Elementary partial differential equations: partial correction of Homework 1

Correction of Exercise 3

Solve the partial differential equation

$$-(1+x^2)\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}=0 \tag{1}$$

by using the method of characteristics, where the unknown function u is a function of two variables $u \equiv u(x,y)$.

1[**st**] **step: derivation of the characteristic curves:** Let $s \mapsto (x(s), y(s))$ be any curve in the space of variables. We are interested in the value $z(s) := u(x(s), y(s))$ taken by the solution u to (1) along this curve. In particular, we would link to find particular curves along which this variation has a simple expression.

To this end, note that the derivative of the value $z(s)$ along any such curve has the following expression:

$$z'(s)=x'(s)\frac{\partial u}{\partial x}(x(s),y(s))+y'(s)\frac{\partial u}{\partial y}(x(s),y(s)), \tag{2}$$

which follows from a use of chain rule.

Now, inspired by the equation (1), a natural choice seems to consider the curves $s \mapsto (x(s), y(s))$ which satisfy the following relations:

$$\begin{cases} x'(s) = & -(1+x^2(s)) \\ y'(s) = & 1 \end{cases}, \tag{3}$$

which are the *characteristic equations* for (1). They consist of a system of two ODE, which we now have to solve. The solution reads:

$$\begin{cases} \arctan(x(s)) & = & -s+c_1 \\ y(s) & = & s+c_2 \end{cases},$$

where c_1, c_2 are arbitrary constants. Now, up to changing the parameter s into $t := s + c_2$, one can rewrite the characteristic equations as:

$$\begin{cases} \arctan(x(t)) & = & -t+c_2+c_1 \\ y(t) & = & t \end{cases},$$

and, now denoting again as s the parameter of curves (so as to keep notations simple), and $c := c_1 + c_2$ the arbitrary constant, we have found the expression for the characteristic curves:

$$\begin{cases} \arctan(x(s)) & = & -s+c \\ y(s) & = & s \end{cases}. \tag{4}$$

Note that this is a system of non intersecting curves in the (x, y) plane, parametrized by a *single* constant c. Now, a point in the (x, y) plane can be equivalently described by means of (see Figure 1)

- either its usual x, y coordinates,
- or the parameter c of the characteristic curves in which it lies, and the parameter s of the point on this particular curve.

Of course, both representations are related by means of the characteristic equations (4). Note also that each characteristic curve (indexed by c) can be represented as a usual $x \mapsto y$ function: $y = -\arctan(x) + c$.

2[**nd**] **step: computation of the value $z(s)$ of the solution u along the characteristic curves:** This step is generally easier than the first one and requires the solution of another ODE. Consider any fixed characteristic curve $s \mapsto (x(s), y(s))$ given by (4) (indexed by the parameter c). Because of the relation (3) for the characteristic curves, and of the general relation (2), we have:

$$z'(s) = 0,$$

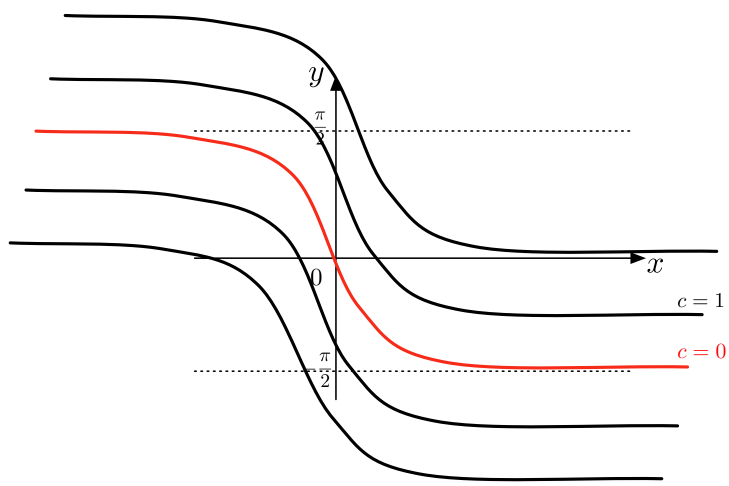

FIGURE 1. Several characteristic curves for equation (1).

an admittedly very simple ODE! Thus, $z(s) = u(x(s), y(s))$ is constant along the considered curve, and its value of course depends on the particular curve we are solving on. Namely, there exists an arbitrary function $f(c)$ such that:

$$u(x(s), y(s)) = f(c).$$

The above relation should be understood as follows: for any point (x, y) in the space of variables, the value $u(x(s), y(s))$ equals $f(c)$, where c is the parameter of the curve (x, y) is lying on.

3rd step: expression of $u(x, y)$ in terms of x, y: We have $u(x, y) = f(c)$, where c is the parameter of the curve (x, y) lies on, and f is an arbitrary function. But we know from (4) that a point $x(, y)$ lies on the characteristic curve with parameter $c = y + \arctan(x)$. Thus,

$$u(x, y) = f(y + \arctan(x)), \tag{5}$$

where f is an arbitrary function, which can only be determined by the specification of boundary conditions, in addition to the considered PDE (1).

Is this over ? Certainly not ! A very easy thing is left to do, and is crucial in practice: you should **check** that the solution found above is indeed solution to (1). To this end, take the expression (5) you just proved, compute the partial derivatives of u using chain-rule, and make sure it does satisfy (1).

Correction of Exercise 5

Solve the partial differential equation

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + u = e^{x+2y} \tag{6}$$

of an unknown function of two variables $u \equiv u(x, y)$, where $c \in \mathbb{R}$ is a fixed parameter, and with the additional 'boundary condition' $u(x, 0) = f(x)$, where f is a given function.

Let us proceed along the lines of the previous corrected exercise (a little bit faster however !).

1st step: derivation of the characteristic curves: Given the form of the considered PDE, the characteristic curves $s \mapsto (x(s), y(s))$ of (6) satisfy the following equations:

$$\begin{cases} x'(s) = & 1 \\ y'(s) = & 1 \end{cases}, \tag{7}$$

whence:

$$\begin{cases} x(s) = & s + c_1 \\ y(s) = & s + c_2 \end{cases},$$

where c_1, c_2 are two arbitrary constants. Up to changing s into $s + c_1$, one can rewrite the characteristic equations as:

$$\begin{cases} x(s) = & s \\ y(s) = & s + c \end{cases}, \tag{8}$$

where c is an arbitrary constant. Note that the characteristic curves are actually parallel straight lines, since combining the two above equations yields: $y = x + c$. This ends this first step.

2^{nd} step: computation of the value $z(s) := u(x(s), y(s))$ of the solution u along the characteristic curves: To this end, considered a given characteristic curve, indexed by the parameter d. We know, from the very definition of the characteristic curves (7) that:

$$\begin{aligned} z'(s) &= x'(s)\frac{\partial u}{\partial x}(x(s), y(s)) + y'(s)\frac{\partial u}{\partial y}(x(s), y(s)) \\ &= \frac{\partial u}{\partial x}(x(s), y(s)) + \frac{\partial u}{\partial y}(x(s), y(s)) \end{aligned};$$

thus, we get:

$$z'(s) + z(s) = e^{x(s)+2y(s)} = e^{3s+2c}. \tag{9}$$

We now have to solve this ODE.

- The solution to the corresponding homogeneous equation $z' + z = 0$ is $z_{\text{hom}}(s) = Ae^{-s}$, where A is an arbitrary constant.
- To find the general solution to the ODE, we rely on the method of variation of constants (or another one if this one is not your favorite !): we search the solutions to (9) under the form: $z_{gen}(s) = A(s)e^{-s}$, where $A(s)$ is a function yet to be determined. For such a function, we have:

  $$z'_{gen}(s) + z_{gen}(s) = A'(s)e^{-s} - A(s)e^{-s} + A(s)e^{-s} = A'(s)e^{-s}.$$

  Hence, if z_{gen} is to solve (9), one should have: $A'(s)e^{-s} = e^{3s+2c}$, that is, $A'(s) = e^{4s+2c}$, whence:

  $$A(s) = \frac{1}{4}e^{4s+2c} + B,$$

  where B is an arbitrary constant. Eventually, the general solution to (9) reads:

$$z_{gen}(s) = A(s)e^{-s} = \frac{1}{4}e^{3s+2c} + Be^{-s}. \tag{10}$$

Let us now go back to our PDE. The value function $z(s) = u(x(s), y(s))$ is thus of the form (10), but the constant B appearing there depends on the particular characteristic curve we are considering, and is consequently a function of the parameter c.

All in all, there exists an arbitrary function, say g, such that the solution $u(x, y)$ to (6) reads:

$$u(x, y) = \frac{1}{4}e^{3s+2c} + g(c)e^{-s}.$$

3^{rd} step: expression of $u(x, y)$ in terms of x, y: This is easy since $s = x$ and $c = y - x$ (see (8)). Consequently:

$$u(x, y) = \frac{1}{4}e^{x+2y} + g(y - x)e^{-y},$$

where g is an arbitrary function. At this point, **you should check your answer by differentiating the obtained expression !**

Eventually, if $u(x, 0) = f(x)$, where f is a given (known) function, then, we must have, as for g:

$$u(x, 0) = \frac{1}{4}e^{x} + g(-x) = f(x).$$

From this relation, one easily gets:

$$g(x) = f(-x) - \frac{1}{4}e^{-x},$$

and:

$$u(x, y) = \frac{1}{4}e^{x+2y} - \frac{1}{4}e^{x-2y} + f(x - y)e^{-y}.$$