\partial x}{\partial \eta}=\frac{4}{3}.$$

Eventually, the new set of variables is defined by:

$$t=\xi,\ \text{ and } x=\frac{5}{3}\xi+\frac{4}{3}\eta,$$

whence, inverting this system to get (ξ,η) in terms of (t,x):

$$\xi=t,\ \text{ and } \eta=\frac{5}{4}t+\frac{3}{4}x.$$

Now, with these new variables, (2) holds by construction, and (1) rewrites:

$$\frac{\partial^2 u}{\partial \xi^2}-\frac{\partial^2 u}{\partial \eta^2}=\frac{1}{3}\sin(t+x)=\frac{1}{3}\sin\left(\frac{8}{3}\xi+\frac{4}{3}\eta\right). \tag{4}$$

(3) Denote $v(\xi,\eta)=A\sin\left(\frac{8}{3}\xi+\frac{4}{3}\eta\right)$, for some constant A to be found. Then,

$$\frac{\partial^2 v}{\partial \xi^2}(\xi,\eta)=-\frac{64}{9}A\sin\left(\frac{8}{3}\xi+\frac{4}{3}\eta\right),\ \text{ and }\ \frac{\partial^2 v}{\partial \eta^2}(\xi,\eta)=-\frac{16}{9}A\sin\left(\frac{8}{3}\xi+\frac{4}{3}\eta\right).$$

Consequently:

$$\frac{\partial^2 v}{\partial \xi^2}(\xi,\eta)-\frac{\partial^2 v}{\partial \eta^2}(\xi,\eta)=-\frac{48}{9}A\sin\left(\frac{8}{3}\xi+\frac{4}{3}\eta\right).$$

Therefore, a particular solution to (4) is $v(t,x)=-\frac{3}{48}\sin\left(\frac{8}{3}\xi+\frac{4}{3}\eta\right)$.

(4) The PDE (1) is linear. Thus, its general solution is given by the sum of the general solution to the associated homogeneous equation

$$\frac{\partial^2 u}{\partial \xi^2} - \frac{\partial^2 u}{\partial \eta^2} = 0 \tag{5}$$

and of a particular solution to (1). By the results of the lectures over the wave equation, we know that the general solution to (5) is of the form:

$$f(\xi - \eta) + g(\xi + \eta),$$

where f and g are two arbitrary twice differentiable functions. Hence, the general solution of (1) is of the form:

$$u(\xi, \eta) = f(\xi - \eta) + g(\xi + \eta) - \frac{3}{48} \sin\left(\frac{8}{3}\xi + \frac{4}{3}\eta\right),$$

or, in term of t, x:

$$u(t, x) = f\left(\frac{1}{4}t + \frac{3}{4}x\right) + g\left(\frac{9}{4}t + \frac{3}{4}x\right) - \frac{3}{48} \sin(t + x),$$

where f and g are two arbitrary twice differentiable functions.

Exercise 5

Our purpose is to solve the PDE:

$$\frac{\partial^2 u}{\partial t^2} - 2\frac{\partial^2 u}{\partial x \partial t} - 3\frac{\partial^2 u}{\partial x^2} = 0. \tag{6}$$

As suggested, let us remark that the differential operator at play here admits the following factorization, for any twice differentiable function u:

$$\frac{\partial^2 u}{\partial t^2} - 2\frac{\partial^2 u}{\partial x \partial t} - 3\frac{\partial^2 u}{\partial x^2} = \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - 3\frac{\partial}{\partial x}\right) u.$$

We now proceed exactly as during the lecture over the wave equation, and remark that, introducing

$$v = \frac{\partial u}{\partial t} - 3\frac{\partial u}{\partial x}, \tag{7}$$

one has:

$$\frac{\partial v}{\partial t} + \frac{\partial v}{\partial x} = 0. \tag{8}$$

Thus, solving the second-order equation (6) boils down to solving two first-order linear equations:

$$\begin{cases} \frac{\partial v}{\partial t} + \frac{\partial v}{\partial x} &= 0 \\ \frac{\partial u}{\partial t} - 3\frac{\partial u}{\partial x} &= v \end{cases}$$

But we know that the general solution to (8) is of the form:

$$v(t, x) = f(x - t),$$

where f is an arbitrary function (we have been solving this transport equation at least a dozen times during the lectures, but if you are not confident with it, use the method of characteristics !). Now, (7) becomes:

$$\frac{\partial u}{\partial t} - 3\frac{\partial u}{\partial x} = f(x - t). \tag{9}$$

As we have seen before, this is an inhomogeneous first-order linear PDE. Its solution is thus the sum of a particular solution, and of the general solution of the the associated homogeneous equation $\frac{\partial u}{\partial t} - 3\frac{\partial u}{\partial x} = 0$.

The general solution to this homogeneous equation is $u_{hom}(t, x) = g(x + 3t)$, where g is an arbitrary function (same as above, use the method of characteristics !). We search now for a particular solution to (9) under the form $u_{part}(t, x) = h(x - t)$, where h is a function to be found. But, under this form:

$$\frac{\partial u_{part}}{\partial t} - 3\frac{\partial u_{part}}{\partial x} = -h'(x - t) - 3h'(x - t) = -4h'(x - t).$$

We may therefore take: $u_{part}(t,x) = -\frac{1}{4}F(x-t)$, where F is a primitive of f over $\mathbb{R}$. But, since f is arbitrary, so is F.

All things considered, the solution $u(t,x)$ to (6) is of the form:

$$u(t,x) = f(x-t) + g(x+3t), \tag{10}$$

where f, g are two arbitrary twice differentiable functions.

If we eventually assume that $u(0,x) = x^2$ and $\frac{\partial u}{\partial t}(0,x) = e^x$, we have, using (10) in combination with chain rule:

$$\forall x \in \mathbb{R},\ \ f(x) + g(x) = x^2,\ \text{and}\ -f(x) + 3g(x) = e^x.$$

Hence,

$$f(x) = \frac{1}{4}\left(3x^2 - e^x\right),\ \text{and}\ g(x) = \frac{1}{4}\left(x^2 + e^x\right),$$

and:

$$u(t,x) = \frac{1}{4}\left(3(x-t)^2 - e^{(x-t)}\right) + \frac{1}{4}\left((x+3t)^2 + e^{(x+3t)}\right).$$