Advanced Calculus I: Workshop 8

Exercise 1

The purpose of this exercise is to prove the following fundamental property of compact sets:

The infimum and the supremum of a non empty compact set $K \subset \mathbb{R}$ belong to K.

(1) Give an example of a (non compact) subset A of $\mathbb{R}$ which does not contain its supremum $\sup(A)$.

Let now $K \subset \mathbb{R}$ be a non empty compact set.

(2) Explain why the supremum and infimum $\sup(K)$ and $\inf(K)$ exist.
(3) Show that there exists a sequence $\{x_n\}$ of elements of K which converges to $\sup(K)$.
(4) Infer that $\sup(K)$ is an element of K.

Exercise 2 The goal of this exercise is to prove the following statements about open subsets of $\mathbb{R}$:

- Any union of open subsets of $\mathbb{R}$ is open.
- An intersection of a *finite* number of open subsets of $\mathbb{R}$ is open.

(1) By using the definition of an open set, show that, if $\{O_\lambda\}_{\lambda \in \Lambda}$ is any family of open subsets of $\mathbb{R}$, then the union $\bigcup_{\lambda \in \Lambda} O_\lambda$ is also open.
(2) Show that if $O_1, ...O_N$ are N open subsets of $\mathbb{R}$, the (finite) intersection $\bigcap_{n=1}^{N} O_n$ is also open.
(3) For any $n \in \mathbb{N}^*$, consider the set $O_n = (-\frac{1}{n}, \frac{1}{n})$. Identify the infinite intersection $\bigcap_{n=1}^{\infty} O_n$. Is it open?
(4) *[This question is subsidiary and is not part of the workshop session]* By using the fact that a set $A \subset \mathbb{R}$ is open if an only if its complement is closed, show that:
   - Any intersection of closed subsets of $\mathbb{R}$ is closed.
   - A union of a *finite* number of closed subsets of $\mathbb{R}$ is closed.