Advanced Calculus I: Workshop 7

Exercise 1

Let $a < b$ be two real numbers, and $f : (a, b) \to \mathbb{R}$ be an *increasing function*, that is:

$$\forall x, y \in \mathbb{R},\ \ x \leq y \Rightarrow f(x) \leq f(y).$$

We also assume that f is bounded over (a, b).

(1) Give an example of such a bounded and increasing function defined over some interval $(a, b) \subset \mathbb{R}$ of your choice.

(2) Recall the proof (seen during the lectures) of the following fact:
'If $\{a_n\}$ is an increasing sequence of real numbers which is bounded from above, then it is convergent.'

(3) Why do the following supremum and infimum exist:

$$\inf\left(\{f(x), x \in (a, b)\}\right),\ \ \sup\left(\{f(x), x \in (a, b)\}\right)?$$

in the following, they are denoted as m and M respectively.

(4) Drawing inspiration from the answer to (2), show that f has limit M at b and limit m at a.

Exercise 2

(1) Show that, for any real numbers x, y, one has:

$$\min(x, y) = \frac{x + y - |x - y|}{2},\ \text{and}\ \max(x, y) = \frac{x + y + |x - y|}{2}.$$

(2) Let $D \subset \mathbb{R}$, and $f : D \to \mathbb{R}$. Define the maximum function $\max(f, g)$ as:

$$\forall x \in D, (\max(f, g))(x) = \max(f(x), g(x)),$$

and similarly for the minimum function $\min(f, g)$. Infer from (1) that, if f and g are two continuous functions on D, then so are $\min(f, g)$, $\max(f, g)$.