Elementary partial differential equations: homework 3

Assigned 02/11/2014, due 02/18/2014.

Warning The fact that several questions may be posed inside a single item does *not* mean that you may choose which one of them you answer to. All of them must be answered.

Exercise 1

This exercise is partly reprinted from [Strauss], §1.6, *Exercises* 1 − 2*.*

(1) What are the types of the following two second-order PDE ?
   (a) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} - 3\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial u}{\partial y} - \frac{\partial^2 u}{\partial y^2} = 0.$
   (b) $9\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial x \partial y} + 3\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial x} = 0.$

(2) Find the regions of the (x, y) plane where the PDE:

$$(1+x)\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 0$$

of a function $u \equiv u(x, y)$ is elliptic, parabolic, and hyperbolic. Sketch those regions.

Exercise 2

This exercise is reprinted from [Strauss], §1.6, *Exercise* 6*.*

Consider the partial differential equation:

$$3\frac{\partial u}{\partial y} + \frac{\partial^2 u}{\partial x \partial y} = 0,$$

where the unknown $u \equiv u(x, y)$ is a function of two real variables.

(1) What is the order of this PDE ? What is its type ?
(2) Find the general solution of this PDE. *(Hint: make the change of unknown function* $v := \frac{\partial u}{\partial y}$*, and solve first the corresponding PDE of the unknown function* v*).*
(3) If this equation is supplemented with the boundary conditions:

$$\forall x \in \mathbb{R},\ \ u(x, 0) = e^{-3x},\ \text{ and } \frac{\partial u}{\partial y}(x, 0) = 0;$$

does it admit a solution ? Is this solution unique ?

Exercise 3

This exercise is partly reprinted from [Strauss], §1.5, *Exercise* 5*.*

Consider the following PDE:

$$\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} = 0,$$

of a function $u \equiv u(x, y)$ of two variables, together with the boundary condition $u(x, 0) = \varphi(x)$, where $\varphi(x)$ is a given function.

(1) What is the general solution to this equation (i.e. without considering boundary conditions) ?
(2) Show that, if $\varphi(x) \equiv x$, the considered system has no solution.
(3) Show that, if $\varphi(x) \equiv 1$, the considered system has an infinity of solutions.

Exercise 4

This exercise is partly reprinted from [Strauss], §1.5, Exercise 4.

Consider the Laplace problem with homogeneous Neumann boundary conditions, posed in a three-dimensional domain Ω:

$$(1) \qquad \left\{ \begin{array}{ll} \Delta u = f & \text{in } \Omega \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega \end{array} \right. ,$$

where $f \equiv f(x,y,z)$ is a given source term, and n is the unit normal vector to $\partial\Omega$, pointing outward Ω.

(1) What can be added to any solution u of the problem (1), so as to end up with another solution of this problem ? Conclude that there is no unicity of solutions to (1).

(2) Assume that there exists a solution u to (1). By using the Green formula, show that *necessarily*, one then has:

$$\int_\Omega f(x,y,z)\, dx\, dy\, dz = 0.$$

Hence, does a solution to (1) always exist ?

(3) Provide a physical interpretation to your answers to questions (1) and (2).

Exercise 5

This exercise is partly reprinted from [Haberman], §1.4, Exercise 3.

Consider a three dimensional rod oriented along the x-axis. The rod is very thin, and lies in the region $(0 < x < 2)$, so that we assume its temperature u only depends on the time t and on x. It is composed of two parts (see Figure 1):

- The first one, lying between $0 < x < 1$, is filled with a homogeneous material with properties:

$$c_1\rho_1 = 1, \;\; \kappa_1 = 1,$$

  and is submitted to a thermal source of intensity $Q = 1$. We denote by $u(t,x)$, $t > 0$, $0 < x < 1$ the temperature in this part.

- The second one, lying between $1 < x < 2$ is filled with a homogeneous material with properties:

$$c_2\rho_2 = 2, \;\; \kappa_2 = 2,$$

  and is submitted to no thermal source. We denote by $v(t,x)$, $t > 0$, $1 < x < 2$ the temperature in this part.

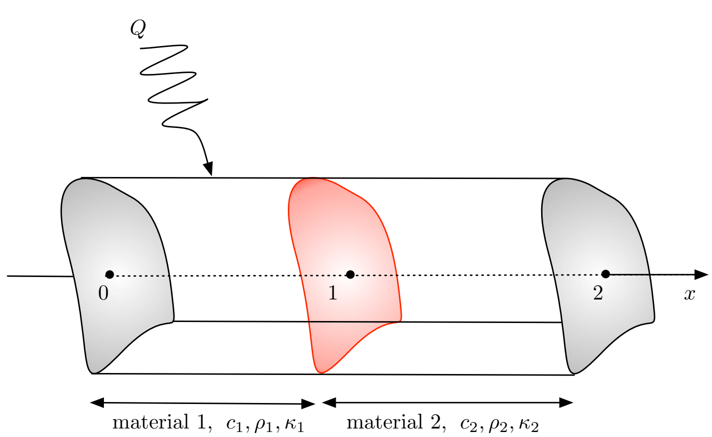

Figure 1. Setting for Exercise 5.

The system is endowed with homogeneous Dirichlet boundary conditions:

$$\forall t > 0, \;\; u(t,0) = 0, \;\; v(t,2) = 0,$$

and the two materials are assumed to be in perfect thermal contact, i.e.:

$$\forall t > 0,\ u(t,1) = v(t,1),\ \kappa_1 \frac{\partial u}{\partial x}(t,1) = \kappa_2 \frac{\partial v}{\partial x}(t,1) \tag{2}$$

(1) Provide a physical interpretation of the conditions (2). In the mechanical literature, such conditions are called *transmission boundary conditions*.
(2) Write the two PDEs satisfied by u and v.
(3) We now assume that a steady state $(u(x), v(x))$ exists for this system. Find the expression of $u(x)$, $0 < x < 1$ and $v(x)$, $1 < x < 2$.